4^{rd} Lecture (2h)

(1) Def'n of differential. Let $f : M \longrightarrow N$ be differentiable, with $q = f(p)$. Then

$$df(p) : T_pM \longrightarrow T_qN \quad \text{is the linear mapping}$$
$$\text{Der}\big(\mathcal{V}(p)\big) \ni Z \longrightarrow df(Z)(u) \in \text{Der}\big(\mathcal{V}(q)\big)$$
$$u \in \mathcal{V}(q) \text{ and } df(Z)(u) = Z(u \circ f) \text{ is a derivation.}$$

(2) Let $(U, \widehat{x})$, $p \in U$ and $(V, \widehat{y})$ be charts such that $f(U) \subset V$ and consider

$$\widehat{y} \circ f \circ \widehat{x}^{-1} : \widehat{x}(U) \longrightarrow \widehat{y}(V),$$

then we represent $df(p)$ as the linear mapping

$$df(p)\big(\sum a^j\, \partial/\partial x_j\big) = \sum a^j\ \partial_{x_j}(\widehat{y}^k \circ f \circ \widehat{x}^{-1})\, \partial/\partial y_k$$

Left as an exercise.

(3) The differential satisfies the chain rule

$$d(f \circ g)(p) = df(g(p)) \circ dg(p)$$

(4) Exm. $f : \mathbb{R} \longrightarrow \mathbb{T}^1$, $f(t) = e^{i(1+t^2)}$, then in local coordinates $\theta \to e^{i\theta}$, we have $\widehat{\theta} \circ f(t) = 1 + t^2 + 2m\pi$ belonging to the coordinate open set of the line

$$df(t_0)(\partial_t) = 2\,t\,\partial_\theta(e^{i(1+t_0^2)})$$

(5) Left and right translations $L_p : G \longrightarrow G$ and $R_p : G \longrightarrow G$ are analytic. Furthermore, $dL_p : T_qG \longrightarrow T_{pq}G$ and $dR_p : T_qG \longrightarrow T_{qp}G$, then for every $Z_e \in T_eG$ and define define $Z(p) = dL_p(Z_e) \in T_pG$. Then Z is an analytic vector field on the Lie group G.

(6) Def'n of left invariant vector field

$$dL_y(x)X(x) = X(L_y x) \quad \text{for every} \quad x, y \in G$$

(7) Def'n. The set of all left invariant vector field of G is denoted by $\mathcal{G}$.

(8) Trivial rmk. $\mathcal{G}$ is a linear subspace of $\mathfrak{X}(M)$

(9) Furthermore. $\mathcal{G}$ is finite dimensional and has the same dimension of G. Precisely, there exists a linear isomorphism

$$T_eG \longrightarrow \mathcal{G},\ Z_e \longrightarrow Z$$

(10) Cor. Any left invariant vector field $X \in \mathcal{G}$ has the form $X(p) = dL_pZ$ for every $p \in G$.

(11) Exm. Find left invariant vector fields in $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$

$$(z,t)(z',t') = (z + z', t + t' + 2\,\text{Im}(\langle z, z'\rangle_{\mathbb{C}^n}).$$

Notice that $T_0\big(\mathbb{C}^n \times \mathbb{R}\big) \simeq \mathbb{R}^{2n+1}$ with basis $(\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}, \partial_t)$ at zero.

(12) Exm. Find left invatiant vector fields in $GL_n(\mathbb{R})$: here $T_I\big(GL_n(\mathbb{R})\big) \simeq M_n(\mathbb{R})$