5^{th} Lecture (3h)

(1) Exm. Find left invatiant vector fields in $GL_n(\mathbb{R})$: here $T_I\big(GL_n(\mathbb{R})\big) \simeq M_n(\mathbb{R})$

(2) Exm. Find left invariant vector fields of $SL_n(\mathbb{R})$. Notice that $\det(e^A) = e^{\mathrm{Tr}(A)}$, then e^{tB}, with $\mathrm{Tr}(B) = 0$, belongs to $SL_n(\mathbb{R})$,...then $X_B(A) = AB$ where $A \in SL_n(\mathbb{R})$ and $\mathrm{Tr}(B)$, here $T_I(SL_n(\mathbb{R})) \simeq \{B \mid \mathrm{Tr}(B) = 0\}$.

(3) Def'n. A derivation $L : C^\infty(M) \longrightarrow C^\infty(M)$ is a linear function satisfying Leibniz $L(uv) = uL(v) + vL(u)$. Denote by $\mathrm{Der}(M)$ the linear space of all derivations on $C^\infty(M)$.

(4) There exists a linear isomorphism between $\mathrm{Der}(M)$ and the space of vector fields $\mathfrak{X}(M)$.
prf. Let $L \in \mathrm{Der}(M)$ and choose $p \in M$. The "induced" derivation at p

$$Z_p(u) := (Lu)(p)$$

defines a section $p \longrightarrow Z_p \in T_pM$ and in local coordinates

$$x \longrightarrow Zu(x) = \sum a^j(x)\,\partial_{x_j} u(x)$$

is smooth for every u, then take $u = x_k$ and get the smoothness of $a^k(x)$. Viceversa, consider a vector field $Z \in \mathfrak{X}(M)$ and for every $u \in C^\infty(M)$ define

$$L_Z : C^\infty(M) \longrightarrow C^\infty(M), \quad (L_Z u)(p) = Z_p(u) \quad \text{for every} \quad p \in M\,.$$

Smoothness of Z_p implies the smoothness of $L_Z(u) : M \longrightarrow \mathbb{R}$, then it is immediate to observe that $L_Z \in \mathrm{Der}(M)$. Then $Z \longrightarrow L_Z$ is well defined and surjective. It is clearly linear and its kernel is 0-dimensional, then it is an isomorphism.

(5) Def'n. L_X denotes the derivation determined by $X \in \mathfrak{X}(M)$

(6) Given derivations L, T, then $L \circ T$ is no longer a derivation, since it is a second order differential operator.
Exm. Consider $L = \partial_x$, $T = \partial_y$ in $\mathbb{R}^2$ and $LT = TL = \partial_x\partial_y$, that is linear but does not satisfy Leibniz.

(7) Def'n of commutator. We define

$$[L, T] = L \circ T - T \circ L : C^\infty(M) \longrightarrow C^\infty(M)$$

(8) It is immediate to check that $[L, T]$ is linear and satisifes Leibniz, namely it is a derivation.

(9) We denote by $[X, Y]$ the unique vector associated to $[L_X, L_Y]$, called Lie bracket of X and Y. Using the definition of commutator one checks immediately the "Jacobi identity"

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

is satisfied.

(10) In local coordinates the Lie bracket becomes

$$\left[\sum_j a^j \partial_{x_j}, \sum_s b^s \partial_{x_s}\right] = \sum_s \left(\sum_j a^j \partial_{x_j} b^s - b^j \partial_{x_j} a^s\right) \partial_{x_s}$$

(11) Exm. Consider $Z = (x^2 - y)\,\partial_x + (y^2 + x)\partial y + (x + \sin\theta)\partial_\theta$ and $T = (\cos\theta + y)\partial_x - \partial_y + \partial_\theta$ on $\mathbb{R}^2 \times \mathbb{T}^1$ and compute $[Z, T]$,... remember that second order terms cancel!

(12) Def'n of real Lie algebra $\mathfrak{g}$.

(13) Def'n of Heisenberg algebra $\mathfrak{h}^n$.

(14) Def'n of flow. Let $X \in \mathfrak{X}(M)$ and let $p \in M$. Consider the Cauchy problem

$$\begin{cases} \dot{\gamma}(p,t) = X(\gamma(p,t)) \\ \gamma(p,0) = p \end{cases}$$

Define $\Phi^X(p,t) = \gamma(p,t)$. By classical results on ODEs there exists an open neigbourhood U of $M \times \{0\}$ in $M \times \mathbb{R}$ such that $\Phi^X : U \longrightarrow M$ and $\Phi^X(p,\tau) = \gamma(p,\tau)$ and $\gamma(p,\cdot)$ solves the Cauchy problem above. Φ^X is the flow associated to X.

(15) The flow Φ^X is also denoted by $\Phi^X_t(q) = \Phi^X(q,t)$.

(16) Def'n. A vector field on M is complete if Φ^X is defined on all of $M \times \mathbb{R}$.

(17) Exm. $Z \in \mathfrak{X}(\mathbb{R})$ and $Z(x) = x^2\partial_x$, then $\Phi(x,t) = 1/(x^{-1} - t)$ if $x \neq 0$ and $\gamma(0,t) = 0$ otherwise. Thus, $U = \{(x,t) \mid tx < 1\}$

(18) Left as Exs. It is not true that commutators of complete vector fields are still complete. Check that the complete vector fields $X = x_2\,\partial_{x_1}$ and $Y = (x_1)^2\,\partial_{x_2}$ on the plane are complete, but their commutator $[X,Y]$ does not.