Surface area on, and the local geometry of, sub-Riemannian manifolds

based on joint work with **Sebastiano Don**, University of Brescia

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The role of surface area in Geometric Measure Theory and Calculus of Variations

[Federer, Bulletin A.M.S., 1978]

It took **five decades**, beginning with Carathéodory's fundamental paper on measure theory in 1914, to develop the intuitive conception of an m dimensional surface as a mass distribution into an efficient instrument of mathematical analysis, capable of significant applications in the calculus of variations.

The **first three decades** were spent learning basic facts on how subsets of $\mathbb{R}^n$ behave with respect to m dimensional Hausdorff measure $\mathcal{H}^m$.

During the next **two decades** this knowledge was fused with many techniques from analysis, geometry and algebraic topology, finally to produce new and sometimes surprising but classically acceptable solutions to old problems. ....

Area of rectifiable sets

The surface area of rectifiable sets can be computed in arbitrary metric spaces.

Theorem [B. Kirchheim, P.A.M.S., 1994], [L. Ambrosio and B. Kirchheim, Math. Ann., 2000]

Consider an injective Lipschitz map $f : A \to X$ from $A \subset \mathbb{R}^k$ to a metric space X. Then we have

$$\mathcal{H}^k(f(A)) = \int_A Jf(x)dx,$$

where the *metric Jacobian $Jf(x)$* is suitably defined in terms of the *metric differential mdf_x* of f at x.

Homogeneous Lie groups as noncommutative extensions of Euclidean spaces

In rough terms, a **homogeneous group** can be seen as $\mathbb{R}^n$ equipped with a *suitable noncommutative group operation*

$$x \cdot y = P(x, y) \in \mathbb{R}^n,$$

a *non-Euclidean distance* $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$, that satisfies

$$d(z \cdot x, z \cdot y) = d(x, y) \quad \text{and} \quad d(\delta_r x, \delta_r y) = r\, d(x, y)$$

for every $x, y, z \in \mathbb{R}^n$ and $r > 0$, where we have

suitable *dilations* $\delta_r : \mathbb{R}^n \to \mathbb{R}^n$ given by a diagonal matrix

$$\begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \ddots & r^j & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & r^N \end{pmatrix}.$$

Fractal nature of noncommutative homogeneous groups

We have the local estimate

$$d(x,y) \leq C|x-y|^{1/N}$$

for x, y in a compact set and $N > 1$
and the exponent cannot be improved!

Unrectifiability of homogeneous groups

The Hausdorff dimension of $(\mathbb{R}^n, d)$ is strictly greater than n. As a consequence

$(\mathbb{R}^n, d)$ **cannot be** n**-rectifiable,**

along with many submanifolds!
Purely k-unrectifiable stratified groups can be precisely characterized, [V.M., Arch. Math. 2004].

Failure of Besicovitch covering theorem

Besicovitch in the Euclidean theory of finite perimeter sets

It is well known from the classical Euclidean theory of finite perimeter sets that Besicovitch's covering theorem naturally appers to convert density estimates into the area formula

$$\|\partial E\| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E.$$

Besicovitch's covering theorem fails to hold

A. Korányi–H. M. Reimann and independently
E. Sawyer–R. L. Wheeden around 1991-1992 first proved a counterexample to the Besicovitch covering property.
S. Rigot and E. Le Donne provide more results, proving the existence of distances with BCP and that for steps higher than two all homogeneous distances do not have BCP.

In a few words, some classical tools of Geometric Measure Theory in Euclidean and metric spaces fail to hold.

Measure-theoretic area formula

Theorem [V.M., P. Royal Soc. Ed., 2015]

Under rather general (but technical) regularity assumptions on the metric space (X, d) and on a Borel measure μ, if $\Sigma \subset X$ is a Borel set and $\mu << \mathcal{S}^\alpha \llcorner \Sigma$, then

$$\mu(B) = \int_B \mathfrak{s}^\alpha(\mu, \cdot)\, d\mathcal{S}^\alpha \llcorner \Sigma \tag{1}$$

for any Borel set $B \subset \Sigma$, where $\mathfrak{s}^\alpha(\mu, p)$ is the *spherical Federer α-density* that has the following explicit formula

$$\mathfrak{s}^\alpha(\mu, p) = \inf_{\varepsilon > 0} \sup \left\{ \frac{2^\alpha\, \mu(\mathbb{B})}{\mathrm{diam}(\mathbb{B})^\alpha} : \mathbb{B} \text{ is a closed ball}, p \in \mathbb{B}, \mathrm{diam}\mathbb{B} \leq \varepsilon \right\}.$$

More recent versions of measure-theoretic area formulas are in [V.M. and G.M. Leccese, A.M.P.A. 2021], with weaker technical assumptions.

