7^{th} Lecture (2h)

(1) Sufficiently many iterated Lie brackets vanish,... what is the precise notion of nilpotent algebra?

(2) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two subsets of a Lie algebra $\mathfrak{g}$. Then we define the following subspace of $\mathfrak{g}$

$$[\mathfrak{a}, \mathfrak{b}] = \left\{ \sum_{j=1}^{n} \lambda_j [X_j, Y_j] \mid \lambda_j \in \mathbb{R},\ X_j \in \mathfrak{a},\ Y_j \in \mathfrak{b} \text{ and } n \in \mathbb{N} \right\}.$$

(3) Exm. Let $\mathfrak{h}^n$ be the Heisenberg algebra with basis $(U_1, \ldots, U_n, V_1, \ldots, V_n, Z)$ and non-trivial brackets $[U_j, U_j] = Z$. Then

$$[\mathfrak{h}^n, \mathfrak{h}^n] = \text{span}\{z\} \quad \text{and} \quad [\mathfrak{h}^n, [\mathfrak{h}^n, \mathfrak{h}^n]] = \{0\}.$$

We say that it is nilpotent of step 2.

(4) Exm. Let $\mathfrak{r} = \{U_1, U_2, U_3\}$ be the *rototranslation algebra* given by the Lie bracket relations

$$[U_1, U_2] = U_3, \quad [U_2, U_3] = U_1, \quad [U_1, U_3] = 0\,.$$

(a) Left as an exs. Check that it is a Lie algebra.
(b) $[\mathfrak{r}, \mathfrak{r}] = \text{span}\{U_1, U_3\}$ and $[\mathfrak{r}, [\mathfrak{r}, \mathfrak{r}]] = \text{span}\{U_1, U_3\}$ and so on ...
(c) iterating preocess we will never get the zero space,... this algebra is not nilpotent

(5) Def'n. Let $\mathfrak{g}$ be a Lie algebra. We define the *descending central series*

$$\mathfrak{g}^{(1)} = \mathfrak{g} \quad \text{and} \quad \mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}] \quad \text{for every integer} \quad k \geq 1.$$

(6) We have the straightforward properties
(a) $\mathfrak{g}^{(k)} \supset \mathfrak{g}^{(k+1)}$ for every k
(b) $\mathfrak{g}^{(n)} = \mathfrak{g}^{(n+1)}$ for some n implies $\mathfrak{g}^{(m)} = \mathfrak{g}^{(n)}$ for every $m \geq n$.

(7) Def'n. A Lie algebra $\mathfrak{g}$ is nilpotent if there exists a positive integer ι such that $\mathfrak{g}^{(\iota)} \neq \{0\}$ and $\mathfrak{g}^{(\iota+1)} = \{0\}$.

(8) Rmk. The Heisenberg algebra $\mathfrak{h}^n$ is a 2-step nilpotent Lie algebra, where the descending central sequence is

$$(\mathfrak{h}^n)^{(2)} = \text{span}\{z\}, \qquad (\mathfrak{h}^n)^{(3)} = \{0\}$$

(9) Def'n. A stratified algebra is a Lie algebra $\mathfrak{g}$ such that
(a) $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\iota$ (direct sum)
(b) $[V_1, V_j] = V_{j+1}$ for every $j = 1, \ldots, \iota - 1$ (Lie bracket generating condition)
(c) $[V_\iota, V_1] = 0$ (nilpotence)

(10) Rmk. The descending central series of a stratified algebra is given by

$$\mathfrak{g}^{(k)} = V_k \oplus \cdots \oplus V_\iota$$

then $\mathfrak{g}$ is nilpotent of step ι.

