8^{th} Lecture (2h)

(1) Qst. When a *nilpotent group* G is isomorphic to its Lie algebra $\mathcal{G}$ equipped with the group operation given by BCH formula?

(2) Exs. The circle $\mathbb{T}^1$ has 1-dimensional Lie algebra and it is clearly nilpotent, since it is commutative and its Lie algebra is isomorphic to the real line. The BCH becomes trivial $c(X,Y) = X+Y$, then its Lie algebra is isomorphic to $(\mathbb{R},+)$, but this cannot be isomorphic to $\mathbb{T}^1$!! Notice that $\mathbb{T}^1$ is not simply connected!

(3) Thm. Let G be a nilpotent Lie group. If G is connected and simply connected, then the exponential mapping $\exp : \mathcal{G} \longrightarrow G$ is a diffeomorphism, that makes

$$(\mathcal{G}, \circledcirc) \quad \text{isomorphic to} \quad G.$$

prf. The exponential mapping is surjective since the group BCH formula is defined for every couple of vectors and $G = \cup_{n\geq 1} U^n$. It is invertible around any point onto the image, namely it is a local homeomorphism, since the differential is everywhere injective. In particular, it is an open mapping. Finally, a surjective local homeomorphism taking values in a connected, simply connected space is injective.

(4) Suppose you have two nilpotent, connected and simply connected Lie groups G_1 and G_2 with the same Lie algebra. Then they are isomorphic groups, namely they are the same group, but with different names for their elements!! We have

$$\exp_1 : \mathcal{G} \longrightarrow G_1, \quad \exp_2 : \mathcal{G} \longrightarrow G_2, \quad \text{then} \quad \exp_2 \circ \exp_1^{-1} : G_1 \longrightarrow G_2$$

is the group isomorphism.

(5) Def'n. The Heisenberg group $\mathbb{H}^n$ is the unique stratified group, up to Lie group isomorphisms.

(6) Rmk. Then as a consequence, we can denote $\mathbb{C}^n \times \mathbb{R}$ equipped with our group operation by $\mathbb{H}^n$, since it represents the "abstract" Heisenberg group in some coordinates.

(7) Unqness. More generally, given a stratified algebra, there exists a unique stratified group up to Lie group isomorphisms, whose Lie algebra is the given stratified algebra.

(8) Thm. Every stratified group is isomorphic to a stratified algebra V equipped with the group operation given by BCH

$$X \circledcirc Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y] - [Y,[X,Y]]\right) + \cdots .$$

prf. It suffices to check that the Lie algebra of V is isomorphic to V itself as a Lie algebra.

(9) Rmk. Then from the algebra we can recover the entire group, namely, the group operation.

(10) A stratified algebra allows for intrinsic dilations

$$\delta_r : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \delta_r\left(\sum_{j=1}^{\iota} X_j\right) = \sum_{j=1}^{\iota} r^j X_j$$

for every $X_j \in V_j$.

(11) Rmk. $(V, \circledcirc, \delta_r)$ *is a universal intrinsic representation of a stratified group that generalizes a vector space* $(V, +, r)$

(12) The finite dimensional vector space V coincides with the abelian case $[X, Y] = 0$, then BCH gives $X \circledcirc Y = X + Y$.

(13) Rmk. *Once we have fixed a basis in the algebra we can identify the group with* $\mathbb{R}^n$*!!*

(14) Exm. From the Heisenberg algebra $\mathfrak{h}^n$ with basis U_j, V_j, Z we easily get the law

$$\exp\Big(tZ + \sum_{j=1}^{n} x_j U_j + y_j V_j\Big)\ \exp\Big(t'Z + \sum_{j=1}^{n} x'_j U_j + y'_j V_j\Big)$$
$$= \exp\Big(\sum_{j=1}^{n}(x_j + x'_j)U_j + (y_j + y'_j)V_j + \big(t + t' + \frac{1}{2}\sum_{j=1}^{n} x_j y'_j - y_j x'_j\big)Z\Big)$$

Then with respect to this basis in $\mathfrak{h}^n$ we have a group law in $\mathbb{R}^n$. This works in general,...

(15) Rmk. Changing the basis in $\mathfrak{h}^n$ would yield another formula for the group operation, but representing the same group.

(16) Rmk. Getting the group operation is in general a not short computation, as for instance in the Engel group. Left as an exs.

(17) Def'n. *Graded basis* $(X_1, \dots, X_n)$ of an n-dimensional stratified algebra $\mathfrak{g}$ equipped with m_j-dimensional layers V_j satisfies the condition

$$(X_1, \dots, X_{m_1}) \quad \text{is a basis of } V_1$$

and more generally

$$(X_{n_{j-1}+1}, \dots, X_{n_j}) \quad \text{is a basis of } V_j$$

with $n_j = \sum_{l=1}^{j} m_l$, where $j \geq 1$.

(18) Def'n. The *graded coordinates* x in $\mathbb{R}^n$ are exactly those coordinates of V associated to a graded basis.

(19) Rmk. In other words the basis respects the grading. The basis we have chosen in the previous example for the Heisenberg group is a graded basis.

(20) Elements of a graded basis can be given a degree. We say that X_j has degree d_j if $X_j \in V_{d_j}$. Then

$$\delta_r X_j = r^{d_j} X_j$$

(21) Once a graded basis is fixed, then any stratified group $(V, \circledcirc, \delta_r)$ can be represented as $(\mathbb{R}^n, p(x, y), \sigma_r)$, where setting

$$\mathcal{I} : V \longrightarrow \mathbb{R}^n, \qquad \mathcal{I}(X_j) = e_j$$

we have

$$p(x, y) = x + y + Q(x, y) = \mathcal{I}\Big(\exp\big(\sum_{j=1}^{n} x_j X_j\big)\exp\big(\sum_{j=1}^{n} y_j X_j\big)\Big)$$

$$\sigma_r(x) = \mathcal{I}\Big(\exp\Big(\sum_{j=1}^{n} r^{d_j} x_j X_j\Big)\Big) = \sum_{j=1}^{n} r^{d_j} x_j e_j$$