Area formulas for some classes of smooth submanifolds

[V. Magnani, Calc. Var. 2019 and JGEA 2022]

Let $\mathbb{G} \approx \mathbb{R}^{\mathrm{q}}$ be a homogeneous group and let Σ be a C^1 smooth submanifold of dimension n. Let us assume that one of the following conditions holds: Σ is a hypersurface, Σ is transversal submanifold, or Σ is of class C^2 and the group has step two.
If d is also *rotationally symmetric*, then we have

$$\mathcal{S}_d^N(\Sigma) = \mu_{SR}(\Sigma),$$

where N is the Hausdorff dimension of Σ with respect to d.

The **sub-Riemannian measure** μ_{SR}, introduced in [V.M. and D. Vittone, J. Reine Angew. Math. 2008], gives the **integral representation of the spherical measure**.

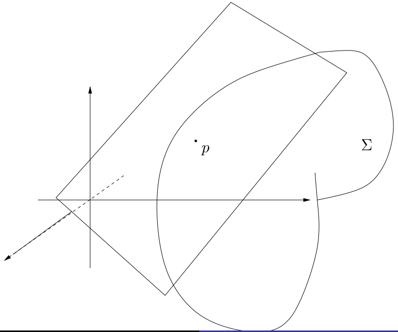
Anisotropic rescaling and upper blow-up

To compute the Federer density, we have to rescale the submanifold with the anisotropic dilations of the group.

We apply the measure-theoretic area formula with $\mu = \mu_{SR}$:

$$\mu(\mathbb{B}) = \mu_{SR}(\mathbb{B} \cap \Sigma)$$

Σ = submanifold, $\mathbb{R}^n \approx \mathbb{G}$ = space, $p \in \Sigma$

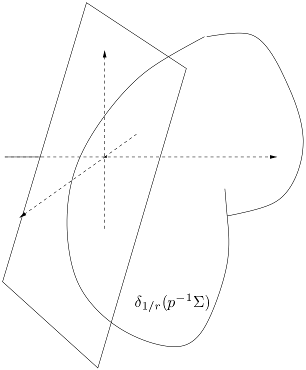


Left translation at the origin and rescaling

We consider the "upper limit" of

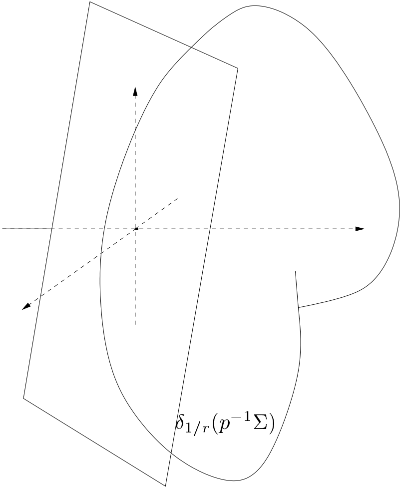
$$\frac{\mu_{SR}(\mathbb{B} \cap \Sigma)}{\mathrm{diam}(\mathbb{B})^N} \quad \text{as} \quad \mathrm{diam}(\mathbb{B}) \to 0^+.$$

$r > 0$ small



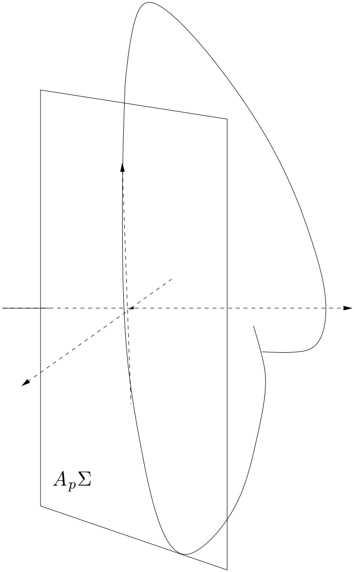
Blow-up and the homogeneous tangent space