(11) Rmk. The Heisenberg algebra $\mathfrak{h}^n$ is a 2-step stratified Lie algebra, where

$$V_1 = \text{span}\{U_1, \ldots, U_n, V_1, \ldots, V_n\}, \quad V_2 = \text{span}\{Z\} \quad \text{and} \quad \mathfrak{h}^n = V_1 \oplus V_2$$

(12) Left as an exs. The Engel algebra is a four dimensional space $\mathfrak{e}_4$ with basis (U_1, U_2, U_3, U_4) with the only nontrivial bracket relations

$$[U_1, U_2] = U_3 \quad \text{and} \quad [U_1, U_3] = U_4$$

(13) The Engel algebra $\mathfrak{e}_4$ is a stratified algebra of step three. How do we find a stratification? We start from the descending central series

$$(\mathfrak{e}_4)^{(2)} = \text{span}\{U_3, U_4\}, \quad (\mathfrak{e}_4)^{(3)} = \text{span}\{U_4\}\,.$$

Then we choose V_1 such that

$$\mathfrak{e}_4 = V_1 \oplus (\mathfrak{e}_4)^{(2)} \quad \text{and} \quad (\mathfrak{e}_4)^{(2)} = V_3 \oplus (\mathfrak{e}_4)^{(3)}$$

then for instance

$$V_1 = \text{span}\{U_1, U_2\}, \quad V_2 = \text{span}\{U_3\} \quad \text{and} \quad V_3 = \text{span}\{U_4\}$$

Infinitely many other choices would be possible!!

(14) Def'ns.
  (a) A *nilpotent group* is any Lie group whose Lie algebra is nilpotent.
  (b) A *stratified group* is a connected and simply connected Lie group whose Lie algebra is stratified.

(15) The group $\mathbb{C}^n \times \mathbb{R}$ is connected and simply connected and its algebra is isomorphic to the Heisenberg algebra, then it represents the $2n+1$ dimensional Heisenberg group $\mathbb{H}^n$ in some coordinates. It is a stratified group.

(16) We want to relate the group operation in G with some operation in the Lie algebra $\mathcal{G}$.

(17) Since $d\exp(0) = \text{Id}_{\mathcal{G}}$, then the exponential mapping yields a bianalytic diffeomorphism $\exp : U_0 \longrightarrow V_e$ and for sufficiently small $X, Y \in \mathcal{G}$ we get

$$\exp^{-1}\big(\exp X \exp Y\big) = c(X, Y) \in \mathfrak{g}\,.$$

(18) $X, Y \longrightarrow c(X, Y)$ is an *analytic function* of (X, Y) that is well defined for for X, Y sufficiently small. Its explicit formula is the celebrated Baker-Campbell-Hausdorff formula.

(19) By analyticity we have

$$c(tX, tY) = \sum_{n=1}^{\infty} t^n\, c_n(X, Y)\,.$$

where by rescaling argument one easily sees that $c_n(tX, tY) = t^n c_n(X, Y)$.

(20) One can write down an iterative formula for $c_{n+1}(X, Y) = F(c_1, c_2, \ldots, c_{n-1})$ of rather high complexity. The first terms are

$$c_1(X, Y), \quad c_2(X, Y) = \frac{1}{2}[X, Y] \quad \text{and} \quad c_3(X, Y) = \frac{1}{12}\big([X, [X, Y]] - [Y, [X, Y]]\big)$$

in general we have

$$(n+1)c_{n+1}(X, Y) = \frac{1}{2}\,[X - Y, c_n(X, Y)] +$$
$$\sum_{1 \le p \le n/2} K_{2p} \sum_{\substack{j_1, \ldots, j_{2p} > 0 \\ j_1 + \cdots j_{2p} = n}} [c_{j_1}(X, Y), [c_{j_2}(X, Y), [\cdots, [c_{j_{2p}}(X, Y), X + Y],], \cdots]$$

numbers K_{2p} are the coefficients of a fixed analytic function that does not depend on the group,...

(21) Rmk. In general c_n are polynomials of degree n in the variables X and Y with respect to the Lie product

(22) Rmk. If the group is ι-nilpotent, then

   (a) $c(X,Y) = \sum_{n=1}^{\iota} c_j(X,Y)$

   (b) $c(\cdot,\cdot)$ is defined on all of $\mathcal{G} \times \mathcal{G}$.

   (c) Set $X \circledcirc Y = \sum_{n=1}^{\iota} c_j(X,Y)$, then $\mathcal{G}$ is also a Lie group!!

(23) Qst. When a *nilpotent group* G is isomorphic to its Lie algebra $\mathcal{G}$ equipped with the group operation given by BCH formula?