Regularity and transversality for Sobolev hypersurfaces

University of Illinois at Urbana-Champaign
Analysis Seminar

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Smooth tangent distribution

We denote by $\mathcal{D}$ a *smooth (tangent) distribution* of k-dimensional subspaces on a smooth manifold M, which are locally spanned by smooth vector fields $X_1, \ldots, X_k$. The fibers of the distribution are

$$\mathcal{D}_x = \mathrm{span}\,\{X_1(x), \ldots, X_k(x)\}\,.$$

We say that the distribution $\mathcal{D}$ is *involutive* if at each point x of the manifold M there holds

$$\mathrm{Lie}_x\mathcal{D} = \mathrm{span}\left\{[X_i, X_j], [[X_i, X_j], X_k], \ldots, \right\} \subset \mathcal{D}_x.$$

We say that a submanifold Σ is *tangent to* $\mathcal{D}$ if $\mathcal{D}_x \subset T_x\Sigma$ for all $x \in \Sigma$.

Frobenius theorem

A k-dimensional tangent distribution $\mathcal{D}$ is involutive if and only if every $x \in M$ has a neighbourhood U that can be foliated by a family of k-dimensional submanifolds that are tangent to $\mathcal{D}$.

We say that $\mathcal{D}$ is *totally nonintegrable* if at each point x of a fixed manifold M there holds

$$\mathrm{Lie}_x \mathcal{D} = \mathrm{span}\left\{[X_i, X_j], [[X_i, X_j], X_k], \ldots, \right\} = T_x M.$$

Suppose Σ is tangent to a totally nonintegrable distribution $\mathcal{D}$. If Σ is a smooth submanifold and X, Y are everywhere contained in $\mathcal{D}$ and are sections of $T\Sigma$ then

$$[X, Y](x) \in T_x\Sigma \quad \text{and} \quad [X, Y](x) \in \mathrm{Lie}_x(\mathcal{D})$$

and iterating these Lie brackets our assumption on $\mathcal{D}$ gives the following *contradiction*

$$\mathrm{Lie}_x \mathcal{D} \subset T_x\Sigma.$$

First regularity problem

If we consider a C^1 smooth hypersurface $\Sigma \subset M$. Any two *tangent vector fields*

$$X \text{ and } Y \text{ are only continuous on } \Sigma,$$

then their commutator

$$[X, Y] \text{ cannot be computed.}$$

Indeed defining $X = \sum a_j \partial_{x_j}$ and $Y = \sum b_i \partial_{x_i}$ we have

$$\begin{aligned}
[X, Y] &= XY - YX \\
&= \sum \left(a_j \partial_{x_j}(b_i \partial_{x_i}) - b_j \partial_{x_j}(a_i \partial_{x_i}) \right),
\end{aligned}$$

therefore second order derivatives are needed.

The previous method already fails to apply.

Question

If $\mathcal{D}$ is a totally nonintegrable smooth distribution how can we show that there are no C^1 smooth submanifolds tangent to $\mathcal{D}$?

The problem has an independent interest since we have two different competing aspects: REGULARITY AND NONINTEGRABILITY.

The tangent distribution of the Heisenberg group

One of the simplest cases of totally nonintegrable distribution is given by the vector fields

$$X_1 = \partial_{x_1} - x_2 \partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3} \quad \text{in } \mathbb{R}^3,$$

that define the 3-dimensional Heisenberg Lie algebra. The distribution $\mathcal{D}$ is such that

$$\mathcal{D}_x = \operatorname{span}\{X_1(x), X_2(x)\}$$

for each $x \in \mathbb{R}^3$. It is the well-known *horizontal distribution*.

From the definition of the vector fields

$$X_1 = \partial_{x_1} - x_2\partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1\partial_{x_3} \quad \text{in } \mathbb{R}^3,$$

one easily checks that

$$\mathrm{Lie}_x\mathcal{D} = \{X_1(x), X_2(x), \partial_{x_3}\} = T_x\mathbb{R}^3,$$

therefore Frobenius theorem shows that there exists no smooth 2-dimensional surface in $\mathbb{R}^3$ that is tangent to $\mathcal{D}$.