$r > 0$ smaller



Blow-up and the homogeneous tangent space

as $r \to 0^+$

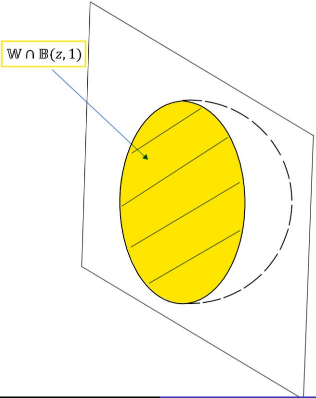


$A_p\Sigma$ is the *homogeneous tangent space* of Σ at p

Spherical factor

Let $\mathbb{W}$ be a p-dimensional subspace of $\mathbb{G}$. Then we define the **spherical factor** of $\mathbb{W}$ with respect to d as

$$\beta_d(\mathbb{W}) = \max_{z \in \mathbb{B}(0,1)} \mathcal{H}^p_E(\mathbb{W} \cap \mathbb{B}(z,1)).$$



The key point to prove the area formula

Objective of the proof

Applying the measure-theoretic area formula, we have only to prove the equality

$$\textbf{spherical factor} \rightarrow \beta_d(A_p\Sigma) = \mathfrak{s}^N(\mu_{SR}, p) \leftarrow \textbf{Federer density}$$

Question

Does the upper blow-up work for **sub-Riemannian manifolds**?

Sub-Riemannian manifolds

We consider

1. a smooth connected n-dimensional manifold M,
2. a smooth distribution of subspaces $\mathcal{D}_p \subset T_pM$ for all $p \in M$,
3. a metric $g_p : \mathcal{D}_p \times \mathcal{D}_p \to \mathbb{R}$ is fixed for all $p \in M$.

Then we define the family of horizontal vector fields as the subclass of vector fields

$$\mathcal{D} \subset \mathfrak{X}(M)$$

such that for every $p \in M$ $\mathcal{D}_p = \mathrm{span}\left\{X(p) \middle| X \in \mathcal{D}\right\}$.
We assume the so-called Chow's condition:

$$\mathrm{Lie}_p(\mathcal{D}) = T_pM \quad \text{for all} \quad p \in M.$$

Then the triple $(M, \mathcal{D}, g)$ is called **sub-Riemannian manifold**.

Sub-Riemannian distance

The Chow's condition $\mathrm{Lie}_p(\mathcal{D}) = T_pM$ for every $p \in M$ implies that for every $p, q \in M$ there exists an absolutely continuous curve $\gamma : [0,1] \to M$ such that

1. $\gamma(0) = p$, $\gamma(1) = q$
2. $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for a.e. $t \in [0,1]$.

Then we say that γ is a horizontal curve.

The sub-Riemannian distance between $p, q \in M$ is

$$d(q,p) = \inf \left\{ \int_0^1 |\dot\gamma(t)|_g dt : \gamma \text{ is horizontal, } \gamma(0) = p,\ \gamma(1) = p \right\}.$$

Volume on sub-Riemannian manifolds

We assume that $(M, \mathcal{D}, g)$ is oriented by a nonvanishing n-form ω.

The volume of a measurable set $E \subset M$ is given by integration:

$$\mathrm{vol}_\omega(E) = \int_E \|\omega\|_g \, d\mathrm{vol}_g .$$

We have defined a sub-Riemannian measure manifold $(M, \mathcal{D}, g, \omega)$.

Equiregular sub-Riemannian manifolds

Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold and define by induction the new families of vector fields

$$\mathcal{D}^1 = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{j+1} = \mathcal{D}^j + [\mathcal{D}^j, \mathcal{D}^1],$$

where we have set

$$[\mathcal{D}^j, \mathcal{D}] = \mathrm{span}\{[X, Y] : X \in \mathcal{D}^j,\ Y \in \mathcal{D}^1\}.$$

For $p \in M$, the fibers associated to these families are

$$\mathcal{D}^j_p = \mathrm{span}\{X(p) : X \in \mathcal{D}^j\}.$$

There exists $s \in \mathbb{N}^+$ such that for every $p \in M$ we have

$$\mathcal{D}^1_p \subset \mathcal{D}^2_p \subset \cdots \subset \mathcal{D}^s_p = T_p(M).$$

A sub-Riemannian manifold $(M, \mathcal{D}, g)$ is **equiregular** if

$$\dim \mathcal{D}^j_p = n_j \in \mathbb{N} \quad \text{for every } p \in M \text{ and } j = 1, \ldots, s.$$

The *Hausdorff dimension Q of an equiregular sub-Riemannian manifold M*, with respect to the sub-Riemannian distance, is given by the formula

$$Q = \dim(\mathcal{D}) + \sum_{i=1}^{\infty} (i+1) \dim(\mathcal{D}^{i+1}/\mathcal{D}^i),$$

see [J. Mitchell, J. Diff. Geo., 1985].

