# A course on the Weil conjectures

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These are the notes from a graduate course at the University of Pisa given by TSz in the first semester of 2019/20. DL typed notes during the class which we have later edited in order to eliminate some mistakes and clarify some points that were not adequately explained in the lectures. Most of the audience had followed an introductory course to étale cohomology before, so we have kept background material to a minimum, focussing on key arguments. Occasionally we have invoked difficult theorems from étale cohomology without reference; the proofs of all of these can be found in the standard textbooks [FK88], [Fu15], [Mil80] and, of course, the ultimate reference SGA4.

TSz has lectured on parts of this material several times before, at the Rényi Institute in Budapest and also at the University of Pennsylvania. This is the most complete version yet and the clarity of the presentation owes a lot to DL.

# 1 Introduction to the Weil conjectures

Let $\mathbb{F}_q$ be the finite field with $q$ elements and $X/\mathbb{F}_q$ be a smooth, projective, geometrically connected variety. We denote by $d$ the dimension of $X$. The starting point of the Weil conjectures is the desire to count the number of rational points of $X$ over all extensions of $\mathbb{F}_q$: we are therefore interested in the positive integers $N_m$ defined by

$$N_m = \#X(\mathbb{F}_{q^m}).$$

In elementary terms, $X$ is defined by equations in projective space, and we are simply counting the number of solutions to such equations over all extensions of $\mathbb{F}_q$. It turns out that the interesting object to look at is the *exponential* generating function of the $N_m$:

**Definition 1.1** (zeta function)**.** We set

$$Z_X(T) := \exp\left(\sum_{m\geq 1} N_m \frac{T^m}{m}\right) \in \mathbb{Q}[[T]]$$

and call it the **zeta function of** $X$.

Why is this called the zeta function? There is a close relationship with the Riemann zeta function which we now discuss. Let $x$ be a closed point of $X$. The **degree** of $x$ is by definition

$$\deg(x) := [\kappa(x) : \mathbb{F}_q],$$

where $\kappa(x)$ is the residue field at $x$ (a finite extension of $\mathbb{F}_q$). We have the following equality of formal series:

$$\begin{aligned}\log\left(\frac{1}{1-T^{\deg(x)}}\right) &= \sum_{n\geq 1} \frac{T^{n\deg(x)}}{n} \\ &= \sum_{n\geq 1} \deg(x)\frac{T^{n\deg(x)}}{n\deg(x)},\end{aligned}$$

which shows that the coefficient of $\frac{T^m}{m}$ is

$$\begin{cases} 0, \text{ if } \deg(x) \nmid m \\ \deg(x), \text{ if } \deg(x) \mid m \end{cases}$$

**Remark 1.2.** The quantity $\deg(x)$ is also the number of points in $X(\mathbb{F}_{q^{\deg(x)}})$ lying above $x$.

Using the previous remark we find

$$\sum_m N_m \frac{T^m}{m} = \sum_{x \text{ closed point}} \log\left(\frac{1}{1-T^{\deg(x)}}\right),$$

because every rational point corresponds to some $x$, and the number of rational points corresponding to a given $x$ is precisely its degree. Exponentiating both sides of the previous identity we get

$$Z_X(T) = \prod_{x \text{ closed point}} \frac{1}{1-T^{\deg(x)}},$$

where the right hand side is now formally very similar to the Euler product for the Riemann zeta function.

### 1.1 Statement of the Weil conjectures

These famous conjectures were originally stated by Weil in his 1949 paper *Number of solutions of equations over finite fields* [Wei49]. All of them are now theorems, and we will see their proof during the course.

**Conjecture 1.3** (Rationality)**.** *The zeta function $Z_X(T)$ is a rational function of $T$, that is, $Z_X(T) \in \mathbb{Q}(T)$. In fact, there exist polynomials $P_0(T), P_1(T), \ldots, P_{2d}(T) \in \mathbb{Q}[T]$ such that*

$$Z_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T).}$$

**Remark 1.4.** We will see later that if we normalise the $P_i(T)$ so that $P_i(0) = 1$, then each $P_i(T)$ has integral coefficients.

**Conjecture 1.5** (Functional equation)**.** *The zeta function satisfies the functional equation*

$$Z_X\left(\frac{1}{q^d T}\right) = \pm q^{d\chi/2} T^{\chi} Z_X(T),$$

*where $\chi$ is the "Euler characteristic of $X$" (to be defined precisely later).*

**Remark 1.6.** The substitution $T \to p^{-s}$ brings the functional equation into a form very close to the functional equation for the usual Riemann zeta function.

**Conjecture 1.7** (Riemann hypothesis)**.** *Using the normalisation $P_i(0) = 1$ and factoring $P_i(T)$ over $\overline{\mathbb{Q}}$ as $P_i(T) = \prod_{j=1}^{\deg P_i(T)} (1 - \alpha_{ij} T)$ we have:*

1. $P_0(T) = 1 - T$
2. $P_{2d}(T) = 1 - qT$
3. $|\alpha_{ij}| = q^{i/2}$ *for all* $i = 0, \ldots, 2d$ *and* $j = 1, \ldots, \deg P_i(T)$.

**Definition 1.8.** A **$q$-Weil number of weight** $i$ is an algebraic number $\alpha$ such that $|\sigma(\alpha)| = q^{i/2}$ for every embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

**Example 1.9.** This is a fairly special property: for example, $\alpha = 1 + \sqrt{2}$ has very different absolute values under the two possible embeddings $\mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{C}$

**Remark 1.10.**

1. We shall see that the $\alpha_{ij}$ are in fact algebraic *integers*, not just algebraic *numbers*.
2. Statement (3) is independent of the choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$: in particular, $\alpha_{ij}$ is a $q$-Weil number of weight $i$.
3. The $\alpha_{ij}$ are the reciprocal roots of the polynomials $P_i(T)$.
4. Consider the special case $d = 1$, that is, $X$ is a smooth, projective, geometrically connected curve. The only interesting polynomial $P_i(T)$ is then $P_1(T)$ (because $P_0(T), P_2(T)$ are independent of the specific choice of curve), and replacing $T$ by $q^{-s}$ we see that the Riemann hypothesis is indeed the statement that all the zeroes of the zeta function (as a function of $s$) have real part $\frac{1}{2}$.

**Conjecture 1.11** (Connection with topology)**.** *Suppose $X$ arises by reduction modulo $\mathfrak{p}$ of a flat projective generically smooth scheme $\mathcal{Y}/\operatorname{Spec}\mathcal{O}_K$, where $K$ is a finite extension of $\mathbb{Q}$ and $\mathfrak{p}$ is a prime of the ring of integers of $K$. Then the degree of $P_i(T)$ is the $i$-th Betti number of the complex variety $(\mathcal{Y} \times_{\mathcal{O}_K} \mathbb{C})(\mathbb{C})$.*

**Remark 1.12.** Starting from $X/\mathbb{F}_q$, one can lift the equations defining $X$ to the ring of integers of some number field, and (with some care) get a $Y$ as in the previous conjecture. One can also run the construction in the other direction: starting from a smooth variety $Y/K$, we can fix a flat model $\mathcal{Y}$ over $\mathcal{O}_K$. Even though $\mathcal{Y}$ will not in general be smooth everywhere, for all but finitely primes $\mathfrak{p}$ of $\mathcal{O}_K$ the fibre $X = \mathcal{Y}_\mathfrak{p}$ will be smooth, and we can then consider the corresponding zeta function.

## 1.2 A bit of history

- Conjecture 1.3 was first proven by Dwork [Dwo60] in 1959, by $p$-adic analytic methods. Expositions of Dwork's proof can be found in Serre's Bourbaki talk [Ser60] as well as Koblitz's book [Kob77] (the latter based on the former).
- Conjectures 1.3, 1.5 and 1.11 were proven by Grothendieck, M. Artin and their collaborators along with the development of étale cohomology theory in SGA during the 1960s.
- As for the main Conjecture 1.7, the elliptic curve case was proven by Hasse (who gave two proofs, in 1934 and 1936, see [Has34],[Has36]).
- The case when $X$ is an algebraic curve was handled by Weil himself in the 1940s (again with two different proofs: see [Wei40],[Wei48]). He also tackled the case of abelian varieties.
- In the paper [Wei49] where he stated his conjectures, Weil also treated the case of diagonal hypersurfaces by elementary methods. Here a *diagonal hypersurface* in $\mathbb{P}^n$ is a hypersurface defined by an equation of the form $\sum_{i=0}^n a_i x_i^r = 0$
- There is a recent argument by Katz [Kat15] that reduces the case of general hypersurfaces to that of diagonal ones. This we shall see later in this course (Section 5). In addition, Scholl [Sch11] shows that the case of hypersurfaces implies the case of general smooth projective varieties. These together constitute the simplest proof of Conjecture 1.7 as of today.
- Conjecture 1.7 was proven by Manin for unirational threefolds [Man68] and by Deligne for K3 surfaces [Del72]. The methods used by Deligne for this special case are completely different from those he used to prove the general case, which he established in 1974 in the fundamental paper [Del74] usually referred to as 'Weil I'.
- Grothendieck had a strategy to prove Conjecture 1.7 that goes through his famous Standard Conjectures(see [Kle68]). Unfortunately, these are still very much open!
- Deligne also obtained [Del80] ('Weil II') vast generalisations of his results that apply to non-smooth and non-projective varieties. We'll give a quick overview of these in section 7.

- A second, simpler proof of Weil II was obtained by Laumon [Lau87] using the $\ell$-adic Fourier transform.
- In Weil II, Deligne also states several open problems, some of which are still open. An important input for those that *have* been solved comes from L. Lafforgue's work on the Langlands correspondence for $\mathrm{GL}_n$.

### 1.3 Grothendieck's proof of Conjectures 1.3 and 1.5

The key point, as observed by Grothendieck (and already implicit in Weil's work), is that there is a cohomological interpretation of the zeta function. Fix a prime number $\ell \neq p$ and let $\mathbb{Q}_\ell$ be the (characteristic zero) field of $\ell$-adic numbers. Let $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. Grothendieck defined certain $\ell$-adic étale cohomology groups $H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$, which are vector spaces over $\mathbb{Q}_\ell$. We briefly recall some properties of these cohomology groups:

1. each $H^i(\overline{X}, \mathbb{Q}_\ell)$ is a finite-dimensional vector space over $\mathbb{Q}_\ell$, and is trivial for $i > 2d$.
2. $\overline{X} \mapsto H^i(\overline{X}, \mathbb{Q}_\ell)$ is a contravariant functor.
3. Poincaré duality: the cup product induces perfect pairings
$$H^i(\overline{X}, \mathbb{Q}_\ell) \times H^{2d-i}(\overline{X}, \mathbb{Q}_\ell) \to H^{2d}(\overline{X}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell.$$
4. Lefschetz fixed point formula: let $f : \overline{X} \to \overline{X}$ be a self-map with isolated fixed points of multiplicity 1. Then
$$\#\{\text{fixed points of } f\} = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}\left(f^* \mid H^i(\overline{X}, \mathbb{Q}_\ell)\right).$$

   Notice that this last formula makes sense since $H^i(\overline{X}, \mathbb{Q}_\ell)$ is a contravariant functor.

**Convention.** From now on, for ease of reading we will omit the superscript $^*$ for the pullback action of a self-map on cohomology. The slight abuse of notation should not cause any confusion.

The idea is now to apply (4) with $f$ given by $F : \overline{X} \to \overline{X}$, where $F$ is the (geometric) Frobenius (namely, the identity on $X$ and $g \mapsto g^p$ on functions). It is a fact (proven along with Poincaré duality) that $F$ acts as multiplication by $q^{2d}$ on $H^{2d}(\overline{X}, \mathbb{Q}_\ell)$. Moreover,
$$X(\mathbb{F}_{q^m}) = \{\text{fixed points of } F^m\}.$$

#### 1.3.1 Proof of Conjecture 1.3

Apply the Lefschetz fixed point formula to $F^m$. It yields
$$N_m = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}\left(F^m \mid H^i\left(\overline{X}, \mathbb{Q}_\ell\right)\right),$$

and therefore

$$\begin{aligned} Z_X(T) &= \exp\left(\sum_{m\geq 1} N_m \frac{T^m}{m}\right) \\ &= \prod_{i=0}^{2d} \exp\left(\sum_{m=1}^{\infty} \operatorname{Tr}\left(F^m \mid H^i(\overline{X}, \mathbb{Q}_\ell)\right) \frac{T^m}{m}\right)^{(-1)^i} \\ &= \prod_{i=0}^{2d} \det(\operatorname{Id} - FT \mid H^i(\overline{X}, \mathbb{Q}_\ell))^{(-1)^{i+1}} \end{aligned}$$

where in the last equality we have used the following well-known lemma:

**Lemma 1.13.** *Let $V$ be a finite-dimensional vector space, $\varphi \in \operatorname{End}(V)$. Then*

$$\exp\left(\sum_{m=1}^{\infty} \operatorname{Tr}\left(\varphi^m \mid V\right) \frac{T^m}{m}\right) = \det(\operatorname{Id} - \varphi T)^{-1}$$

*Proof.* The statement is obvious for 1-dimensional vector spaces, and both sides of the equality are multiplicative in short exact sequences of the form $0 \to V' \to V \to V/V' \to 0$. Since the statement does not depend on the ground field, one can work over the algebraic closure and use the existence of eigenvectors to proceed by induction on the dimension of $V$. ☐

Hence one can take

$$P_i(T) = \det(\operatorname{Id} - FT \mid H^i(\overline{X}, \mathbb{Q}_\ell)),$$

but so far we have only shown that $P_i(T)$ is a polynomial with coefficients in $\mathbb{Q}_\ell$. But this suffices to deduce:

**Theorem 1.14.** *$Z_X(T)$ is a rational function in $\mathbb{Q}(T)$.*

The proof uses the theory of **Hankel determinants** which is summarised as follows. Let $f \in K[[T]]$, where $K$ is any field. Write $f = \sum_{i=0}^{\infty} a_i T^i$ and define

$$H_k = \det(a_{i+j+k})_{0 \leq i,j \leq M}$$

for $k > N$, where both $M$ and $N$ are arbitrary parameters. Then $f \in K(T)$ if and only if $H_k = 0$ for all $M, N$ sufficiently large.

*Proof.* We have

$$Z_X(T) \in \mathbb{Q}[[T]] \cap \mathbb{Q}_\ell(T).$$

Considering $Z_X(T)$ as a formal power series in $\mathbb{Q}_\ell[[T]]$ we find using the above fact with $K = \mathbb{Q}_\ell$ that all Hankel determinants for $M, N \gg 0$ vanish, because $Z_X(T) \in \mathbb{Q}_\ell(T)$. But the entries of these Hankel determinants are coefficients of $Z_X(T)$, hence rational numbers (in fact, even integers by the Euler product description). So the theorem follows by applying the converse statement of the theory of Hankel determinants with $K = \mathbb{Q}$. ☐

The fact that each polynomial $P_i(T)$ has rational coefficients is deeper. However, once one knows the Riemann Hypothesis (Conjecture 1.7) it follows via the following elementary lemma due to Deligne:

**Lemma 1.15.** *Assume we already know that the $\alpha_{ij}$ are algebraic numbers with absolute value $q^{i/2}$. With the same notation as above, every $P_i(T)$ has rational coefficients.*

*Proof.* Write $Z_X(T) = \frac{P(T)}{Q(T)}$, where $P(T), Q(T) \in \mathbb{Q}[T]$. We may assume $P(0) = Q(0) = 1$. Also, write $Z_X(T) = \frac{P_1(T)\cdots P_{2d-1}(T)}{P_0(T)\cdots P_{2d}(T)}$. Since the different $P_i(T)$ have distinct roots (because the roots have different absolute value), they are pairwise coprime, so $P(T) = \prod_{i \text{ odd}} P_i(T)$ and $Q(T) = \prod_{j \text{ even}} P_j(T)$. Finally, since the condition $|\alpha_{ij}| = q^{i/2}$ is invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$, the polynomials $P_i(T)$ must have rational coefficients. □

This concludes the proof of Conjecture 1.3. □

**Remark 1.16.** Since $Z_X(T) \in \mathbb{Z}[[T]] \cap \mathbb{Q}(T)$ with $Z_X(0) = 1$, one can in fact choose the polynomials $P(T), Q(T)$ above so that they have coefficients in $\mathbb{Z}$ and $P(0) = Q(0) = 1$ still holds, using a classical lemma of Fatou (see e.g. [Sta12], Chapter 4, Exercise 2). Then the above proof shows that the polynomials $P_i(T)$ are in fact in $\mathbb{Z}[T]$. The fact that the roots of the $P_i(T)$ are algebraic integers is easier to show and does not rely on the Riemann Hypothesis. We shall do it (following Deligne) in Section 4.2.

