2^{nd} Lecture (3h)

(1) *Before setting clearly the notion of vector field, we clarify the notion of smooth manifold and submanifold (crash course on differentiable manifolds).*

(2) *Recall the notion of C^k atlas* $\{(U_j, \psi_j)\}_{j\in J}$ for a Hausdorff second countable space M, emphasizing that
- J might also be uncountable
- $M = \bigcup_{j\in J} U_j$

(3) *Def'n of C^k function* $f : \Omega_{open} \subset M \longrightarrow \mathbb{R}$ with respect to a C^m atlas $\mathcal{A}$ on M where $k \le m$ are in $\mathbb{N} \cup \{\infty, \omega\}$ and $s \le \infty \le \omega$ for every $s \in \mathbb{N}$.

(4) Exm of atlas $\mathcal{B} = \{(S, \psi)\}$, $\psi(0, y) = y^3$, where $f : S \longrightarrow \mathbb{R}$, $f(0, y) = y$ is not differentiable with respect to $\mathcal{B}$.

(5) The atlas determines the "differentiable structure" of M

(6) Def'n of C^k compatible atlases and of C^k differentiable structure as the union of all C^k compatible atlases, namely the *maximal atlas*.

(7) From Exs: taking $\mathcal{A} = (S, \psi_1)$, $\psi_1(y) = y$, we notice that S has at least two "distinct differentiable structures", since $\mathcal{A}$ is not C^k compatible with $\mathcal{B}$. $(S, \mathcal{A})$ and $(S, \mathcal{B})$ are two distinct manifolds.

(8) Rmk: an open set U belonging to a chart of a smooth manifold M is a coordinate domain. The initial topology of M can be reobtained as the smallest topology containing all possible coordinate domains.

(9) *Def'n of differentiable mapping.* Let $k \le m$ be in $\mathbb{N} \cup \{\infty, \omega\}$, where $s \le \infty \le \omega$ for every $s \in \mathbb{N}$. A mapping $f : M \longrightarrow N$ of C^m manifolds is of class C^k with respect to the differentiable structures $\mathcal{A}$ and $\mathcal{B}$ of M and N, respectively, if for every chart (U, ψ) of $\mathcal{B}$ the mapping $\psi \circ f : f^{-1}(U) \longrightarrow \psi(U)$ is C^k.

(10) *Def'n of diffeomorphism.* Let $k \le m$ be in $\mathbb{N} \cup \{\infty, \omega\}$, where $s \le \infty \le \omega$ for every $s \in \mathbb{N}$. An invertible mapping $f : M \longrightarrow N$ of C^m manifolds is a C^k diffeomorphism if it is of class C^k along with its inverse mapping.

(11) From Exs: Notice that $\mathbb{R}/2\pi$ has an analytic differentiable structure that makes it C^ω diffeomorphic to $\mathbb{T}^1$.

(12) *Def'n of immersion.* A differentiable mapping $f : M \longrightarrow N$ whose differential (with respect to some chart) has rank equal to the dimension of M is called an immersion (not necessarily injective).

(13) $J : \mathbb{R} \longrightarrow \mathbb{R}^2$, $J(t) = (\sin 2t, \sin t)$ is an immersion. The image is the Eight Figure. $x(\sin 2t, \sin t) = t$ with $0 < t < 2\pi$ and $y(\sin 2t, \sin t) = t$ with $-\pi < t < \pi$ determines global charts that give a differentiable structure to E. Notice however that the topology either given to E or induced by the chart makes E homeomorphic to the open interval.

(14) Exm: $J : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, with

$$J(t,\varphi) = \begin{pmatrix} t\cos\varphi \\ t\sin\varphi \\ \varphi \end{pmatrix}$$

is an immersion. Computing the sum of squares of minors one gets $1 + t^2 > 0$.

(15) Def'n. A manifold S that is also a subset of a manifold M is a *submanifold* if the injection $J : S \longrightarrow M$ is an immersion.

(16) Exs. Show that "Eight Figure" E is a submanifold of M.

(17) Rmk. $E \longrightarrow \mathbb{R}^2$ is an immersion with respect to $\mathcal{A} = (E, x)$ but its topology is finer then the induced topology, with respect to which E is compact!!

(18) Def'n of regular submanifold when the two topology coincide. Then J is an homeomorphism onto the image.

(19) Exm. $J : \mathbb{R}/2\pi \times \mathbb{R}/2\pi \longrightarrow \mathbb{R}^3$, with

$$J(\theta,\varphi) = \begin{pmatrix} (R + r\cos\theta)\cos\varphi \\ (R + r\cos\theta)\sin\varphi \\ r\sin\theta \end{pmatrix}$$

and $0 < r < R$ defines an immersion and J that is also open, since the domain is compact and it sends compacts sets into compact sets. $S = \text{Image}(J)$ is a regular submanifold of $\mathbb{R}^3$ and is also diffeomorphic to $\mathbb{T}^2$.

(20) Thm. If $f : M \longrightarrow N$ has differential with rank equal to the dimension of N in the preimage of a fixed point p, then $f^{-1}(p)$ is a regular submanifold (embedded) of M. The proof follows by the implicit function theorem.

(21) Exm: $\mathbb{T}^1 \times \mathbb{T}^1 \times \cdots \times \mathbb{T}^1$ is a manifold since it is the level set $f^{-1}(0)$ where $f : \mathbb{C}^n \longrightarrow \mathbb{R}^n$ and $f(z) = (|z_1|^2 - 1, \ldots, |z_n|^2 - 1)/2$, seen as a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^n$ has jacobian

$$Df = \begin{pmatrix} x_1 & 0 & \cdots & 0 & y_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 & 0 & y_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & x_n & 0 & 0 & \cdots & y_n \end{pmatrix}$$

The rank of Df is maximal on $\mathbb{T}^n$, then we apply the implicit function theorem at any point of $\mathbb{T}^n$ to find the coordinate domain.

(22) Def'n of topological group and Montgomery Zippin's Theorem.

(23) Def'n of Lie group and of Lie subgroup.

(24) $SL_n(\mathbb{R})$ is a Lie subgroup of $GL_(\mathbb{R})$ and it is also a regular submanifold.