Obstacle-type problems for the sub-Laplacian

based on joint work with
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Outline

A quick introduction to the obstacle problem

Obstacle-type problem for vector fields

Some ideas for the proof of the main result

Classical Dirichlet problem

Given $f \in L^\infty(\Omega)$, minimize the Dirichlet energy:

$$u \to \int_\Omega |\nabla u|^2 + 2 \int_\Omega fu$$

among all functions $u \in W^{1,2}(\Omega)$ with a given boundary condition

$$u - \varphi \in W_0^{1,2}(\Omega)$$

for some $\varphi \in W^{1,2}(\Omega)$.

The unique minimum of this functional is well known to satisfy the Poisson equation

$$-\Delta u + f = 0 \quad \text{on } \Omega.$$

Variational formulation of the obstacle problem

The *obstacle problem* appears when we minimize

$$u \to \int_\Omega |\nabla u|^2 + 2\int_\Omega fu$$

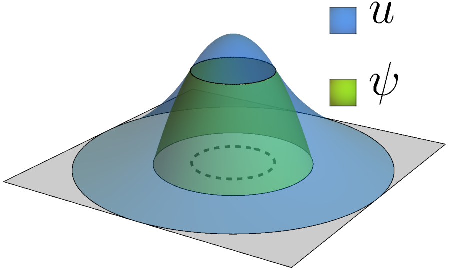
among all functions $u \in W^{1,2}(\Omega)$, with the constraints

$$\mathbf{u} \geq \psi \ \textbf{a.e. in } \Omega \quad \text{and} \quad u - \varphi \in W_0^{1,2}(\Omega).$$

The function $\psi \in C^{1,1}$ represents the constraint,

that is called *obstacle*.

Geometric idea



1. u represents the solution,
2. ψ is the obstacle,
3. the region inside the circle is the coincidence set, where solution and obstacle do coincide.

From the variational problem to PDE

$$\begin{cases} -\Delta u + f \geq 0 & \text{on } \Omega, \\ \Delta u = f & \text{on } \{u > \psi\}, \\ \Delta u = \Delta \psi \ a.e. & \text{on } \{u = \psi\}. \end{cases}$$

The $W^{2,p}$ regularity of the minimum and the regularity of ψ allow us to replace u by $v = u - \psi$ and f by $g = f - \Delta\psi$.
This gives the final equation

$$\Delta v = g\chi_{\{v>0\}}. \tag{1}$$

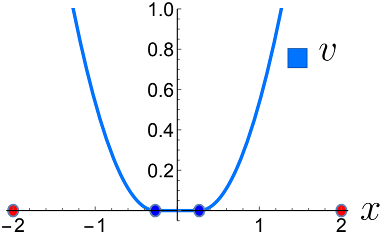
The inequality $\Delta\psi \leq f$ implies that $g \geq 0$.

Reduction to zero level coincidence set

In dimension one, considering $g \equiv 1$, the obstacle equation is

$$v'' = \chi_{\{v>0\}},$$

whose solution v is given by $u - \psi$. Then v is easy to draw:



The *coincidence set* is given by the zero level set of v:

$$\{x : v(x) = 0\} = \{x : u(x) = \psi(x)\}.$$

Our PDE

There are many other variants of the obstacle problem arising from several theoretical and applied problems, like in Engineering, Potential Theory, Geophysics, Superconductivity and Financial Markets for the case of subelliptic obstacle problems.

The no-sign obstacle (type) equation

$$\Delta u = f\chi_{\{u\neq 0\}}. \tag{2}$$

Remark

Being f bounded, the standard Calderon-Zygmund regularity theory implies that the solution to (2) is in $W^{2,p}_{loc}(\Omega)$ for any $p > 1$. Then it is in any $C^{1,\alpha}$ with $0 < \alpha < 1$.

First comments on regularity

Warning

It is however a well known fact that $\Delta u = f$ with $f \in L^\infty$ **does not** imply that $u \in W^{2,\infty}_{loc}$. A nice example is given by

$$u(x,y) = xy\sqrt{-\log(x^2+y^2)} \quad \text{in } B_1,$$

whose Laplacian Δu **is continuous and bounded in** B_1, but

$$u \notin W^{2,\infty}_{loc}(B_1).$$

For this reason we will always consider f such that $f * \Gamma \in C^{1,1}$. For instance this is true if f is Hölder, or Dini, therefore any solution u to $\Delta u = f$ is $C^{1,1}$.