### 1.3.2 Proof of Conjecture 1.5

We now turn to the proof of the functional equation, assuming all the good properties of ($\ell$-adic) étale cohomology. The proof is based on Poincaré duality, that is, the fact that for every $i \in \{0, \dots, 2d\}$ the vector space $H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$ is dual to $H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)$.

**Lemma 1.17.** *Let $H^* := \bigoplus_{i=0}^{2d} H^i$ be a graded algebra over a field, and suppose that each graded piece $H^i$ is finite-dimensional. Assume that for every $i$ there exists a nondegenerate pairing*

$$\langle \, , \rangle : H^i \times H^{2d-i} \to K$$

*induced by the composition of the product map $H^i \times H^{2d-i} \to H^{2d}$ with an isomorphism (the "trace map") $\mathrm{Tr} : H^{2d} \to K$. Let $\varphi = \varphi_0 \oplus \cdots \oplus \varphi_{2d} : H^* \to H^*$ be a graded endomorphism of degree 0 (that is, $\varphi(H^i) \subseteq H^i$ for all $i$). Assume:*

*(i)* $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$

*(ii)* $\varphi_{2d} = \mathrm{id}$

*Then $\varphi$ is an automorphism of $H^*$, so that in particular every $\varphi_i$ is invertible, and $\varphi_i^{-1} = {}^t\varphi_{2d-1}$, where ${}^t$ denotes transposition with respect to $\langle \, , \rangle$.*

*Proof.* Let $a \in H^i$ be any nonzero element. Since the pairing is nondegenerate, there exists $b \in H^{2d-i}$ such that $a \cdot b \neq 0$, so

$$H^{2d} \ni a \cdot b \overset{\text{(ii)}}{=} \varphi(a \cdot b) \overset{\text{(i)}}{=} \varphi(a)\varphi(b),$$

so $\varphi(a)$ is nonzero and $\varphi$ is injective. Since $\dim H^i$ is finite for every $i$, this implies that every $\varphi_i$ (hence also $\varphi$) is an automorphism. Finally, for every $a \in H^i$ and $b \in H^{2d-i}$ we have

$$\begin{aligned}\langle \varphi_i^{-1}(a), b\rangle &= \operatorname{Tr}\left(\varphi_i^{-1}(a)\cdot b\right)\\ &= \operatorname{Tr}\left(\varphi_{2d}\left(\varphi_i^{-1}(a)\cdot b\right)\right)\\ &= \operatorname{Tr}\left(\varphi_i(\varphi_i^{-1}(a))\cdot\varphi_{2d-i}(b)\right) = \langle a, \varphi_{2d-i}(b)\rangle,\end{aligned}$$

so – using the fact that $\langle\ ,\ \rangle$ is nondegenerate – we obtain $\varphi_i^{-1} = {}^t\varphi_{2d-i}$ as desired. $\square$

To prove Conjecture 1.5, we apply the Lemma with $H^i = H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$ and $\varphi_i = \frac{F}{\sqrt{q^i}}$ to obtain

$$\{\text{eigenvalues of } {}^t\varphi_{2d-i}\} = \{\text{eigenvalues of } \varphi_i^{-1}\} = \{\lambda^{-1} : \lambda \text{ eigenvalue of } \varphi_i\}.$$

It follows that

$$\left\{\frac{q^{i/2}}{\alpha_{ij}} : 1 \le j \le \deg P_i(T)\right\} = \left\{\frac{\alpha_{2d-i}}{q^{\frac{2d-i}{2}}} \mid 1 \le j \le \deg P_{2d-i}(T) = \deg P_i(T)\right\},$$

where the equality $\deg P_i(T) = \deg P_{2d-i}(T)$ follows again from Poincaré duality since these degrees are the dimensions of $H^i(\overline{X}, \mathbb{Q}_\ell)$, $H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)$ respectively. Equivalently, if $\alpha_{ij}$ is an eigenvalue of $F$ on $H^i(\overline{X}, \mathbb{Q}_\ell)$, then $\frac{q^d}{\alpha_{ij}}$ is an eigenvalue of $F$ on $H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)$. Using this, the statement is reduced to a direct computation. $\square$

**Exercise 1.18.** *Fill in the details of the previous proof.*

## 2 Diagonal hypersurfaces

In [Wei49], Weil verified his conjectures for the special case of hypersurfaces given by equations of the form

$$\sum_{i=0}^{r} a_i x_i^d = 0,$$

where $(d, q) = 1$. In this section we study the special case of the Fermat curve $x^d + y^d = z^d$. The proof in the general case uses the same argument, only the formulas become a bit more complicated.

**Lemma 2.1** (Lemma Z)**.** *$Z_X(T)$ is a rational function if and only if for some $\alpha_1, \ldots, \alpha_s$ and $\beta_1, \ldots, \beta_r \in \mathbb{C}$ the equality*

$$N_m(X) = \sum_{j=1}^{r} \beta_j^m - \sum_{i=1}^{s} \alpha_i^m$$

*holds for every $m$.*

*Proof.* We have $Z_X(0) = 1$ by definition, so $Z_X(T)$ is rational if and only if we can write $Z_X(T) = \frac{P(T)}{Q(T)}$ with $P(T), Q(T) \in \mathbb{Q}[T]$ with $P(0) = Q(0) = 1$. Assuming rationality, we have

$$Z_X(T) = \frac{\prod_{i=1}^{s}(1-\alpha_i T)}{\prod_{j=1}^{r}(1-\beta_j T)},$$

where the $\{\alpha_i\}, \{\beta_j\}$ are the reciprocal roots of $P(T)$ and $Q(T)$ respectively. Taking the logarithmic derivative of both sides we get

$$\frac{Z_X'(T)}{Z_X(T)} = -\sum_{i=1}^{s} \frac{\alpha_i}{1-\alpha_i T} + \sum_{j=1}^{r} \frac{\beta_j}{1-\beta_j T}.$$

Multiplying by $T$ and expanding the right hand side as a formal power series in $T$ we obtain

$$T\frac{Z_X'(T)}{Z_X(T)} = \sum_{m=1}^{\infty} \left( \sum_{j=1}^{r} \beta_j^m - \sum_{i=1}^{s} \alpha_i^m \right) T^m.$$

On the other hand, recalling that $Z_X(T) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m T^m}{m}\right)$ we also have

$$T\frac{Z_X'(T)}{Z_X(T)} = \sum_{m=1}^{\infty} N_m T^m.$$

Comparing coefficients of $T^m$ we get the desired statement. The converse implication is obtained by reversing the argument. $\square$

### 2.1 Gauss & Jacobi sums

Let $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a multiplicative character, and $\zeta_p$ a primitive $p$-th root of unity.

**Definition 2.2** (Gauss sum)**.** We set

$$g(\chi) = \sum_{a \in \mathbb{F}_q} \chi(a) \zeta_p^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)},$$

where – as is customary in number theory – we have set $\chi(0) = 0$.

**Remark 2.3.** The function

$$\psi(a) := \zeta_p^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}$$

is an additive character of $\mathbb{F}_q$, that is, a homomorphism $(\mathbb{F}_q, +) \to \mathbb{C}^\times$.

We will need some elementary properties of Gauss sums:

**Lemma 2.4** (Lemma G)**.** *Gauss sums enjoy the following properties:*

*(a)* $g(\overline{\chi}) = \chi(-1)\overline{g(\chi)}$*;*

*(b)* $g(\chi)\overline{g(\chi)} = q$ *if* $\chi \neq 1$*;*

*(c)* $g(\chi)g(\overline{\chi}) = \chi(-1)q$ *if* $\chi \neq 1$*.*

*Proof.* Clearly (a) and (b) together imply (c).

(a)

$$\begin{aligned}
\overline{g(\chi)} &= \sum_a \overline{\chi}(a)\overline{\psi(a)} \\
&= \sum_a \overline{\chi}(a)\psi(-a) \\
&= \sum_a \overline{\chi}(-1)\overline{\chi}(-a)\psi(-a) \\
&= \overline{\chi}(-1)g(\overline{\chi}) \\
&= \chi(-1)g(\overline{\chi}),
\end{aligned}$$

where in the last equality we have used the fact that $\chi(-1) \in \{\pm 1\}$ is a real number.

(b)

$$\begin{aligned}
g(\chi)\overline{g(\chi)} &= \sum_{a,b\neq 0} \chi(ab^{-1})\psi(a-b) \\
&= \sum_{b,c\neq 0} \chi(c)\psi(bc-b) && (c = ab^{-1}) \\
&= \sum_{b,c\neq 0} \chi(c)\psi(b(c-1)) \\
&= \sum_{b\neq 0} \chi(1)\psi(0) + \sum_{c\neq 0,1} \chi(c) \sum_{b\neq 0} \psi(b(c-1)) \\
&= (q-1) + \sum_{c\neq 0,1} \chi(c)(-1) \\
&= q,
\end{aligned}$$

where we have repeatedly used the fact that $\sum_{c\in\mathbb{F}_q} \psi(c) = 0$ if $\psi$ is a nontrivial additive character, and similarly for multiplicative characters.

□

Gauss sums involve the multiplicative character $\chi$ and the additive character $\psi$. Other classical objects in number theory, Jacobi sums, are obtained from two multiplicative characters:

**Definition 2.5** (Jacobi sums)**.** If $\chi_1, \chi_2$ are multiplicative characters $\mathbb{F}_q^\times \to \mathbb{C}$, we set

$$J(\chi_1, \chi_2) = \sum_{a\in\mathbb{F}_q} \chi_1(a)\chi_2(1-a).$$

**Lemma 2.6** (Lemma J)**.** *Jacobi sums enjoy the following properties:*

*(a)* $J(1,1) = q-2$

*(b)* $J(1,\chi) = J(\chi,1) = -1$ *for* $\chi$ *nontrivial*

*(c)* $J(\chi,\overline{\chi}) = -\chi(-1)$ *for* $\chi$ *nontrivial*

*(d) If* $\chi_1\chi_2 \neq 1$*, then* $J(\chi_1,\chi_2) = \dfrac{g(\chi_1)g(\chi_2)}{g(\chi_1\chi_2)}$.

**Remark 2.7.** Lemma 2.4 (b) and Lemma 2.6 (d) together imply in particular $|J(\chi_1, \chi_2)| = \sqrt{q}$ whenever $\chi_1, \chi_2, \chi_1\chi_2$ are all nontrivial.

*Proof.* (a) and (b) are easy. We prove (c) and (d) simultaneously:

$$\begin{aligned} g(\chi_1)g(\chi_2) &= \sum_{a,b} \chi_1(a)\chi_2(b)\psi(a+b) \\ &= \sum_{a,c} \chi_1(a)\chi_2(c-a)\psi(c) && (c = a+b) \\ &= \sum_{a\in\mathbb{F}_q, c\in\mathbb{F}_q^\times} \chi_1(a)\chi_2(c-a)\psi(c) + \sum_{a\in\mathbb{F}_q} \chi_1(a)\chi_2(-a). \end{aligned}$$

Let's compute the two sums separately. The second one can be evaluated exactly:

$$\sum_{a\in\mathbb{F}_q} \chi_1(a)\chi_2(-a) = \chi_2(-1)\sum_a (\chi_1\chi_2)(a) = \begin{cases} \chi_1(-1)(q-1), & \text{if } \chi_1\chi_2 = 1 \\ 0, & \text{otherwise,} \end{cases}$$

while the first can we rewritten as

$$\begin{aligned} \sum_{\substack{a\in\mathbb{F}_q \\ c\in\mathbb{F}_q^\times}} \chi_1(a)\chi_2(c-a)\psi(c) &= \sum_{\substack{d\in\mathbb{F}_q \\ c\in\mathbb{F}_q^\times}} \chi_1(cd)\chi_2(c(1-d))\psi(c) && (a = c\cdot d) \\ &= \sum_{c\in\mathbb{F}_q^\times, d\in\mathbb{F}_q} \chi_1\chi_2(c)\psi(c)\chi_1(d)\chi_2(1-d) \\ &= g(\chi_1\chi_2)J(\chi_1, \chi_2). \end{aligned}$$

This finishes the proof of (d). For (c), observe that $g(\chi\overline{\chi}) = g(1) = -1$, so that using the previous computations together with Lemma 2.4 we get

$$-J(\chi, \overline{\chi}) + \chi(-1)(q-1) = g(\chi)g(\overline{\chi}) = \chi(-1)q,$$

which proves (c). ∎

## 2.2 Proof of the Riemann hypothesis for the Fermat curve

**Proposition 2.8** (Main proposition). *Let $X_d = \{x^d + y^d = z^d\} \subset \mathbb{P}^2_{\mathbb{F}_q}$, where $d \geq 2$ and $(d, q) = 1$. Let $e = (q-1, d)$ and $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a character of order $e$. Then*

$$N_1(X_d) = q + 1 + \sum_{\substack{a,b=1 \\ a+b\neq e}}^{e-1} J(\chi^a, \chi^b)$$

*Proof.* For simplicity we do only the case when $q \equiv 1 \pmod{d}$, that is, $e = d$; in fact, one can easily reduce the general case to this one. Recall that for fixed $t \in \mathbb{F}_q$ the number $S(t)$ of solutions of the equation $x^d = t$ is given by $1 + \sum_{a=1}^{d-1} \chi^a(t)$ (indeed, if $t$ is a nonzero $d$-th power this is a sum of terms equal to 1, and therefore equals $d$; for $t = 0$ there is only one nonzero term, equal to 1; and otherwise the sum is 0). We count the rational points on $X_d$ as follows:

1. points at infinity, with $z = 0$. These correspond to the solutions of $(x/y)^d = -1$, hence their number is $S(-1)$.

2. points in the affine open set $z \neq 0$. These correspond to solutions of the equation $x^d + y^d = 1$ with $x, y \in \mathbb{F}_q$, and we may count them as follows:

$$\begin{aligned}
\#\{(x,y) \in \mathbb{F}_q^2 : x^d + y^d = 1\} &= \sum_{t \in \mathbb{F}_q} \#\{(x,y) \in \mathbb{F}_q^2 : x^d = t, y^d = 1-t\} \\
&= \sum_{t \in \mathbb{F}_q} S(t)S(1-t) \\
&= \sum_{t \in \mathbb{F}_q} \left(1 + \sum_{a=1}^{d-1} \chi^a(t)\right)\left(1 + \sum_{b=1}^{d-1} \chi^b(1-t)\right) \\
&= q + \sum_{a=1}^{d-1} \sum_{t \in \mathbb{F}_q} \chi^a(t) + \sum_{b=1}^{d-1} \sum_{t \in \mathbb{F}_q} \chi^b(1-t) + \sum_{a,b=1}^{d-1} \sum_{t \in \mathbb{F}_q} \chi^a(t)\chi^b(1-t).
\end{aligned}$$

For each $a$ (resp. $b$) in $\{1, \dots d-1\}$ the character $\chi^a$ (resp. $\chi^b$) is nontrivial, so the sum $\sum_{t \in \mathbb{F}_q} \chi^a(t)$ (resp. $\sum_{t \in \mathbb{F}_q} \chi^b(1-t)$) vanishes, and we are left with evaluating $q + \sum_{a,b=1}^{d-1} J(\chi^a, \chi^b)$. We observe that for $a+b \equiv 0 \pmod d$ (equivalently, $a+b = d$) we have $\chi^b = \chi^{d-a} = \overline{\chi^a}$, hence by Lemma 2.6 (c) the summand $J(\chi^a, \chi^b) = J(\chi^a, \overline{\chi^a})$ contributes $-\chi^a(-1)$, for a total of $-\sum_{a=1}^{d-1} \chi^a(-1) = 1 - S(-1)$. Thus the number of $\mathbb{F}_q$-rational points in $X_d \cap \{z \neq 0\}$ is $q + 1 - S(-1) + \sum_{\substack{a,b=1 \\ a+b \neq d}}^{d-1} J(\chi^a, \chi^b)$.