Questions

- *What happens if we consider less regular surfaces?*
- *How can we make sense of the notion of tangency if the surface is not smooth?*

The sub-Riemannian Geometry of $\mathbb{H}$

The distribution $\mathcal{D} = \{X_1, X_2\}$ generates a group operation

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2 - x_2 y_1)$$

that makes $\mathbb{R}^3$ a model for the 3-dimensional Heisenberg group $\mathbb{H}$.
The metric structure arises from the *sub-Riemannian distance*

$$d(x, y) = \inf \{\text{R-length}(\gamma) : \gamma \in \mathcal{F}_{x,y}\} \ ,$$

where $\mathcal{F}_{x,y}$ is the family of Lipschitz curves tangent to $\mathcal{D}$ and co x with y in $\mathbb{H} \approx \mathbb{R}^3$. We have set

$$\text{R-length}(\gamma) = \int_I |\dot{\gamma}|_g$$

with respect to a left invariant Riemannian metric g.

Hausdorff dimension of surfaces in $(\mathbb{H}, d)$

If Σ is a 2-dimensional surface of $\mathbb{H}$ and ν is a normal to Σ with respect to a left invariant metric g, we can project $\nu(x)$ onto $\mathcal{D}_x$, getting the

$$\text{horizontal normal} \quad \nu_H(x), \quad x \in \Sigma.$$

The following formula holds

$$\mathcal{S}^3(\Sigma) = \int_\Sigma |\nu_H| \, d\sigma. \tag{1}$$

This formula is well known since the works of P. Pansu 1982 and J. Heinonen 1994. More recently it has been extended to all *homogeneous distances* in stratified groups, [V. M., 2017].

The surface Σ is tangent to $\mathcal{D}$ at x if and only if $T_x\Sigma = \mathcal{D}_x$, that is

$$\nu_H(x) = 0.$$

From Frobenius theorem a smooth surface Σ must be transversal to $\mathcal{D}$, hence (1) *implies that Σ has Hausdorff dimension 3.*

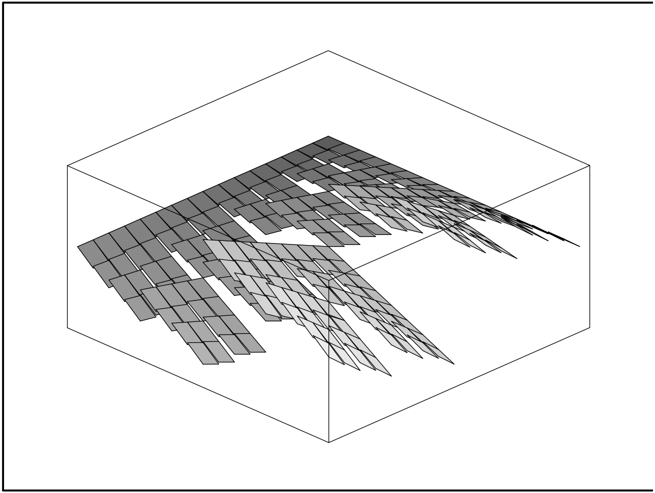
Existence of BV "horizontal" surfaces in $\mathbb{H}$

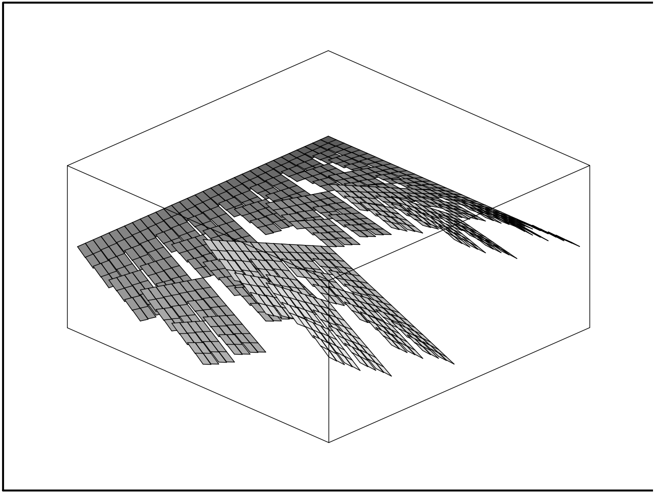
In some sense a surface of Hausdorff dimension 2 should be thought of as horizontal and certainly cannot be smooth.