Our question

Problem

If $f * \Gamma \in C^{1,1}$, then is it true that any solution u to

$$\Delta u = f\chi_{\{u \neq 0\}} \quad \text{is } C^{1,1}?$$

Certainly this regularity cannot be improved

$$u(x, y, t) = -\frac{x^2}{2}\chi_{(-\infty,0)\times\mathbb{R}^2} \quad \text{on } \mathbb{R}^3.$$

Regularity is a **delicate question**, since for instance solutions to

$$\Delta u = -\chi_{\{u>0\}},$$

may not be $C^{1,1}$.

$C^{1,1}$-regularity

Theorem (J. Andersson, E. Lindgren, H. Shahgholian, CPAM 2013)

Let u be a solution to $\Delta u = f\chi_{\{u\neq 0\}}$ and assume that $f * \Gamma \in C^{1,1}$. Then we have

$$\|D^2 u\|_{L^\infty(B_{1/2})} \leq C\left(\|u\|_{L^1(B_1)} + \|D^2(f * \Gamma)\|_{L^\infty(B_1)}\right)$$

For this theorem the authors use a technique from Harmonic Analysis, such as projections of functions into the space of homogeneous harmonic polynomials.

These techniques were extended to fully nonlinear uniformly elliptic equations by A. Figalli and H. Shahgholian, ARMA 2014

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Geometry of stratified groups

Broadly speaking, a stratified group, also well known as Carnot group, can be seen as $\mathbb{R}^n$ and a family of *horizontal vector fields*

$$X_1, \dots, X_m \quad \text{with } m < n.$$

They are left invariant with respect to a Lie group operation

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ (x, y) \to xy := p(x, y),$$

where p is a *polynomial*.

Vector fields replace the classical partial derivatives:

$$\partial_{x_j} u \quad \to X_j u.$$

For a smooth function u, we define the *horizontal gradient*:

$$\nabla_H u = (X_1 u, \dots, X_m u).$$

Geometry of stratified groups

We have dilations $\delta_r : \mathbb{R}^n \to \mathbb{R}^n$, $r > 0$, that satisfy

$$\delta_r(xy) = (\delta_r x)(\delta_r y).$$

The special *anysotropic form* of dilations

$$\delta_r x = (rx_1, \ldots, rx_m, r^2 x_{m+1}, \ldots, r^3 x_{m+n_2+1}, \ldots, r^\iota x_n)$$

yields the homogeneity property

$$X_j\,(u(\delta_r x)) = r\, X_j u(\delta_r x).$$

Hörmander theorem

The celebrated *sub-Laplacian operator* is given by taking the square of of X_j, namely

$$\Delta_H = \sum_{j=1}^{m} X_j^2.$$

Δ_H is then a second order degenerate elliptic operator.

Hörmander condition

The vector fields X_j satisfy the condition $\text{Lie}\,\{X_1, \ldots, X_m\} = \text{Lie}\,\mathbb{G}$.

L. Hörmander, Acta Math., 1967

The previous condition on X_j implies that Δ_H is *hypoelliptic*.

Homogeneous distance

Dilations, group operation, Δ_H, etc... have a natural distance

$$d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty).$$

We say that d is a *homogeneous distance* if it also satisfies:

1. $d(zx, zy) = d(x, y)$ for all $x, y, z \in \mathbb{R}^n$,
2. $d(\delta_r x, \delta_r y) = r d(x, y)$ for all $r > 0$ and $x, y \in \mathbb{R}^n$.

Fractal behavior

For any compact set $K \subset \mathbb{R}^n \approx \mathbb{G}$ there exists $C_K > 0$ such that

$$C_K^{-1}|x - y| \leq d(x, y) \leq C_K |x - y|^{1/2}$$

for all $x, y \in K$.

Function spaces in stratified groups

Sobolev space

We define $W_H^{1,p}(\Omega)$ as the space of functions $u \in L^p(\Omega)$ such that

$$\int_\Omega u \, X_j \phi \, dx = \int_\Omega g_j \, \phi \, dx$$

for $g_j \in L^p(\Omega)$, $\phi \in C_c^\infty(\Omega)$ and $j = 1, \ldots, m$.