Combining (1) and (2) the claim follows immediately. □

To establish the Riemann hypothesis we also need to investigate the number of rational points over extensions of the ground field $\mathbb{F}_q$. Let $\mathbb{F}_{q^m}/\mathbb{F}_q$ be a finite extension, and let

$$\chi_m = \chi \circ N_{\mathbb{F}_{q^m}/\mathbb{F}_q} : \mathbb{F}_{q^m}^\times \to \mathbb{C}^\times.$$

By definition, $g(\chi_m)$ is a Gauss sum for $\mathbb{F}_{q^m}$.

**Proposition 2.9** (Hasse-Davenport relation)**.**

$$-g(\chi_m) = (-g(\chi))^m$$

For a proof, see e.g. [IR90], Theorem 1 in Chapter 11, §3.

**Corollary 2.10.**

$$N_m(X_d) = \#X_d(\mathbb{F}_{q^m}) = q^m + 1 - \sum_{\substack{a,b=1 \\ a+b \neq e}}^{e-1} \left(-J(\chi^a, \chi^b)\right)^m,$$

Using the corollary we can write $N_m(X_d)$ in the form stipulated by Lemma 2.1:

$$N_m = \underbrace{\sum_{i=1}^{q^m+1} 1^m}_{\sum \alpha_i^m} - \underbrace{\sum \Big(-J(\chi^a, \chi^b)\Big)^m}_{\sum \beta_j^m}.$$

The fact that $|\beta_j| = \sqrt{q}$ by Lemma 2.6 then implies the Riemann Hypothesis for Fermat curves.

# 3 A primer on étale cohomology

## 3.1 How to define a good cohomology theory for schemes

The basic example to have in mind is the following. Let $X$ be a topological space, and let $\mathcal{C}_X$ be the category whose objects are open subsets of $X$, with

$$\mathrm{Mor}(V, U) = \{\iota : V \hookrightarrow U \text{ open inclusion}\}.$$

In particular, $\mathrm{Mor}(V, U)$ is nonempty if and only if $V$ is a subset of $U$, and in that case it consists of precisely one element. A **presheaf** (of abelian groups) on $X$ is a contravariant functor $\mathcal{C}_X \to \mathbf{Ab}$, that is,

- for each open subset $U$ of $X$ we have an abelian group $\mathcal{F}(U)$;
- for each inclusion $V \subseteq U$ we have a restriction morphism $U \to V$.

The presheaf $\mathcal{F}$ is a sheaf if and only if for every open cover $\{U_i \to U\}_{i \in I}$ the following sequence is exact:

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j),$$

where the two arrows $\prod \mathcal{F}(U_i) \to \prod_{(i,j)} \mathcal{F}(U_i \cap U_j)$ are induced by the inclusion of $U_i \cap U_j$ into $U_i$ and $U_j$ respectively. Classically, the category of sheaves has enough injectives, and we can define $H^q(X, -)$ as the $q^{\text{th}}$ derived functor of the global sections functor $X \mapsto \mathcal{F}(X)$.

**Definition 3.1.** Let $A$ be an abelian group. The constant sheaf $\mathcal{A}$ associated with $A$ is given by

$$\mathcal{A}(U) := A^{\pi_0(U)},$$

with the obvious restriction maps.

**Problem 3.2.** *The above construction of cohomology is uninteresting if $\mathcal{F}$ is a constant sheaf and $X$ is the underlying topological space of a (reasonable) scheme. The reason is that $\mathcal{A}$ is **flabby**, hence it has trivial cohomology in degree $> 0$.*

**Definition 3.3.** A **Grothendieck topology** consists of:

- a category $\mathcal{C}$ with fibre products;

- a set[1] $\operatorname{Cov}\mathcal{C}$, whose elements are families of morphisms $\{U_i \xrightarrow{\varphi_i} U\}$ ($U$ is fixed, and we consider several $U_i$). These elements are called **coverings**, and are supposed to satisfy the following axioms:

  1. the identity $\{U \xrightarrow{\mathrm{id}} U\}$ is in $\operatorname{Cov}\mathcal{C}$ for every object $U$ in $\mathcal{C}$;
  2. if $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ and $\{V_{ij} \xrightarrow{\psi_{ij}} U_i\}$ are in $\operatorname{Cov}\mathcal{C}$, then so is the family $\{V_{ij} \xrightarrow{\varphi_i\circ\psi_{ij}} U\}$
  3. given $\{U_i \to U\}$ in $\operatorname{Cov}\mathcal{C}$ and $V \to U$ a morphism in $\mathcal{C}$, the family $\{V \times_U U_i \to V\}$ is again in $\operatorname{Cov}\mathcal{C}$.

**Definition 3.4.** In this situation, a **presheaf** is a contravariant functor $\mathcal{F} : \mathcal{C} \to \mathbf{Ab}$. A presheaf is a sheaf (for a Grothendieck topology $\tau$) if

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \times_U U_j)$$

is exact for all coverings $\{U_i \to U\}_{i\in I}$ in the topology $\tau$.

**Theorem 3.5** (Grothendieck)**.** *The category of sheaves has enough injectives, and we can define*

$$H^q(U, \mathcal{F}) := R^q\Gamma(U, \mathcal{F}),$$

*where $\Gamma(U, -) : \mathcal{F} \mapsto \mathcal{F}(U)$ is the global sections functor.*

### 3.1.1 Examples

1. Topological spaces: the category $\mathcal{C}_X$ admits an obvious Grothendieck topology, given by the usual coverings by open subsets in topology.

2. Let $X$ be a scheme and $\mathcal{C}$ be the category of $X$-schemes. Consider the family of coverings given by the collections $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ such that $\bigcup_{i\in I} \varphi_i(U_i) = U$ and every $\varphi_i$ is étale. This gives a Grothendieck topology, called the **étale topology**. We will denote by $H^q(X, -)$ the corresponding cohomology theory, usually called **étale cohomology**.

## 3.2 Comparison of étale cohomology with other cohomology theories

In two important cases, étale cohomology reduces to other well-known cohomology theories:

**Theorem 3.6.**

1. *Let $X = \operatorname{Spec}(k)$, where $k$ is a field with separable closure $k_s$. Then $H^q(X, \mathcal{F}) = H^q(k, \mathcal{F}(k_s))$, where the latter is Galois cohomology.*

2. *Let $A$ be a finite abelian group, and – with a slight abuse of notation – denote by the same letter also the constant sheaf associated with $A$ (Definition 3.1 extends verbatim to the étale setting). Let $X$ be a smooth variety over $\mathbb{C}$. There exists a canonical isomorphism*

$$H^q(X, A) = H^q_{\mathrm{sing}}(X(\mathbb{C}), A).$$

[1] there are nontrivial set-theoretical difficulties here, but we will not discuss them.

Smoothness is not very important here, but finiteness of $A$ is essential: the statement is not true, for example, for cohomology with coefficients in $\mathbb{Z}$. In fact, $H^1(X, \mathbb{Z})$ is already different in the two theories: on the one hand, $H^1_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X(\mathbb{C})), \mathbb{Z})$, while

$$H^1(X, \mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{cont}}\left(\pi_1^{\mathrm{alg}}(X), \mathbb{Z}\right) = (0),$$

because $\pi_1^{\mathrm{alg}}(X)$ is (by construction) a profinite group.

### 3.3 Stalks of an étale sheaf

In the topological case, the stalk of a sheaf $\mathcal{F}$ at a point $x$ is

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U),$$

the direct limit being taken over open neighbourhoods of the point $x$. This definition is modelled over that of the set of germs of functions at a point. In turn, for étale sheaves we have the following analogue. Given a point $\mathrm{Spec}\, k(x) \to X$, an **étale neighbourhood of** $x$ is a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, k(x)_s \arrow[r] \arrow[d] & U \arrow[d, "\text{étale}"] \\ \mathrm{Spec}\, k(x) \arrow[r] & X \end{tikzcd}$$

Given a geometric point $\overline{x} : \mathrm{Spec}\, k(x)_s \to X$ lying above $x$, we define

$$\mathcal{F}_{\overline{x}} := \varinjlim_{\overline{x} \in U} \mathcal{F}(U),$$

where the indexing category is given by étale neighbourhoods of (the topological image of) $\overline{x}$.

### 3.4 Cohomology groups with coefficients in rings of characteristic 0

Let $\ell$ be a prime number. The **ring of $\ell$-adic integers** is

$$\mathbb{Z}_\ell := \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z},$$

and the field of $\ell$-adic numbers is $\mathbb{Q}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathbb{Q}$. When $X$ is a scheme we define

$$H^q(X, \mathbb{Z}_\ell) := \varprojlim_n H^q(X, \mathbb{Z}/\ell^n\mathbb{Z})$$

and

$$H^q(X, \mathbb{Q}_\ell) := H^q(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

**Remark 3.7.** This is *not* the same as the $q$-th étale cohomology group of the constant sheaf of group $\mathbb{Z}_\ell$! In particular, $H^1(X, \mathbb{Z}_\ell)$ (cohomology of the constant sheaf) is often zero, while $\varprojlim_n H^q(X, \mathbb{Z}/\ell^n\mathbb{Z})$ usually captures the 'interesting' information.

In general, one can define "$\ell$-adic sheaves" and take their cohomology. This is a bit technical, so we only give a sketch. Let $(\mathcal{F}_r)_{r\geq 1}$ be a projective system of sheaves such that each $\mathcal{F}_r$ has finite stalks and is moreover **locally constant** (i.e. its pullback to a finite étale cover is constant). Suppose that for every $r \geq 1$ the sheaf $\mathcal{F}_r$ is killed by $\ell^r$, and that $\mathcal{F}_{r+1}/\ell^r \mathcal{F}_{r+1} \cong \mathcal{F}_r$. Such a system is a **locally constant $\ell$-adic sheaf $\mathcal{F}$**. We can then define

$$H^q(X, \mathcal{F}) := \varprojlim_r H^q(X, \mathcal{F}_r).$$

One also gets locally constant $\mathbb{Q}_\ell$-sheaves by tensoring with $\mathbb{Q}_\ell$, or equivalently, by working in the category of $\ell$-adic sheaves up to isogeny. In this category the objects are the same as in the category of locally constant $\ell$-adic sheaves but the Hom-sets are tensored with $\mathbb{Q}_\ell$.

Unfortunately, locally constant sheaves do not behave well with respect to some operations on sheaves, so a larger class is needed: a sheaf $\mathcal{F}$ with finite stalks on $X$ is **constructible** if there is a finite covering of $X$ by locally closed subsets $U_i$ such that the pullback of $\mathcal{F}$ to each $U_i$ is locally constant. By the above procedure one defines constructible $\ell$-adic and $\mathbb{Q}_\ell$-sheaves.

**Remark 3.8.** A good reference for $\ell$-adic sheaves is §12 of [FK88].

**Example 3.9.** If $X$ is a variety over a field $k$ with $(n, \operatorname{char} k) = 1$, then we have an étale sheaf of $n$-th roots of unity,

$$\mu_n(U) := \{f \in \mathcal{O}_U(U) : f^n = 1\}.$$

Notice that $\operatorname{Gal}(k_s/k)$ acts naturally on $\mu_n$. The system $(\mu_{\ell^r})_{r\geq 1}$ defines an $\ell$-adic sheaf which is usually denoted by $\mathbb{Z}_\ell(1)$ and called a **Tate twist**. One also defines $\mathbb{Q}_\ell(1) := \mathbb{Z}_\ell(1) \otimes \mathbb{Q}$ and

$$\mathbb{Z}_\ell(i) := \mathbb{Z}_\ell(1)^{\otimes i}, \quad \mathbb{Q}_\ell(i) := \mathbb{Q}_\ell(1)^{\otimes i}.$$

### 3.5 Operations on sheaves

Let $\varphi : X \to Y$ be a morphism, $\mathcal{F}$ be an étale sheaf on $X$. The **pushforward** $\varphi_*\mathcal{F}$ is the sheaf on $Y$ given by

$$\varphi_*\mathcal{F}(U) := \mathcal{F}(X \times_Y U).$$

The functor $\varphi_*$ has a left adjoint, denoted by $\varphi^*$, which to a sheaf $\mathcal{G}$ on $Y$ associates a pullback sheaf $\varphi^*\mathcal{G}$ on $X$ in such a way that

$$\operatorname{Hom}(\varphi^*\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, \varphi_*\mathcal{G}).$$

The pullback functor is exact. The pushforward functor is left- but not right-exact, and one may consider the right derived functors $R^q\varphi_*$ of $\varphi_*$. If $\varphi$ is **proper** and $\mathcal{F}$ is locally constant with finite stalks or, more generally, constructible, then

$$(R^q\varphi_*)_{\overline{y}} \cong H^q\left(X_{\overline{y}}, \mathcal{F}_{X_{\overline{y}}}\right).$$

**Theorem 3.10** (Proper smooth base change)**.** *If $\varphi : X \to Y$ is proper and smooth, and $\mathcal{F}$ is a locally constant sheaf on $X$ with finite stalks, then the higher direct images $R^q\varphi_*\mathcal{F}$ are locally constant with finite stalks on $Y$.*

The theorem extends to $\ell$-adic and $\mathbb{Q}_\ell$-sheaves by the limit procedure described above.

### 3.6 Cohomology with compact support

Given an open immersion $j : U \hookrightarrow X$, one can define an **extension by zero** functor

$$\begin{array}{ccc} \mathbf{Sh}(U) & \to & \mathbf{Sh}(X) \\ \mathcal{F} & \mapsto & j_!\mathcal{F}, \end{array}$$

which enjoys the following property: for every geometric point $\overline{x}$ of $X$ we have

$$(j_!\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}}, & \text{if the topological image of } \overline{x} \text{ is in } |U| \\ 0, & \text{otherwise} \end{cases}$$

Given $U$ and an étale sheaf $\mathcal{F}$ on $U$, we can then define

$$H^i_c(U, \mathcal{F}) := H^i(X, j_!\mathcal{F})$$

where $X$ is proper and contains $U$ as a dense open subset. For $U$ of finite type over a field, such an $X$ always exist by Nagata's compactification theorem. One shows that the definition does not depend on $X$. Let now $f : X \to Y$ be a compactifiable morphism, i.e., a morphism for which there exists a diagram

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "f"'] & & X^c \arrow[dl, "f^c"] \\ & Y & \end{tikzcd}$$

where $X^c$ is proper over $Y$ and $X$ is a dense open subset of $X^c$. For a sheaf $\mathcal{F}$ on $X$ we can then define the **higher direct images with compact support**

$$R^i f_! \mathcal{F} := R^i(f^c_* \circ j_!)\mathcal{F}.$$

Deligne proves that this construction does not depend on the compactification $X^c$, and that when $\mathcal{F}$ is torsion there are canonical identifications

$$(R^i f_! \mathcal{F})_{\overline{y}} = H^i_c(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$$

for every $i$ and every geometric point $\overline{y}$ of $Y$. Moreover, if $\mathcal{F}$ is **constructible**, then the higher direct images with compact support $R^i f_! \mathcal{F}$ are constructible as well. (This is not true with 'constructible' replaced by 'locally constant'!) All this extends in the usual way to $\mathbb{Z}_\ell$- and $\mathbb{Q}_\ell$-sheaves.