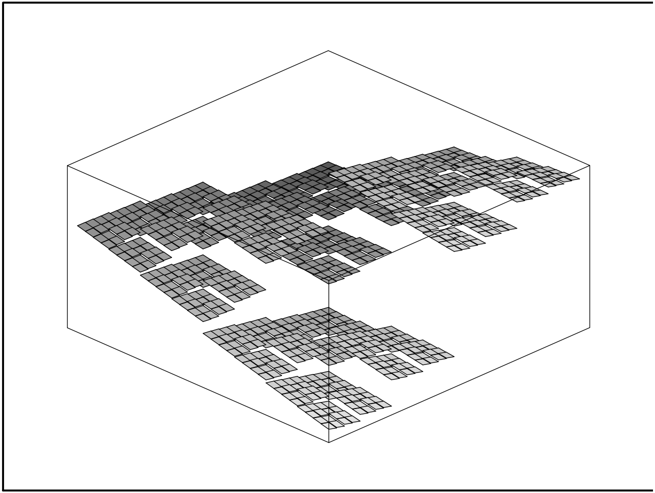
Z. Balogh, J. T. Tyson, 2005
Z. Balogh, R. Hoefer-Isenegger, J. T. Tyson, 2006

There exists a function of *special bounded variation* $u : [0,1]^2 \longrightarrow \mathbb{R}$, whose graph $S_0 \subset \mathbb{H}$ satisfies $\mathcal{H}_\rho$-dim$(S_0) = 2$ and $0 < \mathcal{H}^2_\rho(S_0) < \infty$.

The set S_0, called the *Heisenberg square*, is a *"horizontal fractal"* obtained as *invariant set* of a suitable *affine iterated function system* on the Heisenberg group $\mathbb{H}$.







The Heisenberg square is a countably rectifiable set and its approximate tangent space is a.e. tangent to the horizontal distribution. If u is the Balogh-Tyson function on $(0,1)^2$, Du is the distributional gradient of u, that is a vector measure, and

$$D_a u = \nabla u \mathcal{L}^3$$

is the absolutely continuous part ∇u, then

$$\nabla u = (-y, x)$$

a.e. on $(0,1)^2$. Notice that ∇u can be also seen as the approximate gradient of u in the sense of GMT.

Question

Can we increase a little bit the regularity of the surface mantaining Hausdorff dimension 2? That would mean the a.e. tangency of the surface to the nonintegrable distribution?

Sobolev surface

Let $\Omega \subset \mathbb{R}^k$ be open and let $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ with distributional gradient ∇f of maximal rank almost everywhere. Suppose that and that all negligible sets of $\Omega \subset \mathbb{R}^k$ are sent into $\mathcal{H}^k$-negligible sets. We say that $f(\Omega)$ is a $W^{1,p}$ Sobolev surface of dimension k.

V. M. 2010

If $p \geq 4/3$, then there does not exist any 2-dimensional $W^{1,p}_{loc}$ Sobolev surface which has Hausdorff dimension 2 in $\mathbb{H}$.

Then two independent results give a complete solution to the problem in all Heisenberg groups $\mathbb{H}^n \approx \mathbb{R}^{2n+1}$ with $2n$-dimensional horizontal distribution $\mathcal{D} = \{X_1, \ldots, X_{2n}\}$.

