3^{rd} Lecture (2h)

(1) $H = \{(e^{it}, e^{i\alpha t}) \mid t \in \mathbb{R}\}$ is a nonregular submanifold of $\mathbb{T}^2$. In fact, it is a dense subset. Is suffices to show that $F^{-1}(H)$ is dense in $\mathbb{R}^2$ where $F : \mathbb{R}^2 \longrightarrow \mathbb{T}^2$, $F(x, y) = (e^{ix}, e^{iy})$ and notice that $F^{-1}(H) = \{t + 2k\pi, \alpha t + 2m\pi \mid k, m \in \mathbb{Z}\}$, $y = \alpha t + 2m\pi$ and $x = t + 2k\pi$ yield the line
$$\big(x, \alpha x + 2m\pi - \alpha 2k\pi\big)\,.$$
Let $b \in \mathbb{R}$ and consider $\varepsilon > 0$ and let n_0, m_0 be sufficiently large such that
$$\left|\frac{n_0}{m_0} - \alpha\right| \le \frac{1}{m_0^2} < \varepsilon^2,$$
then there exists a unique integer k such that $k \le b(2\pi)^{-1}(n_0 - m_0\alpha)^{-1} < k + 1$, then
$$0 \le b - k2\pi(n_0 - m_0\alpha) < 2\pi(n_0 - m_0\alpha) < 2\pi\varepsilon\,.$$
Then all these lines intersect the y axis in a dense subset.

(2) Exm. Let S be a submanifold of $\mathbb{R}^3$ and let $(U, \widehat{x})$ be a chart.
 (a) Then $\varphi = \widehat{x}^{-1} : \widehat{x}(U) \longrightarrow U$ is differentiable and $Z_p = \sum a^j \varphi_{x_j}(x)$ is a *vector of* $\mathbb{R}^3$ thought of as applied at the point $p = \varphi(x) \in S$.
 (b) The vector depends on the coordinate chart!
 (c) In intrinsic terms we associate a linear functional to Z_p.
 (d) Let $\mathcal{V}_p$ be the infinite dimensional linear space of differentiable functions u around p on S, then $Z_p : \mathcal{V}(p) \longrightarrow \mathbb{R}$ is the linear mapping defined by
$$Z_p(u) = \sum a^j\, \partial_{x_j}(f \circ \varphi)(x).$$

(3) Rmk. Z_p is linear and satisfies the Leibniz rule: it is a derivation at p.

(4) Def'n of derivation at a point. Let M be a manifold and $p \in M$. $L : \mathcal{V}(p) \longrightarrow \mathbb{R}$ is a derivation at p if L is a linear mapping of vector spaces and satisfies $L(uv) = u(p)Lv + v(p)Lu$

(5) If u and v coincide on a nbd of p, then by a cut-off function ψ around p, we make $\psi u = \psi v$, then $L(u) = L(v)$

(6) u constant equal to one around p, then $L(u^2) = L(u)$, then $Lu = 0$,

(7) General Def'n of tangent space: $T_pM = \{L : \mathcal{V}(p) \longrightarrow \mathbb{R}\}$.

(8) *Thm.* T_mM has dimension n and $\partial/\partial x_j$ is a basis of T_mM.
prf. We have seen that $u \in \mathcal{V}(p)$ is of the form $u = u(p) + \sum_{j=1}^n (x_j - a_j)\, h_j$. Then $L(u) = \sum_{j=1}^n L(x_j - a_j)\, u_{x_j}(p)$, namely $\partial/\partial x_j$ is a basis of T_pM.

(9) Cor. We can express any vector $Z \in T_pM$ as a linear combination $Z = \sum_{j=1}^n a^j\, \partial/\partial x_j$ of derivations $\partial/\partial x_i$. A tangent vector is a directional derivative, then a simple first order differential operator.

(10) Defining a vector field as a smooth section of TM, it follows that locally it is defined as $Z(p) = \sum a^j(p)\, \partial/\partial x_j(p)$ where a^j are smooth on the coordinate domain.

(11) Exm. The vector field $Z(p) = x\, \partial_x + y\partial y + z\partial_z$ is a vector field on $\mathbb{R}^3$ and it is classically denoted by the vector (x, y, z). Notice $Z(p)f = x\partial_x f(p) + y\partial_y f(p) + z\partial_z f(p)$

(12) Exm. $Z(x) = (x^2 - y)\,\partial_x + (y^2 + x)\partial y + (x + \sin\theta)\partial_\theta$ is a vector field on $\mathbb{R}^2 \times \mathbb{T}^1$ defines a vector field on $\mathbb{R}^2 \times \mathbb{T}^1$, but $\tilde{Z}(x) = (x^2 - y + \theta)\,\partial_x + (y^2 + x)\partial y + (x^2 + \sin\theta)\partial_\theta$ does not define a vector field on $\mathbb{R}^2 \times \mathbb{T}^1$ unless we specify another coordinate chart.