### 3.7 Three basic theorems

Let $\mathcal{F}$ be a locally constant sheaf of finite $\mathbb{Z}/n\mathbb{Z}$-modules, or of finite-dimensional $\mathbb{Q}_\ell$-vector spaces.

**Theorem 3.11** (Cohomological dimension, M. Artin)**.** *Let $X$ be a scheme of finite type over a separably closed field $k$, and let $d = \dim X$. Then*

$$H^i(X, \mathcal{F}) = H^i_c(X, \mathcal{F}) = (0)$$

*for $i > 2d$. In addition, $H^i(X, \mathcal{F}) = (0)$ if $i > d$ and $X$ is affine over a separably closed field $k$.*

**Theorem 3.12** (Localisation sequence in compact support cohomology)**.** *Let $Z \hookrightarrow X$ be a closed immersion with complement $U := X \setminus Z$. There is a long exact sequence*

$$\cdots \to H^i_c(U, \mathcal{F}) \to H^i_c(X, \mathcal{F}) \to H^i_c(Z, \mathcal{F}) \to H^{i+1}_c(U, \mathcal{F}) \to \cdots$$

**Theorem 3.13** (Poincaré duality)**.** *Let $X$ be a smooth connected scheme of finite type over a separably closed field. Write $d$ for the dimension of $X$. The following hold:*

1. *There is a "trace map" isomorphism $H^{2d}_c(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-d)$.*
2. *Let $\mathcal{F}$ be a locally constant sheaf of finite-dimensional $\mathbb{Q}_\ell$-vector spaces, and define $\mathcal{F}^\vee := \underline{\mathrm{Hom}}(\mathcal{F}, \mathbb{Q}_\ell)$. There exists a perfect pairing*

$$H^i(X, \mathcal{F}) \times H^{2d-i}_c(X, \mathcal{F}^\vee) \xrightarrow{\cup} H^{2d}_c(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-d).$$

**Corollary 3.14** (Weak Lefschetz)**.** *Let $X$ be a proper smooth scheme over a separably closed field $k$ and let $d = \dim X$. Let $Z \hookrightarrow X$ be a closed immersion such that $U := X \setminus Z$ is affine. Then for every locally constant sheaf $\mathcal{F}$ of $\mathbb{Q}_\ell$-vector spaces the following hold:*

1. *$H^i(X, \mathcal{F}) \cong H^i(Z, \mathcal{F})$ for $i < d - 1$;*
2. *$H^{d-1}(X, \mathcal{F}) \hookrightarrow H^{d-1}(Z, \mathcal{F})$.*

*Proof.* Consider the localisation sequence

$$H^i_c(U, \mathcal{F}) \to H^i_c(X, \mathcal{F}) \to H^i_c(Z, \mathcal{F}) \to H^{i+1}_c(U, \mathcal{F})$$

and observe that for $X$ and $Z$ (that are proper) cohomology with compact support agrees with usual cohomology. In addition, Poincaré duality allows us to identify $H^i_c(U, \mathcal{F})$ with $H^{2d-i}_c(U, \mathcal{F}^\vee)$ and $H^{i+1}_c(U, \mathcal{F})$ with $H^{2d-(i+1)}_c(U, \mathcal{F}^\vee)$. The previous exact sequence can then be rewritten as

$$H^{2d-i}(U, \mathcal{F}^\vee) \to H^i(X, \mathcal{F}) \to H^i(Z, \mathcal{F}) \to H^{2d-(i+1)}(U, \mathcal{F}^\vee).$$

Both claims then follow immediately from Theorem 3.11: for $i \leq d - 1$ we have the cohomological vanishing $H^{2d-i}(U, \mathcal{F}^\vee) = (0)$ since $U$ is affine of dimension $d > 2d - i$, and, for the same reason, for $i < d - 1$ the group $H^{2d-(i+1)}(U, \mathcal{F}^\vee)$ also vanishes. □

### 3.8 Conjecture 1.11: connection with topology

We now discuss how the formalism of étale cohomology leads to a proof of Conjecture 1.11. Suppose $X/\mathbb{F}_q$ lifts to a smooth proper scheme over an open subscheme of $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a number field $K$. The claim is that $\deg P_i(T)$ is the $i$-th Betti number of the corresponding complex variety. Recall that $\deg P_i(T)$ is the dimension of the cohomology space $H^i(\overline{X}, \mathbb{Q}_\ell)$, so the fourth Weil conjecture simply states that the dimension of the étale and Betti cohomology groups agree. For the proof, let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ such that $X \to \mathbb{F}_q$ extends to a smooth proper scheme $\varphi : \mathcal{X} \to \operatorname{Spec} \widehat{\mathcal{O}_{K,\mathfrak{p}}}$, where $\widehat{\cdot}$ denotes completion. We can consider the sheaf $\mathbb{Q}_\ell$ on $\mathcal{X}$ and the higher direct images $R^i\varphi_*\mathbb{Q}_\ell$ on $\widehat{\mathcal{O}_{K,\mathfrak{p}}}$.

Now $\widehat{\mathcal{O}_{K,\mathfrak{p}}}$ is a DVR, so its spectrum consists of two points: a generic point $\eta$, with residue field equal to a finite extension of $\mathbb{Q}_p$, and a special point $s$ with residue field

$\mathbb{F}_q$. Smooth proper base change (Theorem 3.10) gives local constancy of the pushforward sheaves, namely we have

$$\left(R^i\varphi_*\mathbb{Q}_\ell\right)_{\overline{\eta}} \cong \left(R^i\varphi_*\mathbb{Q}_\ell\right)_{\overline{s}},$$

and the two sides of this equality are $H^i(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ and $H^i(\overline{X}, \mathbb{Q}_\ell)$. The right hand side has dimension $\deg P_i(T)$. We claim that for the left hand side we have

$$H^i(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \cong H^i(X_{\mathbb{C}}, \mathbb{Q}_\ell).$$

To see this, one uses the $\mathbb{Q}_\ell$-version of Theorem 3.6 (2). One can either use the axiom of choice to prove that the algebraic closure of $\mathbb{Q}_p$ is isomorphic to $\mathbb{C}$, or use the Lefschetz principle to go down from $\overline{\mathbb{Q}_p}$ to a finitely generated extension of $\mathbb{Q}$, then back up to its algebraic closure, and then from there to $\mathbb{C}$. The claim then follows from the fact that étale cohomology is invariant under extension of the field of definition between algebraically closed fields of characteristic 0.

# 4 Deligne's integrality theorem

## 4.1 Generalised zeta functions

Let $X/\mathbb{F}_q$ be a separated scheme of finite type, and let $\mathcal{F}$ be an étale sheaf on $X$. Denote by $\overline{X}$ the base change $X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ and let $\overline{\mathcal{F}}$ be the pull-back of $\mathcal{F}$ to $\overline{X}$. There is a Frobenius $F : X \to X$, which induces the geometric Frobenius $F : \overline{X} \to \overline{X}$. If $\overline{x}$ is a geometric point of $X$, pulling back along $F$ induces a morphism

$$F_x^* : \mathcal{F}_{F(\overline{x})} \to \mathcal{F}_{\overline{x}},$$

and more generally

$$F_{\overline{x}}^{*n} : \mathcal{F}_{F^n(\overline{x})} \to \mathcal{F}_{\overline{x}}.$$

In particular, if the topological image of $\overline{x}$ is a closed point defined over the degree $n$ extension of $\mathbb{F}_q$, then $F^n(\overline{x}) = \overline{x}$ and $F_{\overline{x}}^{*n}$ is an endomorphism of $\mathcal{F}_{\overline{x}}$.

**Convention.** To reduce notational clutter we will from now on omit the superscript $^*$ and speak simply of the action of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$.

**Definition 4.1.** In this situation we define

$$Z(X, \mathcal{F}, T) := \prod_{\substack{x \in X \\ \text{closed point}}} \det\left(1 - F_{\overline{x}}^{\deg(x)} T^{\deg(x)} \mid \mathcal{F}_{\overline{x}}\right)^{-1}.$$

**Remark 4.2.** For the constant ($\ell$-adic) sheaf $\mathbb{Q}_\ell$ we have $Z(X, \mathbb{Q}_\ell, T) = Z_X(T)$.

There is a generalised Lefschetz trace formula for these zeta functions, which implies in particular

$$Z(X, \mathcal{F}, T) = \prod_{i=0}^{2d} \det\left(1 - FT \mid H_c^i(\overline{X}, \mathcal{F})\right)^{(-1)^{i+1}} \tag{1}$$

**Exercise 4.3.** *Reverse-engineer the general Lefschetz trace formula from the above identity.*

### 4.2 The integrality theorem

We start with a definition:

**Definition 4.4** (Integral sheaves)**.** Let $X$ be a separated scheme of finite type over $\mathbb{F}_q$ and let $\mathcal{F}$ be a constructible sheaf of finite-dimensional $\mathbb{Q}_\ell$-vector spaces. We say that $\mathcal{F}$ is **integral** if for all geometric points $\overline{x}$ the eigenvalues of $F_{\overline{x}}^{\deg(x)}$ acting on $\mathcal{F}_{\overline{x}}$ are algebraic integers.

**Remark 4.5.** There is also a version of this definition where one considers $S$-algebraic integers (that is, algebraic numbers that are integral over $\mathbb{Z}[1/S]$).

**Theorem 4.6** (Deligne)**.** *In the above situation, if $\mathcal{F}$ is integral, then $H_c^i(\overline{X}, \mathcal{F})$ is also integral for every $i \geq 0$ (that is, the eigenvalues of the Frobenius action are algebraic integers).*

**Corollary 4.7.** *The constant sheaf $\mathcal{F} = \mathbb{Q}_\ell$ is integral, so $H_c^i(\overline{X}, \mathbb{Q}_\ell)$ is too. In particular, the zeroes and poles of $Z_X(T)$ are algebraic integers.*

*Proof of Theorem 4.6.* Notice first that we can assume that $X$ is reduced by [Sta18, Tag 03SI]. In the case of curves, this guarantees that the singular locus is 0-dimensional.

The proof is by induction on $d = \dim X$. For $d = 0$, the theorem is trivially true for $H_c^0 = H^0$ (0-dimensional schemes are proper).

Next we consider the case of curves, namely $d = 1$. The first observation is that there exists a closed subscheme $Z \hookrightarrow X$ of dimension zero such that $H^0(\overline{X}, \mathcal{F}) \hookrightarrow H^0(\overline{Z}, \mathcal{F})$. To see this, recall that we have assumed that each $\mathcal{F}_{\overline{x}}$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space, so it suffices to choose sufficiently many points to get the desired injection. The result for $H_c^0(\overline{X}, \mathcal{F})$ then follows from the 0-dimensional case, because we have injections

$$H_c^0(\overline{X}, \mathcal{F}) \hookrightarrow H^0(\overline{X}, \mathcal{F}) \hookrightarrow H^0(\overline{Z}, \mathcal{F})$$

and the eigenvalues on a subspace are a subset of the eigenvalues on the whole space. We now consider the case $d = 1$ and $i = 2$. Choose a closed subscheme $Z \hookrightarrow X$ of dimension 0 containing the singular locus of $X$. The localisation sequence gives

$$(0) = H_c^1(\overline{Z}, \mathcal{F}) \to H_c^2(\overline{U}, \mathcal{F}) \to H_c^2(\overline{X}, \mathcal{F}) \to H_c^2(\overline{Z}, \mathcal{F}) = (0),$$

that is, $H_c^2(\overline{U}, \mathcal{F}) \cong H_c^2(\overline{X}, \mathcal{F})$. So we may assume that $X$ is smooth, which by Poincaré duality (Theorem 3.13) implies that $H_c^2(\overline{X}, \mathcal{F})$ is dual to $H^0(\overline{X}, \mathcal{F}^\vee)$. This concludes the proof in this case, because we already know that the statement holds for $H^0(\overline{X}, \mathcal{F}^\vee)$. Notice that we are taking the reciprocal of the eigenvalues *twice*, once because of the presence of the dual sheaf $\mathcal{F}^\vee$ and once because of the duality between $H_c^2(\overline{X}, \mathcal{F})$ and $H^0(\overline{X}, \mathcal{F}^\vee)$.

To finish the proof for curves we still need to handle the case $d = 1, i = 1$. By Equation (1) we have an equality of formal power series

$$\det(1 - FT \mid H_c^1(\overline{X}, \mathcal{F})) = \det(1 - FT \mid H_c^0(\overline{X}, \mathcal{F})) \det(1 - FT \mid H_c^2(\overline{X}, \mathcal{F})) Z(X, \mathcal{F}, T).$$

As $\mathcal{F}$ is integral, the coefficients of $Z(X, \mathcal{F}, T)$ are algebraic integers, so the same is true for $\det(1 - FT \mid H_c^1(\overline{X}, \mathcal{F}))$. As $\det(1 - FT \mid H_c^1(\overline{X}, \mathcal{F}))$ is a polynomial with constant coefficient 1 and whose other coefficients are (algebraic) integers, the inverse roots of

$\det(1 - FT \mid H^1_c(\overline{X}, \mathcal{F}))$ are also algebraic integers. But these inverse roots are precisely the eigenvalues of Frobenius, and we are done.

Finally, for the induction step, assume that $d > 1$. Choose an open subscheme $U \hookrightarrow X$ such that $\dim(X \backslash U) < d$ and for which there exists a *compactifiable* morphism $f : U \to Y$, with $Y$ a curve, such that the fibres of $f$ have dimension $< d$. In other words, we want a diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & U^c \arrow[dl] \\ & Y & \end{tikzcd}$$

To show that such a $U$ exists, choose $U$ to be a dense open affine subscheme of $X$, embed it in projective space, take its closure, and use a suitable pencil of hyperplanes. Using Theorem 3.12 we get

$$H^i_c(\overline{U}, \mathcal{F}) \to H^i_c(\overline{X}, \mathcal{F}) \to H^i_c(\overline{Z}, \mathcal{F})$$

By the induction hypothesis, it suffices to prove the statement for $H^i_c(\overline{U}, \mathcal{F})$, that is, we can assume $X = U$. In particular, we can assume that there is a compactifiable morphism $f : X \to Y$ with fibres of dimension $< d$, and we may consider $R^i f_! \mathcal{F}$ which is again a constructible sheaf. In this situation, there exists a Leray spectral sequence

$$E_2^{p,q} = H^p\left(\overline{Y}, R^q f_! \mathcal{F}\right) \Rightarrow H^{p+q}_c(\overline{X}, \mathcal{F}).$$

Notice that this is slightly less easy than the usual Leray spectral sequence, because we are working with cohomology with compact support. The spectral sequence is compatible with the action of Frobenius, and $H^{p+q}_c(\overline{X}, \mathcal{F})$ is filtered with graded pieces that are subquotients of $H^p\left(\overline{Y}, R^q f_! \mathcal{F}\right)$. Hence it suffices to prove integrality for $H^p\left(\overline{Y}, R^q f_! \mathcal{F}\right)$. Now recall that

$$\left(R^q f_! \mathcal{F}\right)_{\overline{y}} \cong H^q_c\left(X_{\overline{y}}, \mathcal{F}_{\overline{y}}\right),$$

and by the induction hypothesis we know that $H^q_c\left(X_{\overline{y}}, \mathcal{F}_{\overline{y}}\right)$ is integral. It follows that $R^q f_! \mathcal{F}$ is integral, hence by the case of curves so is $H^p_c(\overline{Y}, R^q f_! \mathcal{F})$. By the previous argument, this implies the theorem for $H^i(\overline{X}, \mathcal{F})$. □

**Remark 4.8.** As we will see later, in the proof of Conjecture 1.7 we will need a much more refined version of the coarse geometric lemma used in this proof. The advantage in the proof of Deligne's integrality theorem is that the fibres of $f : U \to Y$ can have arbitrarily bad singularities.

