9^{th} Lecture (3h)

(1) Def'n. The *graded coordinates* x in $\mathbb{R}^n$ are exactly those coordinates of V associated to a graded basis.

(2) Thm. Let x, y be some graded coordinates of $\mathbb{R}^n$. Then the induced group operation in $\mathbb{R}^n$ has the form $p(x,y) = x + y + S(x,y)$ where $S(x,y)$ is the "noncommutative part". Then
  (a) $S_j = 0$ if $j \leq m$
  (b) $S_j(\sigma_r(x), \sigma_r(x)) = r^{d_j} R_j(x,y)$
  (c) S_j belongs to the ideal of polynomials spanned by $x_p y_s - x_s y_p$, where $p < s$ and $d_p, d_s < d_j$,
  (d) $X_j = \partial_{x_j} + \sum_{l=m+1}^n a_j^l(x)\,\partial_{x_l}$, where $a_j^l(x) = \partial_{y_j} S^l(x,0)$.
  (e) $X_j^R = \partial_{x_j} + \sum_{l=m+1}^n b_j^l(x)\,\partial_{x_l}$, where $b_j^l(x) = \partial_{x_j} S^l(0,x)$

No prf: Just mention that it follows from BCH!,......

(3) Horizontal subbundle: $V_1 = \operatorname{span}\{X_1, \ldots, X_m\}$ and the *horizontal subspace*

$$H_p V = \operatorname{span}\{X_1(p), \ldots, X_m(p)\} \quad \text{for every } p \in V\,.$$

The left invariant vector fields X_j, with $1 \leq j \leq n$ are called *horizontal.*

(4) In $\mathbb{H}^n$ represented by $\mathbb{C}^n \times \mathbb{R}$, we have at the point $p = (x,y,t)$ the *horizontal subspace*

$$H_p\big(\mathbb{C}^n \times \mathbb{R}\big) = \left\{ \sum_{j=1}^n \lambda_j\,\partial_{x_j} + \mu_j\,\partial_{y_j} + 2\Big(\sum_{j=1}^n \lambda_j y_j - \sum_{j=1}^n \mu_j x_j\Big)\partial_t \;\Big|\; \lambda_j, \mu_j \in \mathbb{R} \right\} \subset T_p\big(\mathbb{C}^n \times \mathbb{R}\big)$$

(5) Def'n of *horizontal curve.* An AC curve $\gamma : [a,b] \longrightarrow V$ is horizontal if $\dot\gamma(t) \in H_{\gamma(t)}H$ for a.e. t.

(6) There are plenty of horizontal curves. As an example, considering the variables (x_1, y_1, x_2, y_2, t) in $\mathbb{H}^2$ and the left invariant horizontal vector fields $X_j = \partial_{x_j} + 2y_j\partial_t$, $Y_j = \partial_{y_j} - 2x_j\partial_t$ take for instance the curve

$$\gamma(t) = \big(\cos t, \sin t, t, 0, -2t\big)$$

and notice that it is horizontal, since

$$\dot\gamma(t) = -\sin t\; X_1(\gamma(t)) + \cos t\; Y_1(\gamma(t)) + X_2(\gamma(t))$$

Clearly, not all curves are horizontal. Using the same coordinates, consider for instance

$$\gamma(t) = \big(t, 1, 0, 0, 0\big)$$

then

$$\dot\gamma = \partial_{x_1}(\gamma) = X_1(\gamma) - 2y_1\partial_t(\gamma) = X_1 - 2T\,.$$

(7) Any stratified group is connected by horizontal paths.
prf. Define $[x,y] = xyx^{-1}y^{-1}$ and let $(U_1, \ldots, U_m)$ be a basis of V_1. Then by BCH we get

$$\varphi_U(\exp Y) = [\exp Y, \exp U] + O(|Y|^2), \quad \text{then} \quad d\varphi_U(0)(Y) = [Y, U]$$