New difficulties in sub-Riemannian manifolds

Comparing sub-Riemannian manifolds with homogeneous groups:

1. **there is no group operation**
2. **there are no global dilations**
3. we do not know if $\mathrm{diam}(\mathbb{B}(p,r)) = 2r$ for r small, **locally uniformly in p**
4. **we have to manage a double blow-up, since the blow-up of the submanifold also entails the blow-up of the sub-Riemannian manifold.**

Uniform estimates on the diameter function

The third point is the more delicate and it is related to the application of the measure-theoretic area formula, that requires an estimate of type $\mathrm{diam}(B(p,r)) \approx 2r$, uniform with respect to p.

From the measure-theoretic area formula to local metric properties of the SR manifold

The demanding step to compute the Federer density is to establish a **uniform asymptotic estimate** for the diameter of a sub-Riemannian ball.

We have to prove the following statement

Let $(M, \mathcal{D}, g)$ be an equiregular sub-Riemannian manifold. Then for every $p \in M$ there exists $h_p > 0$ such that

$$\frac{\mathrm{diam} B(q, r)}{2r} \to 1 \quad \text{as } r \to 0 \tag{2}$$

uniformly as q varies in $B(p, h_p)$.

Issue

The uniform asymptotic estimate (2) requires a **uniform nilpotent approximation**.

A general tool for uniform convergence of SR distances

Theorem 1, [S. Don, V.M., 2023]

Let Ξ be a compact metric space and let $\mathbf{X}^{j,q} = (X_1^{j,q}, \dots, X_{\mathrm{m}}^{j,q})$ and $\mathbf{X}^q = (X_1^q, \dots, X_{\mathrm{m}}^q)$ be families of smooth vector fields on $B_E(x_0, R_0)$ satisfying the Chow's condition for all $q \in \Xi$ and $j \in \mathbb{N}$. We assume that

1. $(x, q) \mapsto \partial_x^\alpha X_i^q(x)$ and $(x, q) \mapsto \partial_x^\alpha X_i^{j,q}(x)$ are continuous in $B_E(x_0, R_0)$ for all $i = 1, \dots, \mathrm{m}$, $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{\mathrm{n}}$,
2. for any $i = 1, \dots, \mathrm{m}$ we have the convergence $X_i^{j,q} \to X_i^q$ as $j \to \infty$, w.r.t. the C^∞_{loc}-topology in $B_E(x_0, R_0)$, **and uniformly w. r. t. q** $\in \Xi$.

Let d_j^q and d^q be the SR distances associated to $\mathbf{X}_i^{j,q}$ and $\mathbf{X}^q$ in $B_E(x_0, R_0)$, respectively. Then we have $0 < r_0 < R_0$ such that d_j^q converges to d^q in $B_E(x_0, r_0)$ w.r.t. the L^∞ topology, as $j \to \infty$ and **uniformly as q varies in the compact space** Ξ.

The pointwise case, where $\Xi = \{q\}$ and $B_E(x_0, R)$ is replaced by $\mathbb{R}^{\mathrm{n}}$, was proved in [S. Don and D. Vittone, 2019].

Uniform exponential coordinates of the first kind for equiregular sub-Riemannian manifolds

For an *equiregular sub-Riemannian manifold* $(M, \mathcal{D}, g)$, **locally** we can find a frame of vector fields $\mathbf{X} = (X_1, \ldots, X_n)$ with $\mathcal{D}^j = \mathrm{span}\{X_1, \ldots, X_{n_j}\}$ and $X_{n_{j-1}+1}, \ldots, X_{n_j}$ are $(j-1)$ times iterated Lie brackets of $X_1, \ldots, X_m$.

We say that $(X_1, \ldots, X_n)$ is a **privileged frame**.