### 4.3 An application: Esnault's theorem

The following conjecture was open for a long time:

**Conjecture 4.9** (Manin, Lang). *If $X/\mathbb{F}_q$ is a (smooth projective) Fano variety, then $X(\mathbb{F}_q) \neq \emptyset$; more precisely, $X(\mathbb{F}_q) \equiv 1 \pmod q$.*

**Remark 4.10.** A special case of this is the Chevalley-Warning theorem (which is the case of hypersurfaces of low degree).

**Theorem 4.11** (Esnault [Esn03]). *The conjecture is true more generally for smooth projective varieties $X/\mathbb{F}_q$ such that $\mathrm{CH}_0(X_{\overline{k(X)}}) \cong \mathbb{Z}$.*

**Remark 4.12.** Recall that the Chow group of 0-cycles is

$$\mathrm{CH}_0(X) = \operatorname{coker}\left(\bigoplus_{x\in X_1} k(x)^\times \xrightarrow{\mathrm{div}} \bigoplus_{x\in X_0} \mathbb{Z}\right)$$

where $X_i$ stands for points of dimension $i$ in $X$ and div is induced by the divisor map. The condition on $\mathrm{CH}_0$ appearing in Theorem 4.11 is satisfied for all rationally chain connected varieties over an algebraically closed field. Here *rationally chain connected* means that any two points can be joined by a chain of rational curves.

**Theorem 4.13** (Kollár, Miyaoka, Mori [KMM92])**.** *Fano varieties over an algebraically closed field are rationally chain connected.*

**Remark 4.14.** In Theorem 4.11 one could (equivalently) consider the base change of $X$ to any algebraically closed field $\Omega$ containing the field of rational functions $k(X)$.

**Remark 4.15.** There are generalisations of Theorem 4.11 to singular varieties. For example, Esnault herself has proved that if $Y$ is a smooth projective variety over $\mathbb{Q}_p$, and satisfying $\mathrm{CH}_0\left(Y_{\overline{k(Y)}}\right) \cong \mathbb{Z}$, and $X$ is the (not necessarily smooth) special fibre of a regular flat model of $Y$ over $\mathbb{Z}_p$, then the conclusion holds for $X$.

**Remark 4.16.** Esnault's original proof relies on $p$-adic techniques and rigid cohomology. We will see a modification of the argument, due to Faltings, which allows one to work with 'good old' étale cohomology instead.

The idea of the proof is the following. Let $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. The Lefschetz trace formula gives

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{d} (-1)^i \operatorname{Tr}\left(F \mid H^i\left(\overline{X}, \mathbb{Q}_\ell\right)\right),$$

and $H^0(\overline{X}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$. It follows that the trace of Frobenius on $H^0(\overline{X}, \mathbb{Q}_\ell)$ is 1, hence to prove the theorem it is enough to show that, for $i > 0$, the eigenvalues of $F$ on $H^i(\overline{X}, \mathbb{Q}_\ell)$ are divisible by $q$. Notice that this last statement makes sense because of Deligne's integrality theorem 4.6: the eigenvalues of Frobenius are algebraic *integers*.

We now recall some background material needed for the details of the proof.

### 4.3.1 Cohomology with support in a closed subscheme

If $Z$ is a closed subscheme in $X$, one can define

$$H^0_Z(X, \mathcal{F}) = \ker\left(H^0(X, \mathcal{F}) \to H^0(X \setminus Z, \mathcal{F})\right);$$

this is a left-exact functor, whose right derived functors are by definition $\mathcal{F} \mapsto H^i_Z(X, \mathcal{F})$, cohomology with compact support on $Z$. There is a long exact sequence

$$\cdots \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \cdots$$

where $U := X \setminus Z$, and more generally, if $Y \subseteq Z$ is another closed subset, there is a long exact sequence

$$\cdots \to H^i_Y(X, \mathcal{F}) \to H^i_Z(X, \mathcal{F}) \to H^i_{Z\setminus Y}(X \setminus Y, \mathcal{F}) \to \cdots \tag{2}$$

There is a purity isomorphism: if $\mathcal{F}$ is a locally constant sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules, $Z \subseteq X$ is a smooth pair of varieties over a field, and $c$ is the codimension of $Z$ in $X$, then there is an isomorphism

$$H^i_Z(X, \mathcal{F}) \cong H^{i-2c}(Z, \mathcal{F}(-c))$$

where, as usual, $\mathcal{F}(-c)$ stands for $\mathcal{F} \otimes \mathbb{Z}/n\mathbb{Z}(-c)$. There is of course a $\mathbb{Q}_\ell$-version as well.

**Remark 4.17.** By work of Gabber, the purity isomorphism is now known to exist more generally for regular pairs.

### 4.3.2 Cycle map

Let $X$ be a smooth variety and let $Z^i(X)$ be the group of cycles of codimension $i$ on $X$. We briefly review the construction of the cycle map

$$Z^i(X) \to H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i)).$$

In particular, to any closed subscheme $Z$ of codimension $i$ we want to attach a class in $H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i))$. If $Z$ happens to be smooth, by purity we have

$$H^0(Z, \mathbb{Z}/n\mathbb{Z}) \cong H^{2i}_Z(X, \mathbb{Z}/n\mathbb{Z}(i)) \to H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i)),$$

and we define the cycle class of $Z$ to be the image of $1 \in H^0(Z, \mathbb{Z}/n\mathbb{Z})$ in $H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i))$. If $Z$ is not smooth, let $Y \subset Z$ be the singular locus, and consider the long exact sequence (2) for the pair $Y \subset Z$:

$$H^\alpha_Y(X, \mathbb{Z}/n\mathbb{Z}(\cdot)) \to H^\alpha_Z(X, \mathbb{Z}/n\mathbb{Z}(\cdot)) \to H^\alpha_{Z \setminus Y}(X \setminus Y, \mathbb{Z}/n\mathbb{Z}(\cdot)) \to H^{\alpha+1}_Y(X, \mathbb{Z}/n\mathbb{Z}(\cdot))$$

For $\alpha < 2i = 2\,\mathrm{codim}_X(Z)$ the cohomology groups $H^\alpha_Y(X, \mathbb{Z}/n\mathbb{Z}(\cdot))$ and $H^{\alpha+1}_Y(X, \mathbb{Z}/n\mathbb{Z}(\cdot))$ vanish: if $Y$ is smooth, this follows by purity for dimension reasons. Even if $Y$ is not smooth, one can proceed by (Noetherian) induction by taking $Y = Y_0$ and $Y_{n+1}$ to be the singular locus of $Y_n$ and writing out the above exact sequence for the pairs $(Y_n, Y_{n+1})$; the conclusion is still the same. Now we take $\alpha = 2i$ and $\cdot = i$ to get

$$0 \to H^{2i}_Z(X, \mathbb{Z}/n\mathbb{Z}(i)) \to H^{2i}_{Z \setminus Y}(X \setminus Y, \mathbb{Z}/n\mathbb{Z}(i)) \to 0,$$

while by purity we have

$$H^{2i}_{Z \setminus Y}(X \setminus Y, \mathbb{Z}/n\mathbb{Z}(i)) \cong H^0(Z \setminus Y, \mathbb{Z}/n\mathbb{Z}).$$

From the previous maps we obtain an isomorphism

$$H^0(Z \setminus Y, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} H^{2i}_Z(X, \mathbb{Z}/n\mathbb{Z}(i)).$$

We may now further compose this isomorphism with the canonical maps $H^0(Z, \mathbb{Z}/n\mathbb{Z}) \to H^0(Z \setminus Y, \mathbb{Z}/n\mathbb{Z})$ and $H^{2i}_Z(X, \mathbb{Z}/n\mathbb{Z}(i)) \to H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i))$ to get a morphism

$$H^0(Z, \mathbb{Z}/n\mathbb{Z}) \to H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i)).$$

The image of $1 \in H^0(Z, \mathbb{Z}/n\mathbb{Z})$ through this map is then the value of the cycle class map at $Z$.

**Definition 4.18.** Two cycles $\alpha_1, \alpha_2$ are **rationally equivalent** if $\alpha_1 - \alpha_2$ is a $\mathbb{Z}$-linear combination of cycles of the form $T(0) - T(\infty)$, where $T \subseteq X \times \mathbb{P}^1$ is a closed subscheme.

Here $T(x)$ denotes the scheme-theoretic inverse image of the closed point $x \in \mathbb{P}^1$ by the projection $T \to \mathbb{P}^1$.

The $i$-th Chow group of $X$ is defined by

$$\mathrm{CH}^i(X) := Z^i(X)/\text{rational equivalence}$$

It is known that the cycle map factors through $\mathrm{CH}^i(X) \to H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i))$.

### 4.3.3 Correspondences

Consider the obvious diagram

$$\begin{tikzcd} & \overline{X} \times \overline{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \overline{X} & & \overline{X} \end{tikzcd}$$

We have pullback maps $p_1^*, p_2^* : H^i(\overline{X}, \mathbb{Q}_\ell(j)) \to H^i(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(j))$ and, if $\overline{X}$ is smooth and projective, by Poincaré duality we get push-forward maps

$$p_{1*}, p_{2*} : H^{4d-i}(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(j)) \to H^{2d-i}(\overline{X}, \mathbb{Q}_\ell(j-d)).$$

Now take $\alpha \in \mathrm{CH}^d(\overline{X} \times \overline{X})$, which has a class $[\alpha] \in H^{2d}\left(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(d)\right)$. We can then consider the following chain of maps:

$$H^i(\overline{X}, \mathbb{Q}_\ell(j)) \xrightarrow{p_1^*} H^i\left(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(j)\right) \xrightarrow{-\cup[\alpha]} H^{2d+i}(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(j+d)) \xrightarrow{p_{2,*}} H^i(\overline{X}, \mathbb{Q}_\ell(j)).$$

In particular, given $\alpha$, we obtain $[\alpha]_* : H^i(\overline{X}, \mathbb{Q}_\ell(j)) \to H^i(\overline{X}, \mathbb{Q}_\ell(j))$. The above $\alpha$ is classically called a *correspondence* of $X$ with itself, and its action on cohomology is the correspondence action.

### 4.3.4 Bloch's lemma

**Proposition 4.19** (Bloch's lemma on the decomposition of the diagonal)**.** *Let $X$ be a smooth projective variety over a field $k$, and suppose that $\mathrm{CH}_0(X_{\overline{k(X)}}) \cong \mathbb{Z}$. Let $\Delta \subset X \times X$ be the diagonal. Then, for some positive integer $N$, there exists a decomposition*

$$N \cdot \Delta \sim \Gamma_1 + \Gamma_2 \in \mathrm{CH}^d(X \times X) \quad (\textit{rational equivalence})$$

*such that $\Gamma_1$ is supported on $\star \times X$, where $\star \subseteq X$ is a 0-dimensional closed subscheme, and $\Gamma_2$ is supported on $X \times D$, where $D$ is a divisor.*

For the proof we need:

**Lemma 4.20.** *Let $V$ be a variety over a field $F$, and write $\overline{V}$ for $V \times_F \overline{F}$. The group*

$$\ker\left(\mathrm{CH}_0(V) \to \mathrm{CH}_0(\overline{V})\right)$$

*is torsion.*

*Proof.* If $\alpha$ is an element in the kernel, there exists a finite extension $L/F$ such that $\alpha$ becomes 0 in $\mathrm{CH}_0(X_L)$. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
\bigoplus_{x\in V_1} k(x)^\times \arrow[r] \arrow[d] & \bigoplus_{x\in V_0} \mathbb{Z} \arrow[r] \arrow[d] & \mathrm{CH}_0(V) \arrow[d] \\
\bigoplus_{x\in (V_L)_1} k(x)^\times \arrow[r] \arrow[d, "N_{L/F}"'] & \bigoplus_{x\in (V_L)_0} \mathbb{Z} \arrow[r] \arrow[d] & \mathrm{CH}_0(V_L) \arrow[d] \\
\bigoplus_{x\in V_1} k(x)^\times \arrow[r] & \bigoplus_{x\in V_0} \mathbb{Z} \arrow[r] & \mathrm{CH}_0(V) :
\end{tikzcd}$$

then the composition of the vertical arrows is $[L:F]$, and an easy diagram chasing shows that if $\alpha$ is in $\ker \mathrm{CH}_0(V) \to \mathrm{CH}_0(V_L)$, then it is killed by $[L:F]$. (Notice that the vertical maps at the level of $\bigoplus \mathbb{Z}$ are multiplication by the local degree of the field extensions). $\square$

*Proof of Bloch's lemma.* Let $\eta : \mathrm{Spec}\, k(X) \to X$ be the generic point. Then $\Delta$ induces, by pullback, an element $\Delta_\eta$ of $Z_0(X_{k(X)})$. The assumption implies that $\mathrm{CH}_0(X_{\overline{k(X)}} \setminus \star_{\overline{k(X)}}) = 0$ for some 0-dimensional $\star$ (which, over the algebraic closure, becomes a bunch of points, one of which is such that one can concentrate the support of any 0-cycle on it, up to rational equivalence).

By the lemma above, some multiple $N\Delta_\eta$ of $\Delta_\eta$ vanishes in $\mathrm{CH}_0(X_{k(X)} \setminus \star_{k(X)})$. Let $\star \times X$ be the closure of $\star_{k(X)}$ in $X \times X$. There exists some $\Gamma_1$, supported on $\star \times X$, such that $N \cdot \Delta - \Gamma_1$ maps to 0 in $\mathrm{CH}_0(X_{k(X)})$. Now it is an easy fact that

$$\mathrm{CH}_0(X_{k(X)}) = \varinjlim_{U \text{ open in } X} \mathrm{CH}_d(X \times U),$$

and open subsets of the form $X \setminus D$ are cofinal in the system of all open subsets, so

$$\mathrm{CH}_0(X_{k(X)}) = \varinjlim_{D} \mathrm{CH}_d(X \times (X \setminus D)).$$

The fact that $N\Delta - \Gamma_1$ is 0 in the direct limit now means precisely that, for some $D$, the cycle $N\Delta - \Gamma_1$ is of the desired form $\Gamma_2$. This follows from the exact sequence

$$\mathrm{CH}_d(X \times D) \to \mathrm{CH}_d(X \times X) \to \mathrm{CH}_d(X \times (X \setminus D)).$$

$\square$

### 4.3.5 Proof of Theorem 4.11

Proposition 4.19, applied to the diagonal of $X \times X$, implies that for the correspondence action of the diagonal $\Delta$ we have

$$N[\Delta]_* = [\Gamma_1]_* + [\Gamma_2]_* \text{ on } H^i(\overline{X}, \mathbb{Q}_\ell);$$

moreover, $[\Delta]_*$ is the identity, while $[\Gamma_1]_* = 0$, because the action factors via the $d$-dimensional cohomology of the 0-dimensional variety $\star$. As $\Gamma_2$ is concentrated on $X \times D$, for every $\alpha \in H^i(\overline{X}, \mathbb{Q}_\ell)$ we then have

$$N\alpha \in \ker \left( H^i(\overline{X}, \mathbb{Q}_\ell) \to H^i(\overline{X \setminus D}, \mathbb{Q}_\ell) \right)$$

But $N : H^i(\overline{X}, \mathbb{Q}_\ell) \to H^i(\overline{X}, \mathbb{Q}_\ell)$ is an isomorphism, so the same is true for $\alpha$ itself, which therefore comes from a class in $H^i_{\overline{D}}(\overline{X}, \mathbb{Q}_\ell)$. Recall once more that we have a purity isomorphism: if $Z \subset \overline{X}$ is smooth,

$$H^i_Z(\overline{X}, \mathbb{Q}_\ell(j)) \cong H^{i-2c}(Z, \mathbb{Q}_\ell(j-c)),$$

where $c$ is the codimension of $Z$ in $X$. If moreover we are over $\mathbb{F}_q$, this implies that the action of Frobenius on $H^i_Z(\overline{X}, \mathbb{Q}_\ell(j))$ is $q^c$ times the action on $H^{i-2c}(Z, \mathbb{Q}_\ell(j))$. Applying this with $Z = \overline{D}$ (we are assuming, for now, that $D$ is smooth) and $c = 1$, we have seen that $\alpha$ comes from $H^i_{\overline{D}}(\overline{X}, \mathbb{Q}_\ell) \cong H^{i-2}(\overline{D}, \mathbb{Q}_\ell(-1))$. The eigenvalues of the action of Frobenius on this latter space are of the form $q^2 \times$ eigenvalues on $H^{i-2}(\overline{D}, \mathbb{Q}_\ell)$, hence they are divisible by $q^2$ (by theorem 4.6). Finally, if $D$ is not smooth, we can find a chain of subschemes $\{\text{pt}\} \subset D_0 \subset D_1 \subset \cdots \subset D_r = D$ such that $D_g \setminus D_{g-1}$ is smooth. The long exact sequence

$$\cdots \to H^i_{\overline{D_{g-1}}}(\overline{X}, \mathbb{Q}_\ell) \to H^i_{\overline{D_g}}(\overline{X}, \mathbb{Q}_\ell) \to H^i_{\overline{D_g \setminus D_{g-1}}}(\overline{X} \setminus \overline{D_{g-1}}, \mathbb{Q}_\ell)$$

then implies the theorem by Noetherian induction.