Z. M. Balogh, P. Hajłasz, K. Wildrick 2014, V. M., J. Malý, S. Mongodi 2015

If $n < k \leq 2n$, then there do not exist k-dimensional $W^{1,1}_{loc}$ Sobolev surfaces having Hausdorff dimension k.

Sobolev hypersurfaces in stratified groups

A *stratified group* $\mathbb{G} \approx \mathbb{R}^n$ can be seen as graded finite dimensional real nilpotent Lie algebra that is the direct sum of subspaces

$$\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota.$$

There are *intrinsic dilations* $\delta_r : \mathbb{G} \to \mathbb{G}$, $\delta_r x = r^j x$, whenever $x \in H^j$ and $j = 1, \ldots, \iota$, that are Lie group homomorphisms.

A distance function $d : \mathbb{G} \times \mathbb{G} \to [0, +\infty)$ which satisfies

$$d(xz, xw) = d(z, w) \quad \text{and} \quad d(\delta_r z, \delta_r w) = r\, d(z, w)$$

for each $x, z, w \in \mathbb{G}$ and $r > 0$ is a *homogeneous distance*.

The Hausdorff dimension of $\mathbb{G}$ with respect to d is

$$Q = \sum_{j=1}^{n} j \dim(H_j) > \mathrm{top}_{dim}\mathbb{G} = n.$$

The graded structure of $\mathbb{G} = H^1 \oplus H^2 \oplus \cdots H^\iota$ yields the grading of the Lie algebra

$$\mathrm{Lie}(\mathbb{G}) = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_\iota. \tag{2}$$

In order to have a stratified group we assume the condition

$$[\mathcal{V}_1, \mathcal{V}_j] = \mathcal{V}_{j+1} \qquad \text{for every} \quad i \geq 1$$

and $\mathcal{V}_j = \{0\}$ whenever $j > \iota$.

If $X_1, \ldots, X_m$ is a basis of left invariant vector fields spanning the subspace $\mathcal{V}_1 \subset \mathrm{Lie}(\mathbb{G})$, the assumption (2) implies that the so-called *horizontal distribution*

$$\mathcal{D} = \mathrm{span}\,\{X_1, \ldots, X_m\} \qquad \text{is } \textit{totally nonintegrable}.$$

The tangency question we consider in $\mathbb{G}$ can then be formulated as follows: are there hypersurfaces $\Sigma \subset \mathbb{G}$ of low regularity such that

$$\mathrm{Hausd}_{dim}\Sigma < Q - 1?$$

Gromov's dimension comparison

M. Gromov in his seminal work of 1996 on "Carnot-Carathéodory spaces seen from within" shows the following estimate

$$\mathrm{Hausd}_{dim}(S) \geq Q - 1,$$

whenever $S \subset \mathbb{G}$ is a topological submanifold.

The Balogh-Tyson's Heisenberg square $Q \subset \mathbb{H}$ offers a counterexample to this estimate when the topological manifold is replaced by a rectifiable set.
Indeed, for the Heisenberg group we have

$$\mathrm{Hausd}_{dim}(Q) = 2 < Q - 1 = 3.$$

Question

Does the previous counterexample persist also in the case of Sobolev hypersurfaces, that are special classes of countably rectifiable sets?

V. M., A. Zapadinskaya, 2017

Let $S \subset \mathbb{G}$ be an $(n-1)$-dimensional Sobolev hypersurface of regularity $W^{1,p}_{loc}$. If the following conditions hold

$$\begin{cases} p > n - m & \text{if } n - m > 1 \\ p = 1 & \text{if } n - m = 1 \end{cases} . \tag{3}$$

Then we have

$$\mathrm{Hausd}_{dim}(S) \geq Q - 1.$$

Ideas about the proof, 1

The metric formulation of the problem allows us to treat less regular hypersurfaces. *Indeed the question is whether Σ is or not a.e. tangent to the horizontal distribution.*