$C^{1,\alpha}$ space

We define $C_H^{1,\alpha}(\Omega)$ as the space of continuous functions such that for every $j = 1, \ldots, m$ there exists $X_j u \in C(\Omega)$ and

$$|X_j u(z) - X_j(y)| \leq C d(z, y)^\alpha.$$

Obstacle-problem in stratified groups

D. Danielli, N. Garofalo and S. Salsa, Indiana Univ. Math. J., 2003

For an obstacle $\psi \in C_H^{1,1}(\overline{\Omega})$, let us define

$$K_\psi = \left\{ v \in W_0^{1,2}(\Omega) : v \geq \psi \text{ a.e. in } \Omega \right\}.$$

If $u \in K_\psi$ solves the obstacle problem

$$\int_D \langle \nabla_H v, \nabla_H (v-u) \rangle \geq 0$$

for all $v \in K_\psi$, then $u \in C^{1,1}_{H,loc}(\Omega)$.

There are other important related papers by Danielli, Garofalo, Frentz, Nyström, Pascucci, Petrosyan and Polidoro.

Main result

V. M. and Andreas Minne, 2019

There exists a universal constant $C > 0$ such that for any $f \in L^\infty(B_1)$ such that $f * \Gamma \in C_H^{1,1}(B_1)$ and any solution $u \in W^{2,p}(B_1)$ to the equation

$$\Delta_H u = f\chi_{\{u\neq 0\}}$$

with $p > Q$, we obtain

$$\|u\|_{C_H^{1,1}(B_{1/2})} \leq C\left(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right).$$

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BMO spaces

We say that a function $u \in L^1_{loc}(\Omega)$ is *bounded mean oscillation* if

$$[u]_{BMO(\Omega)} = \sup_{x\in\Omega, r>0} \fint_{B_r(x)} |u - u_{B_r(x)}|$$

and denote by $BMO(\Omega)$ the space of all of these functions.

Remark

The important feature of BMO spaces is that they are somehow intermediate between all L^p and L^∞:

$$L^\infty(\Omega) \subset BMO(\Omega) \subset L^p_{loc}(\Omega).$$

The horizontal Hessian

For $u \in W^{2,p}_H(\Omega)$ and $x_0 \in \Omega$ a Lebesgue point of second horizontal derivatives, we can define

$$D^2_h u(x_0) = \left(\begin{array}{cccc} X_1X_1u(x_0) & X_1X_2u(x_0) & \cdots & X_1X_mu(x_0) \\ X_2X_1u(x_0) & X_2X_2u(x_0) & \ddots & X_2X_mu(x_0) \\ \vdots & \ddots & \ddots & \vdots \\ X_mX_1u(x_0) & \cdots & \cdots & X_mX_mu(x_0) \end{array} \right).$$

We simply denote by $|D^2_h u(x_0)|$ the matrix norm of $D^2_h u(x_0)$.

Our point

We wish to estimate the L^∞- norm of $x \to |D^2_h u(x)|$.

BMO estimates

Theorem (M. Bramanti and L. Brandolini, Rev. Mat. Ibero., 2005)

Let $u : B_1 \to \mathbb{R}$ *be locally summable such that* $\Delta_H u \in BMO(B_1)$*. Then we have*

$$\|D_h^2 u\|_{BMO(B_{1/2})} \leq C(Q) \left(\|\Delta_H u\|_{BMO(B_1)} + \|u\|_{BMO(B_1)} \right).$$

Horizonal Hessian and coincidence set

The *averaged trace free horizontal Hessian* is defined as

$$P_r^{x_0} := (D_h^2 u)_{B_r(x_0)} - \frac{1}{m}(\Delta_H u)_{B_r(x_0)} I_m.$$

We have a second order homogeneous polynomial $p_r^{x_0}$ with

$$D_h^2 p_r^{x_0} = P_r^{x_0}.$$

Sub-quadratic growth

If $u \in W^{2,1}_{H,loc}(B_1)$, $\Delta_H u \in L^\infty(B_1)$, $u(x_0) = 0$, $\nabla_H u(x_0) = 0$ and $\lambda, \sigma \in (0, 1)$, $x_0 \in B_\lambda$, for all $r > 0$ sufficiently small, there exists $C > 0$, such that

$$\sup_{y \in B_{\sigma r}(x_0)} |u(y) - p_r^{x_0}(x_0^{-1} y)| \le C \left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{loc}(B_1)} \right) r^2.$$

The ALS's dichotomy argument

Dichotomy statement

We can show that **either** $P_r^{x_0}$ stays bounded around x_0, **or** the coincidence set decays exponentially.