# 5 Katz's proof of the Riemann Hypothesis for hypersurfaces

In this section we discuss Katz's 2015 proof [Kat15] of the following special case of the Riemann hypothesis:

**Theorem 5.1.** *Let $X \subset \mathbb{P}^{d+1}_{\mathbb{F}_q}$ be a smooth projective hypersurface. The eigenvalues of Frobenius acting on $H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$ all have absolute value $q^{i/2}$, for $i = 0, \dots, 2d$.*

**Convention.** For the course of this whole section, *absolute value* means *absolute value with respect to all embeddings of $\mathbb{Q}_\ell$ into $\mathbb{C}$.*

**Remark 5.2.** Scholl [Sch11] shows that the case of hypersurfaces implies the case of general smooth projective varieties.

**Remark 5.3.** The weak Lefschetz theorem (Corollary 3.14) implies that $H^i(\overline{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(\mathbb{P}^d, \mathbb{Q}_\ell)$ is an isomorphism for $i < d$. Moreover, if $k$ is a separably closed field,

$$H^i(\mathbb{P}^d_k, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell\left(-\frac{i}{2}\right), & 0 \le i \le 2d, i \text{ even} \\ 0, i \text{ odd} \end{cases}$$

To see this, notice that for $k = \mathbb{C}$ this is known from topology (up to the Galois action). Since cohomology does not change by extension of algebraically closed fields of characteristic 0, the result also holds for every $k$ of characteristic 0. By smooth proper base change, as in Section 3.8, this implies the result for arbitrary separably closed $k$. This proves that the group is $\mathbb{Q}_\ell$. As for the Galois action, we have a Galois-equivariant map

$$\mathbb{Z}/n\mathbb{Z} \cong \operatorname{Pic}(\mathbb{P}^n)/n \operatorname{Pic}(\mathbb{P}^n) \cong H^1\left(\mathbb{P}^n, \mathbb{G}_m\right)/nH^1\left(\mathbb{P}^n, \mathbb{G}_m\right) \to H^2\left(\mathbb{P}^n, \mu_n\right)$$

which is an isomorphism of abelian groups by comparison with the case $k = \mathbb{C}$. Hence it is an isomorphism of Galois modules, which gives the action of Galois on $H^2\left(\mathbb{P}^n, \mu_n\right)$. For $i > 2$, one similarly considers the Galois-equivariant map

$$\begin{array}{ccc} H^2(\mathbb{P}^d, \mu_n) \otimes \cdots \otimes H^2(\mathbb{P}^d, \mu_n) & \to & H^{2i}(\mathbb{P}^d, \mu_n^{\otimes d}) \\ \omega_1 \otimes \cdots \otimes \omega_i & \mapsto & \omega_1 \cup \cdots \cup \omega_i, \end{array}$$

which is again an isomorphism by comparison with topology.

By the previous arguments, for $i < d$ we get

$$H^i\left(\overline{X}, \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(-\frac{i}{2}\right), i \text{ even} \\ 0, i \text{ odd} \end{cases}$$

By Poincaré duality, the same holds for $2d \geq i > d$, hence in particular the Riemann hypothesis holds for all $H^i$ except at most for $i = d$.

**Goal.** The eigenvalues of Frobenius on $H^d\left(\overline{X}, \mathbb{Q}_\ell\right)$ are of absolute value $q^{d/2}$.

**Lemma 5.4** (Key Lemma). *Let $U \subseteq \mathbb{P}^1_{\mathbb{F}_q}$ be a nonempty open subscheme of the projective line, and let $\mathcal{F}$ be a locally constant sheaf of finite $\mathbb{Q}_\ell$-vector spaces on $U$. Assume that, for every closed point $x \in U$ the inverse characteristic polynomial $P_x := \det(1 - F_{\overline{x}}^{\deg(x)} T \mid \mathcal{F}_{\overline{x}})$ has real coefficients. Assume furthermore that there exists a closed point $x_0 \in U$ such that the polynomial $P_{x_0}$ has roots of absolute value 1. Then the same holds for all closed points.*

**Remark 5.5.** Here *real* means *real in all embeddings* in $\mathbb{C}$, absolute value means absolute value with respect to all embeddings and $\overline{x}$ is a fixed geometric point above $x$.

We now show that the Key Lemma implies the Riemann Hypothesis:

*Proof.* Let $F$ be an equation for $X$, let $n$ be its degree, and let $G$ be the equation of a hypersurface (of the same degree) for which the result is known. For example, if $n = \deg F$ is prime to $q$, then we can take $G = \sum_{i=0}^{d} a_i x_i^n$ (see Section 2). If $(n, q) \neq 1$, then one can take

$$G = x_0^n + \sum_{i=0}^{d} x_i x_{i+1}^{n-1},$$

and similar methods involving Gauss sums allow one to prove the Riemann hypothesis directly. Consider now the deformation $tF + (1-t)G = 0$. We consider this as a fibration over the $t$-line. This fibration may have singular fibres, but the fibres at $t = 0$ and at $t = 1$ are smooth by construction and by assumption respectively. Let $U \subseteq \mathbb{P}^1_{\mathbb{F}_q}$ be the non-empty open locus where the fibres are smooth. The statement follows by applying Lemma 5.4 to $R^d f_* \mathbb{Q}_\ell\left(-\frac{d}{2}\right)$ and $x_0 = 1$.

We check that the assumptions of the lemma are satisfied. The sheaf $R^d f_* \mathbb{Q}_\ell$ is locally constant by Theorem 3.10. We also need to know that the coefficients of the relevant characteristic polynomials are real. To see this, let $x$ be a closed point of $U$, and denote the Frobenius associated with $\kappa(x)$ by $F_x$. We use the fact that

$$\det(1 - F_{\overline{x}}^{\deg(x)} T \mid (R^d f_* \mathbb{Q}_\ell)_{\overline{x}}) = \det(1 - T F_x \mid H^d(X_{\overline{x}}, \mathbb{Q}_\ell)),$$

where $X_{\overline{x}}$ is the geometric fibre of $f$ over $\overline{x}$. This is a projective hypersurface of dimension $d$ defined over $\kappa(x)$, so by our previous arguments we know that $\det(1 - T^{\deg(x)} F_x \mid H^i(X_{\overline{x}}, \mathbb{Q}_\ell))$ has rational coefficients for $i \neq d$. On the other hand, the cohomological expression for the zeta function shows that

$$\prod_{i=0}^{2d} \det(1 - T^{\deg(x)} F_x \mid H^i(X_{\overline{x}}, \mathbb{Q}_\ell))^{(-1)^{i+1}}$$

is a formal power series with rational coefficients. So $\det(1 - T^{\deg(x)} F_x \mid H^d(X_{\overline{x}}, \mathbb{Q}_\ell))$ has rational coefficients as well, and therefore $\det(1 - F_{\overline{x}}^{\deg(x)} T^{\deg(x)} \mid (R^d f_* \mathbb{Q}_\ell)(-d/2)_{\overline{x}})$

has real coefficients (when $d$ is odd, we may need to extract a square root of $|\kappa(x)|$). This polynomial has the same coefficients as $\det(1 - F_{\overline{x}}^{\deg(x)} T \mid (R^d f_* \mathbb{Q}_\ell)_{\overline{x}})$, with some 0's inserted. $\square$

**Remark 5.6.** Provided that one knows that the coefficients of the characteristic polynomials of Frobenius are real, the argument also works for any $R^j f_* \mathbb{Q}_\ell \left(-\frac{j}{2}\right)$. In particular, if one could find a deformation from an arbitrary variety $X$ to one for which the Riemann hypothesis is known to hold, the Key Lemma would prove the Riemann Hypothesis for $X$.

## 5.1 Reminder on the arithmetic fundamental group

Let $X$ be a connected space and let $\overline{x} \to X$ be a geometric point. One can define a profinite fundamental group $\pi_1(X, \overline{x})$ such that there is a correspondence

$$\left\{ \begin{array}{c} \text{finite étale} \\ \text{covers } Y \to X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{finite sets endowed} \\ \text{with a continuous action of } \pi_1(X, \overline{x}) \end{array} \right\}$$
$$Y \qquad \mapsto \qquad Y_{\overline{x}}$$

and a similar one

$$\left\{ \begin{array}{c} \text{locally constant sheaves of} \\ \text{finite-dimensional } \mathbb{Q}_\ell\text{-vector space} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{finite-dimensional continuous} \\ \text{representations of } \pi_1(X, \overline{x}) \text{ over } \mathbb{Q}_\ell \end{array} \right\}$$
$$\mathcal{F} \qquad \mapsto \qquad \mathcal{F}_{\overline{x}}.$$

The objects in the second correspondence are usually called *lisse* or *smooth* sheaves, and (essentially) correspond to continuous homomorphisms $\pi_1(X, \overline{x}) \to \mathrm{GL}_n(\mathbb{Z}_\ell)$.

Let $X$ be a geometrically connected scheme of finite type over $\mathbb{F}_q$. There is an exact sequence

$$1 \to \pi_1(\overline{X}, \overline{x}) \to \pi_1(X, \overline{x}) \to \mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \to 1.$$

The group $\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)$ is isomorphic to $\hat{\mathbb{Z}}$, with topological generator denoted by $F_q$. To simplify the notation, from now on we drop explicit reference to the base point $\overline{x}$. Let $x$ be a closed point of $X$ of degree $m$,

$$x : \mathrm{Spec}\left(\mathbb{F}_{q^m}\right) \to X.$$

As $\pi_1$ is a covariant functor, $x$ induces a map $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m}\right) \to \pi_1(X)$ which sends the canonical generator $\mathrm{Frob}_q^m$ of $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m}\right)$ to an element of $\pi_1(X)$ which we denote by $F_x$. (In particular, when $x$ is a rational point, we obtain a splitting of the exact sequence.)

**Remark 5.7.** If $\mathcal{F}$ is a lisse sheaf on $X$, its pullback to $\overline{X}$ corresponds to the restriction of the representation of $\pi_1(X)$ on $\mathcal{F}_{\overline{x}}$ to the geometric fundamental group $\pi_1(\overline{X})$. Moreover,

$$H^0\left(\overline{X}, \overline{\mathcal{F}}\right) = \mathcal{F}^{\pi_1(\overline{X})}.$$

If $U/\mathbb{F}_q$ is an affine curve and $\mathcal{F}$ is lisse on $U$, then $H_c^0(\overline{U}, \overline{\mathcal{F}}) = 0$ (either by definition, or because by Poincaré duality this is the same as $H^2(\overline{U}, \overline{\mathcal{F}})$, which vanishes for an *affine* curve). Finally, Poincaré duality also implies

$$H_c^2(\overline{U}, \overline{\mathcal{F}}) = \mathcal{F}_{\pi_1(\overline{X})}(-1),$$

where $\mathcal{F}_{\pi_1(\overline{X})}$ denotes the co-invariants of the action of $\pi_1(\overline{X})$.

**Remark 5.8.** The crucial observation is the following: for every closed point $x \in U$, the Frobenius $F_x$ acts on $\mathcal{F}_{\pi_1(\overline{X})}$ via $F_q^{\deg(x)}$: indeed, if we work in a quotient where $\pi_1(\overline{X}, \overline{x})$ acts trivially, then the action of an element in $\pi_1(X, \overline{x})$ (coming from a point $x$) depends only on its image in $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, which depends only on the degree of $x$.

## 5.2 Katz's proof

The inspiration for Katz's proof is the following result of Deligne:

**Proposition 5.9** (Deligne)**.** *Let $U/\mathbb{F}_q$ be an affine curve and let $\mathcal{F}$ be a $\mathbb{Q}_\ell$-lisse sheaf over $U$ such that $P_x$ has real coefficients for all closed points $x \in U$. Suppose that there exists a closed point $x_0 \in U$ such that all the eigenvalues of $F_{x_0}$ on $\mathcal{F}$ are of absolute value $\leq 1$. Then the same holds for all closed points.*

**Lemma 5.10** (Rankin's trick)**.** *Under the above assumptions, for all $k \geq 1$ the eigenvalues of $F_q$ on $\left(\mathcal{F}^{\otimes 2k}\right)_{\pi_1(\overline{X})}$ are of absolute value $\leq 1$.*

*Proof.* If $\beta$ is an eigenvalue of $F_q$ on $\left(\mathcal{F}^{\otimes 2k}\right)_{\pi_1(\overline{X})}$, then for $m = \deg(x_0)$ the number $\beta^m$ is an eigenvalue of $F_{x_0}$. But then $\beta^m$ is a product of $2k$ eigenvalues of $F_{x_0}$ on $\mathcal{F}$, which means $|\beta^m| \leq 1 \Rightarrow |\beta| \leq 1$. $\square$

*Proof of Proposition 5.9.* We have already seen that

$$H_c^2\left(\overline{U}, \overline{\mathcal{F}}^{\otimes 2k}\right) \cong \left(\mathcal{F}^{\otimes 2k}_{\pi_1(\overline{U})}\right)(-1),$$

so the Lemma implies that $F_q$ acts on it with eigenvalues of absolute value $\leq q$. Consider now the zeta function associated with $\mathcal{F}^{\otimes 2k}$:

$$Z(U, \mathcal{F}^{\otimes 2k}, T) := \prod_{x \in U_{(0)}} \det\left(1 - T^{\deg x} F_x \mid \mathcal{F}^{\otimes 2k}\right)^{-1},$$

which by the Grothendieck-Lefschetz trace formula we know to be equal to

$$\frac{\det(1 - TF \mid H_c^1(\overline{U}, \overline{\mathcal{F}}^{\otimes 2k})}{\det(1 - TF \mid H_c^2(\overline{U}, \overline{\mathcal{F}}^{\otimes 2k})},$$

where the term corresponding to $H_c^0$ is trivial by Remark 5.7. The fact that the absolute value of the eigenvalues of Frobenius is bounded by $q$ implies that the zeta function converges for $|T| < \frac{1}{q}$, because the denominator has no poles there. By assumption, every factor $\det\left(1 - T^{\deg x} F_x \mid \mathcal{F}\right)$ has real coefficients. The identity

$$\begin{aligned}\det\left(1 - T^{\deg x} F_x \mid \mathcal{F}^{\otimes 2k}\right)^{-1} &= \exp\left(\sum_{n \geq 1} \mathrm{Tr}(F^n \mid \mathcal{F}^{\otimes 2k}) \frac{T^n}{n}\right) \\ &= \exp\left(\sum_{n \geq 1} \mathrm{Tr}(F^n \mid \mathcal{F})^{2k} \frac{T^n}{n}\right)\end{aligned}$$

shows that $\det\left(1 - T^{\deg x} F_x \mid \mathcal{F}^{\otimes 2k}\right)$ has *positive* real coefficients (this is why it is important to work with *even* tensor powers!). In particular, if we fix a factor in the Euler product, each of its coefficients is bounded above by the corresponding coefficient in

$Z(X, \mathcal{F}^{\otimes 2k}, T)$. This implies that every fixed Euler factor for $\mathcal{F}^{\otimes 2k}$ converges for $|T| < \frac{1}{q}$. Hence for every closed point $x$ of $U$ and every eigenvalue $\alpha$ of $F_x$ on $\mathcal{F}$ the number $\alpha^{2k}$ is an eigenvalue of $F_x$ on $\mathcal{F}^{\otimes 2k}$, and by the previous estimate this implies $|\alpha^{2k}| \leq q^{\deg(x)}$ or else $|\alpha| \leq q^{\deg(x)/2k}$. Passing to the limit $k \to \infty$ gives $|\alpha| \leq 1$. $\square$

**Lemma 5.11.** *Let $U \subseteq \mathbb{P}^1$ be a nonempty affine open subscheme, and let $\mathcal{F}$ be a $\mathbb{Q}_\ell$-lisse sheaf **of rank 1** on $U$. Then there exists $m > 0$ such that $\pi_1(\overline{U})$ acts trivially on $\mathcal{F}^{\otimes m}$.*

We briefly postpone the proof of this lemma, and show that Lemma 5.11 and Proposition 5.9 together imply Lemma 5.4.