Definition of Sobolev surface tangent to $\mathcal{D}$

We say that a Sobolev surface $\Sigma = f(\Omega) \subset \mathbb{G}$, with $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ and a.e. maximal rank is *a.e. tangent to the horizontal distribution $\mathcal{D}$ of $\mathbb{G}$* if for a.e. $x \in \Omega$ the approximate differential $df(x)$ of f at x satisfies

$$\mathcal{D}_{f(x)} \subset df(x)(T_x\Omega).$$

Lemma

Let $S \subset \mathbb{G}$ be an $(n-1)$-dimensional $W^{1,1}_{loc}$ Sobolev hypersurface and let $\tilde{\Sigma} \subset \Sigma$ be a rectifiable set. If there exists a "transversal subset"

$$\Sigma_T = \left\{ x \in \tilde{\Sigma} : H_x\mathbb{G} \not\subset \operatorname{approx}(T_x\Sigma) \right\} \quad \text{such that} \quad \mathcal{H}^{n-1}_E(\Sigma_T) > 0,$$

then we have $\mathcal{H}^{Q-1}_d(\Sigma_T) > 0$.

Ideas about the proof, 2

The rectifiable set $\tilde{\Sigma}$ can be seen by Whitney's estension theorem as a subset of a C^1 smooth hypersurface Σ_0 for which we have

$$0 < \int_{\Sigma_T} |\nu_H(x)| d\mathcal{H}_E^{n-1}(x) \leq C\, \mathcal{S}^{Q-1}(\Sigma_T).$$

This estimate is a consequence of results in V.M., JEMS, 2006.

As in the case of surfaces in $\mathbb{H}$, the horizontal normal ν_H satisfies

$$|\nu_H(x)| > 0 \quad \text{if and only if} \quad \mathcal{D}_x \not\subset \operatorname{approx}(T_x\Sigma).$$

We wish to show the existence of a transversal rectifiable set Σ_T, hence we assume by contradiction that this is not the case.

Ideas about the proof, 3

Algebraic fact

The condition $\mathrm{span}\,\{X_1(x),\dots,X_m(x)\} \subset \mathrm{approx}(T_x\Sigma)$ for $\mathcal{H}^{n-1}_E$-a.e. $x \in \Sigma$ can be translated in terms of differential forms

$$f^*(\eta_{m+1} \wedge \cdots \wedge \eta_n) = 0$$

for a.e. $x \in \Omega$, where $f \in W^{1,p}(\Omega, \mathbb{R}^n)$.

We have defined the left invariant differential 1-forms

$$\eta_1, \eta_2, \dots, \eta_n$$

as the dual basis of the *graded basis* $X_1, X_2, \dots, X_n$ of $\mathrm{Lie}(\mathbb{G})$. Essentially each $X_i \in \mathcal{V}_{d_i}$ for a unique integer $1 \le d_i \le \iota$, called the *degree of* X_i.

Ideas about the proof, 4

The a.e. differentiation of the $\mathcal{D}$-tangency condition $f^*(\eta_{m+1} \wedge \cdots \wedge \eta_n) = 0$ a.e. yields

$$f^*\left(\sum_{j=m+1}^{n} (-1)^{j-m-1} \eta_{m+1} \wedge \cdots \wedge \eta_{j-1} \wedge d\eta_j \wedge \eta_{j+1} \wedge \cdots \eta_n \right) = 0 \quad a.e.$$

From Maurer-Cartan equations

$$d\eta_k = - \sum_{\substack{1 \le i < j \le q \\ d_j < d_k}} c_{ij}^k \, \eta_i \wedge \eta_j, \qquad where \quad [X_i, X_j] = \sum_{k=1}^{q} c_{ij}^k X_k,$$

multiplying by a suitable pullback $f^*\theta_s$, we get

$$f^*(\eta_1 \wedge \cdots \wedge \eta_{s-1} \wedge \eta_{s+1} \wedge \cdots \wedge \eta_n) = 0$$

for each s such that $m < s \le n$.