Fix $p \in M$ and two open neighborhoods $U \subset M$ of p and $V \subset \mathbb{R}^n$ of 0 such that $F_{\mathbf{X}} : U \times V \to M$, $x = (x_1, \ldots, x_n) \in V$ and

$$F_{\mathbf{X}}(q, x) = F_{\mathbf{X},q}(x) = \exp(x_1 X_1 + \cdots + x_n X_n)(q) = \Phi_1^{x_1 X_1 + \cdots + x_n X_n}(q)$$

is smooth and $F_{\mathbf{X},q} : V \to F_q(V)$ is a smooth diffeomorphism for each $q \in U$. The coordinates $(q, x_1, \ldots, x_n)$ are called **uniform canonical coordinates of the first kind**.

Nilpotent approximation I

Consider a "privileged frame" $(X_1, \ldots, X_n)$ and around $p \in M$ and read it by the exponential coordinates $F_{\mathbf{X},p}$, getting the **local frame of vector fields around** p:

$$\widetilde{X}_1^p = (F_{\mathbf{X},p})_*^{-1}(X_1),\ \widetilde{X}_2^p = (F_{\mathbf{X},p})_*^{-1}(X_2), \ldots, \widetilde{X}_n^p = (F_{\mathbf{X},p})_*^{-1}(X_n)$$

around an open neighborhood $V \subset R^n$ of 0.

The **nilpotent approximation of M at p** is the frame

$$(\widehat{X}_1^p, \ldots, \widehat{X}_n^p),$$

whose elements are **homogeneous of degree -1** and satisfy

$$\widetilde{X}_i^p = \widehat{X}_i^p + R_i^p.$$

Nilpotent approximation II

The frame of vector fields

$$(\widehat{X}_1^p, \dots, \widehat{X}_n^p) \quad \text{on} \quad \mathbb{R}^n \tag{3}$$

defines **the tangent Lie group to the SR manifold M at p**, that is a stratified Lie group structure, that represents the **metric tangent cone of M at p**, with respect to the **Gromov-Hausdorff convergence of metric spaces,**
see [J. Mitchell, 1985], [A. Bellaïche, 1996],...
[F. Jean, 2014], [A. Agrachev, D. Barilari, U. Boscain, 2020],
[R. Monti, A. Pigati, D. Vittone, 2018].

Local SR distance and tangent distance

The local frame $(\widetilde{X}_1^p, \ldots, \widetilde{X}_n^p)$ defines an "**induced local SR distance**" $\widetilde{\mathrm{d}}_{\mathrm{p}}(\mathrm{x}, \mathrm{y})$ on the coordinate open set $V \subset \mathbb{R}^n$, where $x, y \in V$ **and the connecting curves are in V**. For this reason, we have

$$\widetilde{d}_p(x, y) \geq d(F_{\mathbf{X},p}(x), F_{\mathbf{X},p}(y)),$$

where $F_{\mathbf{X},p}; V \to F_{\mathbf{X},p}(V)$.

The "**tangent SR distance**" is the the sub-Riemannian distance

$$\widehat{d}_p(x, y) = \widehat{d}_{p,\mathbf{X}}(x, y)$$

for every $x, y \in \mathbb{R}^n$, associated to the nilpotent approximation $\widehat{\mathbf{X}}^p = (\widehat{X}_1^p, \ldots, \widehat{X}_n^p)$. It is the distance of the tangent group.

SR distance and rescaled vector fields

For fixed $p \in M$, at **"small scales"** we may compare the *local distance* $\widetilde{d}_p$ with the *SR distance* d around p, where we can verify that

$$\widetilde{d}_p(x, y) = d(F_{\mathbf{X},p}(x), F_{\mathbf{X},p}(y)),$$

where $F_{\mathbf{X},p}; V \to F_{\mathbf{X},p}(V)$.

Rescaled vector fields

We consider a privileged orthonormal frame $\mathbf{X} = (X_1, \ldots, X_{\mathrm{n}})$ around regular point $p \in M$ and let $F_{\mathbf{X}}: U \times V \to W$ be the system of uniform exponential coordinates.

We define the rescaled vector fields

$$\widetilde{X}_i^{q,r} = r^{w_i}(\delta_{1/r})_* \widetilde{X}_i^q$$

for $i = 1, \ldots, \mathrm{n}$, $q \in U$ and $r > 0$, where $\widetilde{X}_i^q = (F_{\mathbf{X}}(q, \cdot)^{-1})_* X_i$

The main application of Theorem 1 is the following.