In the **first case** we have a control of the second derivatives at small scales:

From BMO estimates, we can conclude that if we can control

$$|P_r^{x_0}| \leq C$$

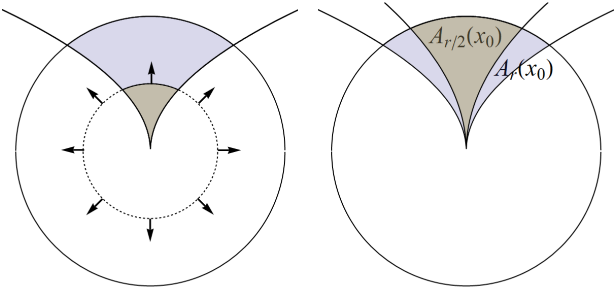
uniformly in x_0 for all $r > 0$ small, then we get a control

$$\|D_h^2 u\|_{L^\infty} \leq C$$

The ALS's dichotomy argument

In the **second case** we must have $|P_r^{x_0}|$ "large" and then the following decay of the coincidence set

$$|A_{r/2}(x_0)| \leq \frac{|A_r(x_0)|}{2^{\beta Q}}$$



We *suitably rescale* the solution u to u_0 and f to f_0. Then we consider the decomposition

$$u_0 = v_0 - w_0$$

Such that we have the Dirichlet problems:

$$\begin{cases} \Delta_H v_0(x) = f_0 & \text{in } B_{1/2}, \\ v_0 = u_0, & \text{on } \partial B_{1/2} \end{cases}$$

and

$$\begin{cases} \Delta_H w_0 = f_0 \chi_{A_{2^{-k_0}}(x_0)} & \text{in } B_{1/2}, \\ w_0 = 0 & \text{on } \partial B_{1/2}. \end{cases}$$

Caffarelli's iteration techniques by harmonic polynomials

We use an idea from "Interior *a priori* estimates for solutions of fully non-linear equations", by L. Caffarelli, Ann. Math. 1989.

We set

$$w_k(x) := \frac{w_0(\delta_{\lambda^k} x) - q_k(\delta_{\lambda^k} x)}{\lambda^{(2+\alpha)k}}$$

so that

$$\Delta_H w_k = \lambda^{-\alpha k} f(x_0 \delta_{2^{-k_0}\lambda^k} \cdot) \chi_{A_{2^{-k_0}\lambda^k}(x_0)}$$

very rapidly goes to zero.

We consider the harmonic function h_k by

$$\begin{cases} \Delta_H h_k = 0 & \text{in } B_1, \\ h_k = w_k & \text{on } \partial B_1 \end{cases}$$

that is close to w_k.

Caffarelli's iteration techniques by harmonic polynomials

The Taylor expansion of h_k yields the harmonic polynomial q_k:

$$\|w_0 - q_k\|_{L^\infty(B_{2^{-k}}(0))} \leq C\left(\|D^2(\Gamma * f)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right) 2^{-k(2+\alpha)}$$

We characterize the limit of q_k as the Taylor expansion of w_0 by the following Calderon-type differentiability theorem.

V. M., Studia Math. 2005

If $u \in W^{2,1}_{H,loc}(\Omega)$, then for a.e. $x \in \Omega$ there exists a second order polynomial p^x such that

$$\frac{1}{r^2} \fint_{B_r(x)} |u(z) - p^x(z)| dz \to 0.$$

Turning from w_0 to w, and then to $u = v - w$, we reach the estimate

$$\begin{aligned} |D_h^2 u(x_0)| &\le |D_h^2 u_0(0)| + |P_{2^{-k_0}}| \\ &\le C(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}), \end{aligned}$$

The bound on the norm of the matrix $P_{2^{-k_0}}$ comes from the threshold constant selected for the dichotomy argument.
This concludes the proof.

Thanks a lot for your attention.