6^{th} Lecture (2h)

(1) Geometric idea of push-forward of vector fields: picture

(2) Def'n of f_*X and meaning reasoning with integral curves

(3) Exm. $f : \mathbb{R} \longrightarrow]-\pi/2, \pi/2[$, $f(x) = \arctan(x)$, $X(t) = a(t)\partial_t$. Then for $y \in]-\pi/2, \pi/2[$, we have $f_*X(y)$....

(4) Rmk. $(f \circ g)_* = f_* \circ g_*$

(5) Left as Exs. Show that X is left invariant if and only if $(L_p)_*X = X$ for every $p \in G$.

(6) Thm. Let $X, Y \in \mathfrak{X}(M)$ and let $f : M \longrightarrow N$ be a diffeomorphism. Then

$$f_*\big([X,Y]\big) = [f_*X, f_*Y].$$

(7) Cor. Let X, Y be left invariant. Then $[X,Y]$ is left invariant, namely $\mathcal{G}$ is a Lie algebra w.r.t. Lie product of vector fields.

(8) Denote by $\mathfrak{gl}_n(\mathbb{R})$ the Lie algebra of $GL_n(\mathbb{R})$. Notice that (after some computations)

$$[X_B, X_C] = X_{[B,C]}$$

where $B, C \in M_n(\mathbb{R})$ and $[B,C] = BC - CB$. Then we *identify* $gl_n(\mathbb{R})$ with $(M_n(\mathbb{R}), [\cdot,\cdot])$.

(9) Denote by $\mathfrak{sl}_n(\mathbb{R})$ the Lie algebra of $SL_n(\mathbb{R})$. Then we have seen

$$\mathfrak{sl}_n(\mathbb{R}) \simeq (\{B \mid \mathrm{Tr}(B) = 0\}, [\cdot,\cdot])$$

where again $[B,C] = BC - CB$.

(10) Exm. Lie algebra of $\mathbb{C}^n \times \mathbb{R}$. By yesterday's computations we have know that

$$[X_j, Y_j] = -4T, \quad [X_i, T] = 0 \quad \text{and} \quad [X_i, X_j] = 0 \quad \text{if } i \neq j.$$

by bilinearity of $[\cdot,\cdot]$ it suffices to define the Lie product on a basis!

(11) Prop'n. Let $X \in \mathfrak{X}(M)$. Then the following holds
(a) $\Phi^X_{t+\tau} = \Phi^X_t \circ \Phi^X_\tau$
(b) $\Phi^{sX}_{t/s} = \Phi^X_t$.
In particular, we have

$$\Phi^{tX}_1(p) = \Phi^X_t(p) \quad \text{and} \quad \Phi^{\alpha X}_t(p) = \Phi^X_{\alpha t}(p)\,.$$

(12) Prop'n. Let $f : M \longrightarrow N$ be a diffeomorphism and let $X \in \mathfrak{X}(M)$. Then

$$f\big(\Phi^X(q,t)\big) = \Phi^{f_*X}(f(q), t)\,.$$

for every $q \in M$ and every $t \in \mathbb{R}$ such that $\Phi^X(q,t)$ is defined.
prf. by uniqueness of Cauchy problem

(13) Every $X \in \mathcal{G}$ is complete and also

$$y\Phi_t^X(e) = \Phi_t^X(y)$$

prf. Consider $\phi_t^X(e)$ initially defined on $]-2\delta, 2\delta[$, hence define

$$\gamma(t) = \begin{cases} \phi_t^X(e) & -2\delta < t < 2\delta \\ y\phi_{t-\delta}^X(e) & -\delta < t < 3\delta \end{cases}$$

check that it is smooth and then repeat on the next intervals,..

(14) Def'n. Exponential mapping $\exp : \mathcal{G} \longrightarrow G$, $\exp X = \Phi_1^X(e)$

(15) If $G = GL_n(\mathbb{R})$, then $\exp B = e^B$.
prf. notice that $\Phi_t^B(I) = e^{tB}$, since they solve the same Cauchy problem

(16) $\exp : \mathrm{Lie}(\mathbb{T}^1) \longrightarrow \mathbb{T}^1$ is not injective, $i\beta \in \mathrm{Lie}(\mathbb{T}^1)$, $\exp i\beta = e^{i\beta}$.

(17) $\exp : \mathfrak{o}(2) \longrightarrow O(2)$ is not surjective. $\exp B = e^B$, $\det(e^B) = e^{\mathrm{Tr}(B)} > 0$.