Theorem 2, (Uniform blow-up of SR manifold)

For every bounded open set $A \subset \mathbb{R}^{\mathrm{n}}$, the following holds

1. The rescaled frame $\widetilde{\mathbf{X}}^{q,r} = (\widetilde{X}_1^{q,r}, \ldots, \widetilde{X}_{\mathrm{n}}^{q,r})$ converges to $\widehat{\mathbf{X}}^{q} = (\widehat{X}_1^{q}, \ldots, \widehat{X}_{\mathrm{n}}^{q})$ on $A \subset \mathbb{R}^{n}$ in the C^∞_{loc}-topology as $r \to 0$, **uniformly w.r.t. q varying in any compact set of U.**
2. The induced distance $\widetilde{d}_q^r$ w.r.t. $\widetilde{\mathbf{X}}_h^{q,r} = (\widetilde{X}_1^{q,r}, \ldots, \widetilde{X}_{\mathrm{m}}^{q,r})$ converges to $\widehat{d}_q$ in $L^\infty(A \times A)$ as $r \to 0$, **uniformly as q varies in any compact set of U**, where $\widehat{d}_q : \mathbb{R}^n \times \mathbb{R}^{\mathrm{n}} \to [0, +\infty)$ is the distance associated to the nilpotent approximation.

Local uniform estimates diameters of SR balls

Theorem 2, namely the uniform nilpotent approximation, yields the following uniform estimate.

Theorem 3, [S. Don, V.M., 2023]

Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold and let $p \in M$ be a regular point. Then there exist a compact neighborhood U of p such that for every $0 < \varepsilon < 1$ there exists $r_\varepsilon > 0$ such that

$$1 - \varepsilon \leq \frac{\mathrm{diam} B(q, r)}{2r} \leq 1$$

for every $q \in U$.

Main idea to prove the uniform diameter estimate

It suffices to apply the uniform blow-up of Theorem 2, hence

$$\sup_{x,y\in A} |\widetilde{d}_q^r(x,y) - \widehat{d}_q(x,y)| \to 0 \quad \text{as} \quad r \to 0^+,$$

uniformly as q varies in a compact set of U.

Question

Who can ensure that the domain of the local rescaled distance $\widetilde{d}_q^r$ contains a SR ball of "uniform radius", as q varies in a compact set?

Topological existence of uniform radius

Theorem 4, [S. Don, V.M., 2023]

Let M be a length metric space, $p \in M$ and fix two open sets $\mathcal{U} \subset M$ and $\mathcal{V} \subset \mathbb{R}^{\mathrm{n}}$, with $p \in \mathcal{U}$ and $0 \in \mathcal{V} \subset \mathbb{R}^n$.

We assume that there exists a mapping $E : \mathcal{U} \times \mathcal{V} \to M$ such that:

1. E is continuous,
2. $E(q, 0) = q$ for every $q \in \mathcal{U}$,
3. the mapping $E(q, \cdot) \colon \mathcal{V} \to E(q, \mathcal{V})$ is a homeomorphism for every $q \in \mathcal{U}$.

Then there exist open sets $V \subset \mathcal{V}$ and $U \subset \mathcal{U}$ with $0 \in V$ and $p \in U$ such that $q \mapsto \sup\{t > 0 : B(q,t) \subset E(q, V)\} \in (0, +\infty)$ is well defined and lower semicontinuous.

In particular, there exist $r_0 > 0$ and $\varepsilon_0 > 0$ such that $B(q, r_0) \subset E(q, V)$ for every $q \in B(p, \varepsilon_0)$.

Perimeter measure in sub-Riemannian measure manifolds, [L. Ambrosio, R. Ghezzi and V.M., Ann. Inst. H. Poincaré, 2015]

We say that a measurable set $E \subset M$ has finite perimeter with respect to $\mathcal{D}$ if the following

$$\sup \left\{ \int_{E\cap\Omega} \mathrm{div}_\omega(\varphi X)\,\omega \colon X \in \mathcal{D}, \|X\|_g \le 1, \varphi \in C_c^\infty(\Omega), |\varphi| \le 1 \right\}$$

is finite for $\Omega = M$ and for every open set $\Omega \subset M$ and it defines the perimeter measure

$$\|D_{\omega,g}\mathbf{1}_E\|(\Omega),$$

that extends to a Borel regular measure on M.