*Proof.* By Proposition 5.9, all eigenvalues are of absolute value $\leq 1$. To show that they are equal to 1, it suffices to show that their product is 1. Now this product is an eigenvalue of Frobenius acting on $\Lambda^r \mathcal{F}$, where $r = \operatorname{rk} \mathcal{F}$. By Lemma 5.11, there exists $m > 0$ such that $(\Lambda^r \mathcal{F})^{\otimes m}$ has trivial $\pi_1(\overline{U})$-action. It follows that all $F_x$ act on $(\Lambda^r \mathcal{F})^{\otimes m}$ via $F_q^{\deg x}$. In particular, if one of them acts with eigenvalues of absolute value 1, then all do. $\square$

*Proof of Lemma 5.11.* The sheaf $\mathcal{F}$ corresponds to a representation $\rho : \pi_1(U) \to \mathbb{Z}_\ell^\times = \mathrm{GL}_1(\mathbb{Z}_\ell) \cong \mathbb{F}_\ell^\times \times (1 + \ell\mathbb{Z}_\ell)$. Assume for simplicity that $\ell > 2$: then $1 + \ell\mathbb{Z}_\ell \cong \ell\mathbb{Z}_\ell$ via the logarithm map, hence we can consider $\rho^{\ell-1}$ as a map from $\pi_1(U)$ to $\ell\mathbb{Z}_\ell \hookrightarrow \mathbb{Q}_\ell$. By the étale version of Hurewicz's theorem, this map corresponds to an element of $H^1(U, \mathbb{Q}_\ell)$. In particular, $\rho^{\ell-1} \big|_{\pi_1(\overline{U})}$ corresponds to an element of $H^1(\overline{U}, \mathbb{Q}_\ell)$ fixed by Frobenius (because it comes by pullback from something defined over $U$). Now observe that $H^1(\overline{U}, \mathbb{Q}_\ell)$ is dual to $H^1_c(\overline{U}, \mathbb{Q}_\ell)(-1)$, and – setting $Z := \mathbb{P}^1 \setminus U$ – by the long exact sequence for compact support cohomology we get

$$H^0\left(\overline{Z}, \mathbb{Q}_\ell\right) \to H^1_c(\overline{U}, \mathbb{Q}_\ell) \to H^1(\mathbb{P}^1_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell) = (0).$$

Since Frobenius acts trivially on $H^0\left(\overline{Z}, \mathbb{Q}_\ell\right)$, and we have just shown that $H^1_c(\overline{U}, \mathbb{Q}_\ell)$ is a quotient of $H^0\left(\overline{Z}, \mathbb{Q}_\ell\right)$, we get that Frobenius acts trivially on $H^1_c(\overline{U}, \mathbb{Q}_\ell)$, hence it has no invariants when acting on $H^1_c(\overline{U}, \mathbb{Q}_\ell)(-1)$, because of the twist. Hence $\rho^{\ell-1}|_{\pi_1(\overline{U})}$ must be trivial, which is what we wanted to show. $\square$

# 6 Deligne's original proof of the Riemann Hypothesis

## 6.1 Reductions in the proof of the Weil Conjectures

We now only have one remaining conjecture, namely the Riemann Hypothesis 1.7:

**Theorem 6.1.** *The eigenvalues of Frobenius on $H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$ have absolute value $q^{i/2}$.*

**Remark 6.2.** The proof will be by induction on the dimension of $X$, starting from $\dim X = 0$.

**Remark 6.3.** We continue with our running convention that *absolute value* means *absolute value under every embedding.* Also recall that $X/\mathbb{F}_q$ is a smooth projective variety.

**Lemma 6.4.** *It is enough to prove the theorem after finite extension of the base field.*

*Proof.* Let $\mathbb{F}_{q'}/\mathbb{F}_q$ be a finite extension. The hypothesis is that $F^{[\mathbb{F}_{q'}:\mathbb{F}_q]}$ has eigenvalues of absolute value $(q')^{i/2}$, which implies that the eigenvalues of $F$ have absolute value $q^{i/2}$. $\square$

**Lemma 6.5.** *It is enough to prove the theorem for $i = d = \dim X$.*

*Proof.* By Poincaré duality it is enough to prove the statement for $i \leq d$. Let $\overline{Y} \hookrightarrow \overline{X}$ be a smooth connected hyperplane section (which exists by Bertini's theorem). By Lemma 6.4, we may assume that such a smooth section is defined over $\mathbb{F}_q$. The weak Lefschetz theorem (Corollary 3.14) yields

$$H^i(\overline{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(\overline{Y}, \mathbb{Q}_\ell) \quad \text{for } i < d-1$$

and

$$H^{d-1}(\overline{X}, \mathbb{Q}_\ell) \hookrightarrow H^{d-1}(\overline{Y}, \mathbb{Q}_\ell),$$

so the induction hypothesis on $Y$ gives the statement for $H^i(X, \mathbb{Q}_\ell)$ with $i < d$. Hence – as claimed – only the case $i = d$ remains. □

**Lemma 6.6.** *It is enough to prove the following for $d$ **even:** for every smooth projective $d$-dimensional variety $X$, the eigenvalues $\alpha$ of $F$ on $H^d(\overline{X}, \mathbb{Q}_\ell)$ satisfy*

$$q^{\frac{d}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}. \tag{3}$$

*Proof.* If $k$ is even, $\alpha^k$ is an eigenvalue of Frobenius on $H^{kd}(X^k, \mathbb{Q}_\ell)$ by the Künneth formula. The hypothesis implies

$$q^{kd/2-1/2} \leq |\alpha|^k \leq q^{kd/2+1/2},$$

which by taking $k$-th roots and passing to the limit $k \to \infty$ yields $|\alpha| = q^{d/2}$. □

### 6.2 Geometric and topological ingredients: Lefschetz pencils

Let $\mathbb{P}$ be a projective space of dimension $> 1$ over an algebraically closed field $k$. Let $\check{P}$ be the dual projective space: points of $\check{P}$ correspond to hyperplanes in $\mathbb{P}$. We denote the bijection (on closed points) by $t \in \check{P} \longleftrightarrow H_t \subseteq \mathbb{P}$. If $A \subseteq \mathbb{P}$ is a subspace of codimension 2, the hyperplanes $H \supset A$ are parametrised by the points of a line $D \subseteq \check{P}$. These hyperplanes form a **pencil** with axis $A$. Let $\overline{X} \subseteq \mathbb{P}$ be a smooth projective variety of dimension $d = n + 1$. Define the incidence variety

$$\tilde{X} := \{(x, t) \in \overline{X} \times D : x \in H_t\},$$

which as usual comes equipped with maps

$$\begin{tikzcd} \overline{X} & \tilde{X} \arrow[l, "\pi"'] \arrow[d, "f"] \\ & D. \end{tikzcd}$$

Assume that $\overline{X} \cap A$ is smooth and has codimension 2 in $\overline{X}$. The fibre of $f$ over $t \in D$ is $\overline{X} \cap H_t$. The fibre of $\pi$ over $x \in \overline{X}$ is:

- $(x, D) \cong D$ if $x \in \overline{X} \cap A$;
- a single point $(x, H_x)$ if $x \notin A$.

This remark suggests (and the definition of $\tilde{X}$ shows) that $\tilde{X}$ is the blowup of $\overline{X}$ in $\overline{X} \cap A$. The map $f : \tilde{X} \to D$ is a proper surjective map (a fibration over the line, in the sense of algebraic geometry).

**Theorem 6.7.** *By choosing $A$ appropriately (in fact, sufficiently generally), and after performing some Veronese embedding in characteristic $p > 0$, we may arrange for the following to hold:*

1. *there exists a finite set of points $S \subseteq D$ such that the fibre $\tilde{X}_t$ is smooth for all $t \notin S$*
2. *the fibres $\tilde{X}_s$ for $s \in S$ have only one singularity, and it is quadratic (see Definition 6.8).*

**Definition 6.8.** A singularity is **quadratic** if the completion of the corresponding local ring is isomorphic to $k[[t_1, \ldots, t_n]]/(Q(t_1, \ldots, t_n))$ with $Q$ a nondegenerate quadratic form.

Before moving on, we give a general overview of Deligne's argument to prove the estimate in Equation (3).

### 6.3 Strategy of proof of estimate (3)

Let $k = \overline{\mathbb{F}_q}$ and let $X/\mathbb{F}_q$ be a smooth projective variety. Take a Lefschetz pencil on $\overline{X}$. Up to extending $\mathbb{F}_q$, we may assume the following objects to all be defined over $\mathbb{F}_q$:

- the base $D$ and axis $A$ of the pencil, hence the pencil itself;
- the points of $S$;
- some other point $u_0 \in D \setminus S$;
- a smooth hyperplane section $\overline{Y}$ of the fibre $\overline{X}_{u_0}$.

Notice that $\dim \overline{Y} = d - 2 = n - 1$: we wish our induction to only involve varieties of *even* dimension.

**Theorem 6.9** (Blow-up formula for étale cohomology)**.** *There is a Frobenius-equivariant isomorphism*

$$H^i\left(\tilde{X}, \mathbb{Q}_\ell\right) = H^i\left(\overline{X}, \mathbb{Q}_\ell\right) \oplus H^{i-2}\left(A \cap \overline{X}, \mathbb{Q}_\ell(-1)\right).$$

**Remark 6.10.** The proof of the blow-up formula goes through the projective bundle formula and some localisation sequences, so it exists for almost any cohomological theory (in particular, for étale cohomology).

The blow-up formula (together with its compatibility with the Frobenius action) shows that it is enough to prove (3) for $\tilde{X}$ instead of $\overline{X}$, hence we may assume that there is a Lefschetz pencil $\overline{X} \to D$. We have a Frobenius-equivariant Leray spectral sequence (for usual étale cohomology: everything is proper, so we don't need compact support cohomology)

$$H_2^{p,q} = H^p\left(D, R^q f_* \mathbb{Q}_\ell\right) \Rightarrow H^{p+q}\left(\overline{X}, \mathbb{Q}_\ell\right)$$

We know that the stalks of $R^q f_* \mathbb{Q}_\ell$ at points $t \in D$ are the cohomology groups $H^q(\overline{X}_t, \mathbb{Q}_\ell)$. Assume for now that all fibres of $f$ are smooth. Notice that this almost never happens, and avoiding this assumption is a substantial difficulty; for now, however, we just discuss this toy case.

By proper smooth base change, the sheaves $R^q f_* \mathbb{Q}_\ell$ are locally constant on $D \cong \mathbb{P}^1$, hence they are constant (because $\mathbb{P}^1$ has no finite connected étale covers). It follows that $R^q f_* \mathbb{Q}_\ell$ is the constant sheaf associated with the abelian group $H^q(\overline{X}_{u_0}, \mathbb{Q}_\ell)$ (because this is the fibre at the point $u_0$, hence at every point by constancy). The groups $E_2^{p,q}$ contributing to $H^{n+1}(\overline{X}, \mathbb{Q}_\ell)$ are

$$H^0(D, R^{n+1} f_* \mathbb{Q}_\ell), \quad H^1(D, R^n f_* \mathbb{Q}_\ell), \quad H^2(D, R^{n-1} f_* \mathbb{Q}_\ell),$$

hence it is enough to prove that (3) holds for the eigenvalues of Frobenius acting on these groups.

1. $H^1(D, \mathcal{F})$ vanishes for any constant sheaf of finite-dimensional $\mathbb{Q}_\ell$-vector spaces: one reduces first to $\mathbb{Z}_\ell$-sheaves, and then to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$, for which it is well-known that $H^1$ vanishes (because $\mathbb{P}^1$ has no nontrivial covers, for example).

2. Using Poincaré duality we find

$$H^2(D, R^{n-1} f_* \mathbb{Q}_\ell) = H^2(D, H^{n-1}(\overline{X}_{u_0}, \mathbb{Q}_\ell)) = H^0(D, H^{n-1}(\overline{X}_{u_0}, \mathbb{Q}_\ell)^\vee)^\vee(-1),$$

   and since $\dim \overline{X}_u = n$ we obtain $H^{n-1}(\overline{X}_{u_0}, \mathbb{Q}_\ell) \hookrightarrow H^{n-1}(\overline{Y}_{u_0}, \mathbb{Q}_\ell)$, so the claim follows from the case $d-2$ by induction.

3. A similar argument works for $H^0\left(D, R^{n+1} f_* \mathbb{Q}_\ell\right) \cong H^{n+1}\left(\overline{X}_{u_0}, \mathbb{Q}_\ell\right)$ (applying Poincaré duality and weak Lefschetz).

Of course, the difficulty is that when $f$ has singular fibres the sheaves $R^{n+1} f_* \mathbb{Q}_\ell$ are in general not locally constant. The sheaf $H^1(D, R^n f_* \mathbb{Q}_\ell)$ will prove to be the most difficult one to understand. For its study we need the monodromy theory of Lefschetz pencils, which is coming up next.

## 6.4 Lefschetz theory over $\mathbb{C}$

Lefschetz theory is the study of the cohomology of Lefschetz pencils. This was first developed over $\mathbb{C}$ by Lefschetz by topological methods, and the results were later transposed in SGA7 for étale cohomology. As Deligne does in Weil I, we first present the theory over $\mathbb{C}$ for singular cohomology with $\mathbb{Z}$-coefficients as a motivation. We shall start with the local picture – what happens around a single bad fibre – and then move on to the global aspects.

### 6.4.1 Local theory

Consider a small complex disc $D$ around a singular fibre.

$$\begin{tikzcd}
X_t \arrow[r, hook] \arrow[d, "\text{smooth}"'] & X \arrow[d] & X_0 \arrow[l, hook'] \arrow[d, "\text{one quadratic singularity}"] \\
\{t \neq 0\} \arrow[r, hook] & D & \{0\} \arrow[l, hook']
\end{tikzcd}$$

As $X_0$ is a deformation retract of $X$, we have an isomorphism $H^i(X_0, \mathbb{Z}) \xleftarrow{\sim} H^i(X, \mathbb{Z})$, and we also have a restriction map $H^i(X, \mathbb{Z}) \to H^i(X_t, \mathbb{Z})$. By composing (the inverse of) the former with the latter we obtain a (co)specialisation map

$$\mathrm{cosp} : H^i(X_0, \mathbb{Z}) \to H^i(X_t, \mathbb{Z}).$$

There is a monodromy action of the fundamental group of the punctured disc,

$$\pi_1(D \setminus \{0\}, t) \cong \mathbb{Z},$$

on $H^i(X_t, \mathbb{Z})$. We shall fix a generator $\gamma$ of $\pi_1(D \setminus \{0\}, t)$, so that studying the monodromy action reduces to the study of the action of $\gamma$.