Take a multi-index $\alpha \in I_m^j$ and define

$$\varphi_\alpha^j = \varphi_{U_{\alpha_j}} \circ \varphi_{U_{\alpha_{j-1}}} \circ \cdots \circ \varphi_{U_{\alpha_2}}$$

observing by the chain rule that

$$d\varphi^j_\alpha(0) = [,[,[\cdots,[Y,U_{\alpha 2}],U_{\alpha 3}],\cdots,U_{\alpha_j}]$$

Since $\mathcal{G}$ is stratified, we have

$$V_j = [V_1, V_{j-1}] = [V_1,[V_1,V_{j-2}]] = \underbrace{[V_1,[V_1,[\cdots,[V_1,V_1],\cdots]}_{j-\text{times}}$$

any vector of V_j can be written as a linear combination of elements of the form

$$U_\alpha = [,[\cdots,[U_{\alpha_1},U_{\alpha 2}],U_{\alpha 3}],\cdots,U_{\alpha_j}]$$

for suitable multi-indexes $\alpha = (\alpha_1,\dots,\alpha_j) \in I^j_m$.

Now we select families $\mathcal{A}_j \subset I^j_m$ such that

$$\{U^j_\alpha\}_{\alpha\in\mathcal{A}_j} \quad \text{is a basis of} \quad V_j \quad \text{for every} \quad j = 1,\dots,\iota$$

notice that for $j = 1$, we have $\mathcal{A}_1 = I_m$, then consider

$$\Phi(x^j_\alpha) = \prod_{j=1}^{\iota}\prod_{\alpha\in\mathcal{A}_j} \varphi^j_\alpha\big(\exp(x^j_\alpha U_{\alpha_1})\big)$$

then

$$\partial_{x^j_\alpha}\Phi(0) = U^j_\alpha$$

is a basis of T_eV!! It follows that Φ is surjective from a neighbourhood of the origin in $\mathbb{R}^N$ for some $N \in \mathbb{N}$ onto a neighbourhood of 0 in V.

Then a neighbourhood of 0 in V is connected by horizontal curves and this implies global connectedness.

(8) Sub-Finsler metric. Fix a norm on $|\cdot|_F : H_0V \longrightarrow \mathbb{R}$. Then the corresponding left invariant Finsler norm. Let $X \in H_pV$ and set

$$|X|_p = |dL_{p^{-1}}X|_F\,.$$

Then $p \longrightarrow |\cdot|_p$ is a continuous and left invariant Finsler norm, namely

left translating a vector does not affect its norm

by construction, for $A \in H_qV$ we have

$$|dL_pA|_{pq} = |A|_q$$

(9) Length of horizontal curves

$$\text{length}(\gamma) = \int_a^b |\dot\gamma(t)|_{\gamma(t)}\,dt$$

(10) Rmk. If $|\cdot|_F = \sqrt{\langle\cdot,\cdot\rangle}$, then we have a SR metric

(11) We defined the SF distance between p and q to be

$$d(p,q) = \inf\{\text{length}(\gamma) \mid \gamma \text{ is horizontal and connects } p \text{ with } q\}$$

that is finite due to connectivity by horizontal curves.

(12) It is also called the Carnot-Carathéodory distance of the group V

(13) To show that d is a distance, triangle inequality and symmetry are rather simple. For the first one, just join "quasi geodesics",...

(14) Lemma. For every p in V there exists $c_p > 0$ such that for every $q \in V$ we have

$$d(p,q) \geq c_p \, \min\{1, |p-q|_E\}$$

prf. Take any horizontal curve $\gamma : [a,b] \longrightarrow V$ joining p with q. If $\gamma([a,b]) \subset \overline{B^E_{p,1}}$, then

$$\int |\dot\gamma|_\gamma = \int |dL_{\gamma(t)^{-1}}\dot\gamma(t)| \geq c \int |\dot\gamma(t)|_E = c\,|p-q|_E$$