Theorem 5, [S. Don, V.M., 2023]

If $\Omega \subset M$ is an open set with C^1 boundary and $\bar{g}$ is any Riemannian metric on M such that $\bar{g}|_{\mathcal{D}} = g$, then for every open set $U \subset M$, we have

$$\|D_{\omega,g} 1_\Omega\|(U) = \int_{\partial\Omega\cap U} \|\omega\|_{\bar{g}} \, \|\nu_{\mathcal{D}}\|_{\bar{g}} \, d\sigma_{\bar{g}}. \tag{4}$$

1. $\nu_{\mathcal{D}}$ denotes the projection on $\mathcal{D}$ with respect to $\bar{g}$ of the outer normal ν to $\partial\Omega$.
2. $\sigma_{\bar{g}}$ is the Riemannian surface measure associated with $\bar{g}$ on the boundary $\partial\Omega$.

Motivated by the previous integral representation of the perimeter measure, we give the following definition.

Definition of sub-Riemannian surface area

1. Let $(M, \mathcal{D}, g, \omega)$ be an equiregular sub-Riemannian measure manifold.
2. $\bar{g}$ is any Riemannian metric on M such that $\bar{g}|_{\mathcal{D}} = g$
3. $\Sigma \subset M$ be a C^1 smooth hypersurface.
4. $\nu_{\mathcal{D},\Sigma}$ denotes the projection on $\mathcal{D}$ with respect to $\bar{g}$ of a normal ν to Σ.

Then we define the **SR surface measure of Σ** as

$$\sigma_{\Sigma}^{SR}(U) = \int_{\Sigma \cap U} \|\omega\|_{\bar{g}} \, \|\nu_{\mathcal{D}}\|_{\bar{g}} \, d\sigma_{\bar{g}}.$$

Spherical factor on a sub-Riemannian manifold

Definition, [S. Don, V. M., 2023]

1. Let $p \in M$ be a regular point of a sub-Riemannian measure manifold $(M, \mathcal{D}, g, \omega)$ and denote by the same symbol g a Riemannian metric that extends g.
2. Let $\mathbf{X} = (X_1, \dots, X_{\mathrm{n}})$ be an adapted orthonormal frame in a neighborhood of p and let $F_{p,\mathbf{X}} : V \to M$ be its associated system of coordinates of the first kind, centered at p.
3. Let $\nu \in \mathcal{D}_p \setminus \{0\}$ be a unit vector with respect to g and denote by $\Pi(\nu) \subset T_pM$ its orthogonal subspace.

The *spherical factor* of Σ at p is

$$\beta_{d,\omega}(\nu) = \|\omega(p)\|_g \max_{z \in \widehat{\mathbb{B}}_p(0,1)} \mathcal{H}_E^{\mathrm{n}-1}((dF_p)(0)^{-1}(\Pi(\nu)) \cap \widehat{\mathbb{B}}_p(z, 1)),$$

where $\mathcal{H}_{d_E}^{\mathrm{n}-1}$ denotes the Euclidean Hausdorff measure. The closed metric unit ball $\widehat{\mathbb{B}}_p(z, 1)$ refers to the SR distance $\widehat{d}_p$ associated to the *nilpotent approximation* $(\widehat{X}_1^p, \dots, \widehat{X}_{\mathrm{m}}^p)$.

Central technical result

Theorem 6, [S. Don and V.M., 2023]

Let $(M, \mathcal{D}, g, \omega)$ be a sub-Riemannian measure manifold, with a regular point $p \in M$ and a sub-Riemannian distance d.
If Σ is a C^1 smooth hypersurface, p is a noncharacteristic point and σ_Σ^{SR} denotes the sub-Riemannian surface area, then

$$\mathfrak{s}^{Q-1}(\sigma_\Sigma^{SR}, q) = \beta_{d,\omega}(\nu_{\mathcal{D}}(p)).$$

Area formula

Theorem 7, [S. Don and V.M., 2023]

Let $(M, \mathcal{D}, g, \omega)$ be an equiregular sub-Riemannian measure manifold and fix a sub-Riemannian distance d. If Σ is a C^1 hypersurface with $\nu_{\mathcal{D},\Sigma} \neq 0$ $\mathcal{S}^{Q-1}$-a.e., then for every Borel set $B \subset \Sigma$ we have

$$\sigma_{\Sigma}^{SR}(B) = \int_B \|\omega\|_{\bar{g}} \, \|\nu_{\mathcal{D}}\|_{\bar{g}} \, d\sigma_{\bar{g}} = \int_B \beta_{d,\omega}(\nu_{\mathcal{D},\Sigma}) d\mathcal{S}_d^{Q-1}.$$

THANK YOU FOR YOUR ATTENTION