Ideas about the proof, 5

The fact that $\mathcal{D}_x \subset \mathrm{approx}(T_x\Sigma)$ implies by a simple algebraic fact that

$$f^*(\eta_1 \wedge \cdots \wedge \eta_{s-1} \wedge \eta_{s+1} \wedge \cdots \wedge \eta_n) = 0$$

whenever $1 \leq s \leq m$. By the definition of pullback differential form, all $(n-1)$ minors of the approximate differential Df are a.e. vanishing.

Conclusion 1

Df is not of maximal rank a.e., giving a contradiction with the definition of Sobolev hypersurface.

Conclusion 2

We have shown that the Sobolev surface $S = f(\Omega)$ cannot be a.e. tangent to $\mathcal{D}$, in the class of 2 step groups. Then $\mathrm{Hausd}_{dim}(S) \geq Q - 1$.

Subtlety

In two step groups we have proved a stronger result: if we consider a weaker notion of Sobolev surface $S = f(\Omega)$, where f *has maximal rank only on a set of positive measure*, then we can still conclude that

$$\mathrm{Hausd}_{dim}(S) \geq Q - 1.$$

Question

In *higher step* stratified Lie groups the validity of this stronger conclusion is an open problem.

Indeed for these groups we use an argument *by induction*, iterating the exterior differentiation. This indeed requires the stronger assumption of maximal rank a.e. in order to apply another key algebraic lemma.

"Very weak" exterior differentiation

Up to this point, we have assumed the possibility to perform the exterior differentiation of the equality

$$f^*(\eta_{m+1} \wedge \cdots \wedge \eta_n) = 0 \quad \text{a.e.} \tag{4}$$

where $f \in W^{1,p}(\Omega, \mathbb{R}^n)$.

In fact, the equality (4) cannot be seen in the distributional sense, hence even distributional exterior differentiation is not possible.

We extend the technique of V. M., J. Malý, S. Mongodi 2015 in the Heisenberg group to this general setting, showing that

V. M., A. Zapadinskaya 2017

Let $\Omega \subset \mathbb{R}^n$ open, $1 \leq k < n$, and $k \leq m$. If $p > k > 1$, or $p = k = 1$ and $f \in W^{1,p}_{loc}(\Omega, \mathbb{R}^m)$, η is a C^1 smooth k-form in $\mathbb{R}^m$, then the condition $f^*\eta = 0$ a.e. implies that

$$f^*(d\eta) = 0 \quad \text{a.e.}$$

Due to the assumptions we have to assume for the previous "very weak" exterior differentiation, our minimal regularity assumptions on the Sobolev surface $f \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ are

$$\begin{cases} p > n - m & \text{if } n - m > 1 \\ p = 1 & \text{if } n - m = 1 \end{cases} . \tag{5}$$

Indeed we have to differentiate a $k = n - m$ differential form.

Question

Can we get the Gromov's dimensional estimate for all Sobolev hypersurfaces surfaces of regularity $p \geq 1$ in all stratified Lie groups?

We could interpret this question as asking whether the image $f(\Omega)$ as a set *locally separate the space into two parts* also for $p \geq 1$.
This separation property is typical of topological hypersurfaces and it is one of the main points in the Gromov's proof.

Graph form for Sobolev hypersurfaces

Our Sobolev regularity does not allow for an implicit function theorem, that could transform the set into a graph.

On the other hand, one could consider a set S as a Sobolev graph with respect to a $W^{1,1}_{loc}$ Sobolev function

$$\Sigma = \{(x, u(x)) : x \in \mathcal{U}\}$$

for an open set $\mathcal{U} \subset \mathbb{R}^{n-1}$.