where we have defined

$$c_p = \min_{p\in\overline{B^E_{p,1}}} \inf_{v\in T_pV\setminus\{0\}} \left\{ \frac{|dL_{p^{-1}}v|_F}{|v|_E} \right\},$$

then taking the infimum, it follows that $d(p,q) \geq c\,|p-q|_E$. If $\gamma([a,b]) \nsubseteq \overline{B^E_{p,1}}$, then there exists $\tilde b < b$ such that $\gamma(\tilde b) \in \partial B^E_{p,1}$, then applying the previous argument

$$d(p,q) \geq c_p\,|\gamma(\tilde b) - p|_E = c_p\,.$$

(15) By this lemma, it follows immediately that $d(p,q) = 0$ yields $p = q$. The opposite being trivial, we have shown that d is a distance on the stratified group V.

(16) Where does the stratification appear in this distance? Where does appear the choice of the left invariant metric?

(17) We have that d is left invariant and homogeneous, namely

$$d(px,py) = d(x,y), \qquad d(\delta_r x, \delta_r y) = r\,d(x,y)$$

(18) Define an auxiliary "homogeneous norm", define $x = \big(x^{(1)},\dots,x^{(\iota)}\big)$

$$\|x\|_G = \sum_{j=1}^{\iota} |x^{(j)}|_E^{1/j}$$

it is homogeneous, quasi-triangle inequality,......bla bla

(19) Let $K = \{|p|_G \leq 1\}$ be compact and let K^* be its closed convex envelop, then letting $\gamma(t) = tp$ with $p \in K$, one gets

$$d(p) := d(p,0) \leq \int |\dot\gamma|_\gamma \leq \big(\max_{p\in K^*}\max_{|v|_E=1} |dL_{p^{-1}}v|_F\big)\,|p|_E \leq c$$

then by homogeneity of both d and $\|\cdot\|_G$, for every $p,q \in V$, we get

$$d(p,q) \leq c\|p^{-1}q\|_G$$

(20) For p,q in a compact set E of V we have

$$\gamma(E)^{-1}\,|p-q|_E \leq d(p,q) \leq \gamma(E)\,|p-q|_E^{1/\iota}$$

prf. for the sake of simplicity in in step 2 stratified groups: we have

$$\|p^{-1}q\|_G = \|-p+q-\frac{1}{2}\,[p,q]\|_G$$

and also

$$\begin{aligned}
&\big|\big(p^{-1}q\big)^{(2)}\big|_E = |-p^{(2)}+q^{(2)}-\frac{1}{2}\,[p^{(1)},q^{(1)}]| \\
&\leq |-p^{(2)}+q^{(2)}| + c_1\,|q^{(1)}|\,|-p^{(1)}+q^{(1)}| \leq K_q\,|p-q|_E\,,
\end{aligned}$$

then for q running in a compact set

$$\|p^{-1}q\|_G \leq K\ |p-q|_E^{1/2}\,.$$

the opposite inequality follows from coercivity inequality seen before.

(21) Then the topology of the SF-distance coincides with the topology of V.

(22) One one has the distance it can construct a measure!!!

(23) In some fixed graded coordinates, defining the Lebesgue measure, then it defines a left invariant measure. This follows by the properties of $Q(x, y)$ that make the left and right translations a preserving volume mappings.

(24) Since the change of graded coordinates is just a linear mapping of $\mathbb{R}^n$, the invariance does not depend on the chosen coordinates. Then we have found the Haar measure of the group.

(25) Furthermore, taking into account of the form of dilations σ_r in $\mathbb{R}^n$, we get

$$\mu(B_{p,r}) = \mu(B_r) = \mu(\delta_r(B_1)) = r^Q\mu(B_1)$$

then standard covering theorems imply that the Hausdorff dimension of V with respect to d is exactly Q (it is an exercise using Vitali's covering theorem)

(26) This is the natural dimension of the group, that is strictly bigger than its topological dimension and makes it a fractal object.

(27) Then problems like studying the Hausdorff dimension and Hausdorff measure of smooth submanifolds naturally arise,....