Question

For this special graph form of the Sobolev hypersurface, can we expect the validity of the Gromov's dimensional estimate

$$\mathrm{Hausd}_{dim}(\Sigma) \geq Q - 1?$$

Isoperimetric inequality in stratified Lie groups

For a Sobolev hypersurface $\Sigma = \{(x, u(x)) : x \in \mathcal{U}\}$, we can introduce an inner and outer part of the space by defining

$$E = \{(x,t) : t < u(x),\ x \in \mathcal{U}\}\,,$$

that is a set of locally finite perimeter in the Euclidean sense.
We can then apply the local sub-Riemannian isoperimetric inequality

$$\min\{|B(x,r)\cap E|, |B(x,r)\setminus E|\} \le C_{iso}\,|\partial_H E|(B(x,r))^{Q/(Q-1)},$$

where $|\partial_H E|$ is the homogeneous perimeter on $\mathbb{G}$.
Since points of the reduced boundary of E have half spaces as blow-ups, one may easily expect that

$$0 < \min\{|B(x,r)\cap E|, |B(x,r)\setminus E|\}\,.$$

This is the property of the Sobolev surface S to locally separate the space into two parts.

If $|\partial E|$ is the Euclidean perimeter, then

$$|\partial_H E|\llcorner\mathcal{F}E = |\nu_H||\partial E|\llcorner\mathcal{F}E,$$

where $\mathcal{F}E$ is the Euclidean reduced boundary. This implies that

$$|\partial_H E|(B(x,r)) = \int_{B(x,r)\cap\mathcal{F}E} |\nu_H|\, d|\partial E| > 0.$$

By L. Ambrosio, Adv Math 2001, there holds

$$|\partial_H E| = \omega\, \mathcal{S}^{Q-1}\llcorner\mathcal{F}E$$

for a $\mathcal{S}^{Q-1}$ measurable $\omega : \mathcal{F}_H E \to [c_1, c_2]$, with $0 < c_1 \le c_2$.

Then there exists a subset $F_0 \subset \mathcal{F}E$ with $\mathcal{S}^{Q-1}(F_0) > 0$.

It is also reasonable to expect that

$$\Sigma_0 = \{(x, u(x)) \in \Sigma : \ u \text{ is approximately differentiable at } x\ \}$$

is contained in $\mathcal{F}E$ up to $\mathcal{H}^{n-1}_{|\cdot|}$ negligible sets. It follows that

$$\mathcal{S}^{Q-1}(\Sigma_0 \cap F_0) > 0 \quad \text{and} \quad \Sigma_0 \cap F_0 \subset \Sigma.$$

Comments on the "very weak" exterior differentiation

Sobolev embedding on spheres

For $p > n-1$ e $u \in W^{1,p}(\partial B(x,r))$ we have

$$\operatorname{diam}\big(u(\partial B(x,r))\big)^p \le C(n,p)\, r^p \fint_{\partial B(x,r)} |\nabla u|^p \, d\mathcal{H}^{n-1}$$

up to selecting the continuous representative of u in $\partial B(x,r) \subset \mathbb{R}^n$.

Thus, up to technical details, we can show that the *rescaled function*

$$y \to f_{z,r}(y) = \frac{f(z+ry) - f(z)}{r}$$

at a Lebesgue point $z \in \Omega$ both for f and ∇f, up to selecting a suitable sequence of radii and for a.e. t, satisfies

$$f_{z,r_j}|_{\partial B(0,t)} \to g|_{\partial B(0,t)} \quad \text{in } L^\infty(\partial B(0,t)),$$

where $g = df(z) : \mathbb{R}^n \to \mathbb{R}^m$.

In heuristic terms, our assumption $f^*\eta = 0$ implies on the rescaled functions that

$$\int_{\partial B(0,t)} f^*_{z,r_j}\eta = 0.$$

By the previous convergence we have

$$\begin{aligned}
\int_{\partial B(0,t)} f^*_{z,r_j}\eta \quad &\to \quad \int_{\partial B(0,t)} (g^*\eta)\big(f(z)\big) \\
\text{(by Stokes theorem)} \quad &= \quad \int_{B(0,t)} (g^* d\eta)(f(z)) \\
&= \quad \mathcal{L}^n\big(B(0,t)\big)\ f^*(d\eta)(z)(e_1 \wedge \cdots \wedge e_n),
\end{aligned}$$

therefore we conclude that

$$f^*(d\eta)(z) = 0.$$