**Theorem 6.11** (Lefschetz)**.** *The following hold:*

1. *for $i \neq n, n+1$ the cospecialisation map* cosp *is an isomorphism, and the monodromy action is trivial;*

2. *there exists a canonical element $\delta \in H^n(X_t, \mathbb{Z})$, called the* ***vanishing cycle*** *and well-defined up to sign, such that we have an exact sequence*

$$0 \to H^n(X_0, \mathbb{Z}) \xrightarrow{\mathrm{cosp}} H^n(X_t, \mathbb{Z}) \xrightarrow{\alpha} \mathbb{Z} \to H^{n+1}(X_0, \mathbb{Z}) \xrightarrow{\mathrm{cosp}} H^{n+1}(X_t, \mathbb{Z}) \to 0,$$

*where the map $\alpha$ sends $\xi \in H^n(X_t, \mathbb{Z})$ to $\xi \cup \delta \in H^{2n}(X, \mathbb{Z}) = \mathbb{Z}$.*

The action of $\pi_1(D \setminus \{0\}) = \mathbb{Z}$ on $H^{n+1}(X_t, \mathbb{Z})$ is trivial, and on $H^n(X_t, \mathbb{Z})$ is given by:

**Theorem 6.12** (Picard–Lefschetz formula)**.** *The action of $\gamma$ is given by*

$$\gamma \cdot x = x \pm (x, \delta)\delta,$$

*where $(\cdot, \cdot)$ is the Poincaré duality pairing $H^n(X_t, \mathbb{Z}) \times H^n(X_t, \mathbb{Z}) \to H^{2n}(X_t, \mathbb{Z}) \cong \mathbb{Z}$. The sign is determined by $n$ mod 4 (once $\gamma$ is fixed): if $n \equiv 0, 1 \pmod 4$ then the sign is $+$, and if $n \equiv 2, 3 \pmod 4$ the sign is $-$.*

Further interesting properties of this situation:

- The monodromy action on $H^n(X_t, \mathbb{Z})$ is compatible with the Poincaré duality pairing $(\cdot, \cdot)$.
- The orthogonal of $\langle \delta \rangle$ with respect to the Poincaré pairing is $H^n(X_t, \mathbb{Z})^{\pi_1(D \setminus \{0\}, t)}$: this is an immediate consequence of the Picard–Lefschetz formula.

### 6.4.2 Global theory

We will now consider a Lefschetz pencil $X \to D \cong \mathbb{P}^1$ and let $U = D \setminus S$, where $S \subseteq D$ is the set of points over which the fibre is singular. Fix $u \in U$ and consider the fundamental group $\pi_1(U, u)$: it is generated by loops $\gamma_s$ around the points $s \in S$. The group $\pi_1(U, u)$ acts by monodromy on $H^i(X_u, \mathbb{Z})$ for all $i$, and the local theory describes the action of every $\gamma_s$: for each $s \in S$ we have a vanishing cycle $\delta_s \in H^n(X_u, \mathbb{Z})$, and $\gamma_s \in \pi_1(U, u)$ acts by

$$\gamma_s \cdot x = x \pm (x, \delta_s)\delta_s. \tag{4}$$

More information on the global action of the fundamental group are provided by the following result:

**Theorem 6.13** (Lefschetz). *The following hold:*

- $\pi_1(U, u)$ *acts trivially on* $H^i(X_u, \mathbb{Z})$ *for* $i \neq n$;
- *the elements* $\delta_s$ *are conjugate under the action of* $\pi_1(U, u)$.

**Proposition 6.14.** *Let $E$ be the subspace of $H^n(X_u, \mathbb{Q})$ generated by the vanishing cycles $\delta_s$. Then $E$ is stable by the action of $\pi_1(U, u)$, and its orthogonal $E^\perp$ (with respect to Poincaré duality) is $H^n(X_u, \mathbb{Q})^{\pi_1(U,u)}$.*

*Proof.* This is an immediate consequence of the Picard-Lefschetz formula (see for example Equation (4)): if $x \in E$ then $\gamma_s \cdot x \in E$, and $x$ is orthogonal to all the $\delta_s$ if and only if it is stable under the action of every $\gamma_s$. $\square$

**Proposition 6.15.** *The action of $\pi_1(U, u)$ on $E/E \cap E^\perp$ is absolutely irreducible.*

**Remark 6.16.** The subspace $E \cap E^\perp$ is in fact 0, but this is a consequence of the Hard Lefschetz theorem. Over $\mathbb{C}$, one can prove Hard Lefschetz independently of what we are doing, but the known proofs of Hard Lefschetz in étale cohomology rely on the Weil conjectures, so we will not be able to assume that $E \cap E^\perp = (0)$ in our applications.

*Proof.* If $F \subseteq E \otimes \mathbb{C}$ is stable by $\pi_1(U, u)$, and $F$ is not contained in $(E \cap E^\perp) \otimes \mathbb{C}$, then there exists $x \in F$ and $s \in S$ such that $(x, \delta_s) \neq 0$. By Picard-Lefschetz we have $\gamma_s x - x = \pm(x, \delta_s)\delta_s$, and this is an element of $F$. Furthermore, the vanishing cycles are all conjugate, so all $\delta_s$ are in $F$, which proves $F = E \otimes \mathbb{C}$. $\square$

Let $n$ be odd (this will be the case in our applications, where the total space is even-dimensional and the fibres are odd-dimensional). The duality pairing on $\left(E/E \cap E^\perp\right) \otimes \mathbb{C}$ is non-degenerate. Hence we get a representation

$$\rho : \pi_1(U, u) \to \mathrm{Sp}\left((E/E \cap E^\perp) \otimes \mathbb{C}\right)$$

whose analogue in étale cohomology will play an important role.

**Remark 6.17.** When $n$ is even, it is interesting to study the self-product $(\delta_s, \delta_s)$, which (with the appropriate normalisations) turns out to be equal to 2.

## 6.5 Lefschetz theory in étale cohomology

### 6.5.1 Local theory

Consider a Lefschetz pencil

$$\begin{array}{c} \overline{X} \\ \big\downarrow \\ D \cong \mathbb{P}^1 \end{array}$$

defined over an algebraically closed field $k$. Write $d = \dim \overline{X} = n + 1 = 2m + 2$ and let $s \in D$ be a point for which the fibre $\overline{X}_s$ is singular. The completion of the local ring $\mathcal{O}_{D,s}$ is $k[[t]]$. Let $B = \operatorname{Spec} k[[t]]$ and consider the Cartesian diagram.

$$\begin{tikzcd} \overline{X}_B \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ B \arrow[r] & D \cong \mathbb{P}^1 \end{tikzcd}$$

Let $\overline{\eta}$ be a geometric generic point of $B$ and let $s$ be the closed point of $B$. For any $\ell \neq \mathrm{char}(k)$ we have a cospecialisation map

$$H^i(\overline{X}_s, \mathbb{Q}_\ell) \xleftarrow{\sim} H^i(\overline{X}, \mathbb{Q}_\ell) \to H^i(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell),$$

where the first arrow is an isomorphism by the proper base change theorem (which replaces the argument with deformation retracts). There exists a vanishing cycle $\delta \in H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)(m)$, well-defined up to sign.

**Theorem 6.18.** *The cospecialisation map*

$$\mathrm{cosp} : H^i\left(\overline{X}_s, \mathbb{Q}_\ell\right) \to H^i\left(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell\right)$$

*is an isomorphism for $i \neq n, n+1$. Furthermore, there is an exact sequence*

$$0 \to H^n(\overline{X}_{\overline{s}}, \mathbb{Q}_\ell) \to H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(m-n) \to H^{n+1}(\overline{X}_{\overline{s}}, \mathbb{Q}_\ell) \to H^{n+1}(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \to 0 \tag{5}$$

*where the second map is given by*

$$x \mapsto x \cup \delta \in H^{2n}(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)(m) \cong \mathbb{Q}_\ell(m-n).$$

To state the Picard–Lefschetz formula, we need the following facts. Set

$$I = \mathrm{Gal}\left(\overline{k((t))}/k((t))\right).$$

The action of $I$ on $H^i(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ is the analogue of the local monodromy action. Recall that the formula

$$t_\ell(\sigma) := \frac{\sigma(t^{1/\ell^n})}{t^{1/\ell^n}}$$

associates an $\ell^n$-th root of unity $t_\ell(\sigma) \in \mu_{\ell^n}(k)$ with each $\sigma \in I$. Passing to the inverse limit over $n$ we obtain a map $t_\ell : I \to \mathbb{Z}_\ell(1)$ called the cyclotomic character. It induces an isomorphism of the maximal pro-$\ell$ quotient of $I$ with $\mathbb{Z}_\ell(1)$. Moreover, the action of $I$ on $H^i(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ is trivial for $i \neq n$ and factors through $t_\ell$ for $i = n$.

**Theorem 6.19** (Picard-Lefschetz formula)**.** *For all $x \in H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ and $\sigma \in I$ we have*

$$\sigma(x) = x \pm t_\ell(\sigma)(x, \delta)\delta,$$

Notice that $(x, \delta)$ is an element in $\mathbb{Q}_\ell(m-n)$ and $\delta$ is in $H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$, so the product carries a Galois action with twist $2m - n = -1$, to which the cyclotomic character contributes a twist $+1$.

**Remark 6.20.** The Picard-Lefschetz formula is proven in SGA7 by reduction to the complex case. Illusie has later given an independent, purely algebraic proof, which however is not easier than the original one.

**Corollary 6.21.**

1. *Suppose that the vanishing cycle $\delta$ is nonzero. There is a canonical isomorphism $H^n(\overline{X}_s, \mathbb{Q}_\ell) \cong H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)^I$, and this space has codimension 1 in $H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$.*

   *Furthermore, there is a canonical isomorphism $H^{n+1}(\overline{X}_s, \mathbb{Q}_\ell) \cong H^{n+1}(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$.*

2. *Suppose $\delta = 0$. There is a canonical isomorphism $H^n(\overline{X}_s, \mathbb{Q}_\ell) \cong H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ and an exact sequence*

   $$0 \to \mathbb{Q}_\ell(m-n) \to H^{n+1}(\overline{X}_{\overline{s}}, \mathbb{Q}_\ell) \to H^{n+1}(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \to 0. \tag{6}$$

   *Furthermore, the action of $I$ on $H^{n+1}(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ is trivial.*

### 6.5.2 Global theory

Consider again

$$\begin{array}{c} \overline{X} \\ \Big\downarrow \\ D \end{array}$$

and let $S = \{s \in D : \overline{X}_s \text{ is singular}\}$. Also define $\overline{U} := D \setminus S$ and let $\overline{u}$ be a geometric generic point of $\overline{U}$. There is a problem in translating the classical theory to the étale setting: $\pi_1(\overline{U}, \overline{u})$ is huge in characteristic $p$. For example, $\pi_1(\mathbb{A}^1)$ admits every finite $p$-group as a quotient!

To remedy this, instead of the full $\pi_1(\overline{U}, \overline{u})$, one considers its **tame quotient** $\pi_1^t(\overline{U}, \overline{u})$, which classifies étale covers $\overline{V} \to \overline{U}$ extending to $\overline{W} \to D$ that are tamely ramified at the points of $S$ (i.e. the ramification indices at these points are prime to $p$).

**Remark 6.22.** There is a general definition of the tame fundamental group for the complement of any divisor in any proper variety. In our case it is easily defined using the Galois theory of discrete valuations.

**Good news.** As a consequence of the Picard-Lefschetz formula, the action of $\pi_1(\overline{U}, \overline{u})$ on $H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)$ factors via $\pi_1^t(\overline{U}, \overline{u})$. Furthermore, $\pi_1^t(\overline{U}, \overline{u})$ is generated by the inertia groups $I_s$ for $s \in S$.

For every $s \in S$, there is a vanishing cycle $\delta_s \in H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)$ (after twisting by $-m$). If $E$ is the subspace generated by the vanishing cycles, then $E$ is stable by the action of $\pi_1^t(\overline{U}, \overline{u})$, and

$$E^\perp = H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)^{\pi_1^t(\overline{U}, \overline{u})}.$$

This is proven in the same way as the classical case, starting from the Picard-Lefschetz formula. As in the complex case, one proves that the vanishing cycles $\delta_s$ are conjugate under the action of $\pi_1^t(\overline{U}, \overline{u})$. In particular, either all of them are zero, or none of them are. The conjugacy of the $\delta_s$ under the action of $\pi_1(\overline{U}, \overline{u})$ implies that $E/E \cap E^\perp$ is an absolutely irreducible representation, and we get (for odd $n$) a representation

$$\rho : \pi_1^t(\overline{U}, \overline{u}) \to \mathrm{Sp}\left(E/E \cap E^\perp\right).$$

**Theorem 6.23** (Kazhdan-Margulis)**.** *The image of $\rho$ is Zariski dense and $\ell$-adically open in* $\mathrm{Sp}\left(E/E \cap E^\perp\right)$.

This result is a not too hard piece of algebra based on the Picard–Lefschetz formula, and is proven by comparing Lie algebras. We shall only use the Zariski version.

We now start deriving consequences for the sheaves $R^i\pi_*\mathbb{Q}_\ell$, where $\pi : \overline{X} \to D$ is the projection. The first is an immediate consequence of parts of Corollary 6.21:

**Corollary 6.24.** *If the vanishing cycles are nonzero, the sheaves $R^i\pi_*\mathbb{Q}_\ell$ are locally constant for $i \neq n$. If the vanishing cycles are zero, they are locally constant for $i \neq n+1$.*

Now we draw consequences from global Picard–Lefschetz theory.

**Corollary 6.25.** *Consider the inclusion $j : \overline{U} \hookrightarrow D$ and let $\pi : \overline{X} \to D$ be the Lefschetz pencil. Assume the vanishing cycles are* **nonzero**.

1. *The natural adjunction map*

$$R^n\pi_*\mathbb{Q}_\ell \to j_*j^*R^n\pi_*\mathbb{Q}_\ell$$

*is an isomorphism.*

2. $j^*R^n\pi_*\mathbb{Q}_\ell$ *is locally constant on* $\overline{U}$, *and has a filtration by locally constant subsheaves*

$$0 \subseteq \mathcal{E}\cap\mathcal{E}^\perp \subseteq \mathcal{E} \subseteq j^*R^n\pi_*\mathbb{Q}_\ell$$

*where* $\mathcal{E}$ *is the sheaf with stalks* $\mathcal{E}_{\overline{u}} = E_{\overline{u}}$. *Moreover,* $\mathcal{E}\cap\mathcal{E}^\perp$ *and* $j^*R^n\pi_*\mathbb{Q}_\ell/\mathcal{E}$ *are constant sheaves.*

*Proof.* Statement 1 can be checked on stalks: for points in $\overline{U}$ they are the same, so there is nothing to prove. For $s\in S$ we have $(j_*j^*R^n\pi_*\mathbb{Q}_\ell)_{\overline{s}} = H^n\left(\overline{X}_{\overline{\eta}},\mathbb{Q}_\ell\right)^{I_s}$, and by the Picard-Lefschetz formula (more precisely, by Corollary 6.21 (1)) we have $H^n\left(\overline{X}_{\overline{\eta}},\mathbb{Q}_\ell\right)^{I_s} = H^n\left(\overline{X}_s,\mathbb{Q}_\ell\right)$. For 2, note that $\pi_1^t(\overline{U},\overline{\eta})$ acts trivially on the subsheaf $\mathcal{E}\cap\mathcal{E}^\perp$ (because $E_{\overline{u}}^\perp = H^n\left(\overline{X}_{\overline{\eta}},\mathbb{Q}_\ell\right)^{\pi_1^t(\overline{U},\overline{u})}$), whence the constancy of $\mathcal{E}\cap\mathcal{E}^\perp$. Similarly, the monodromy action on $j^*R^n\pi_*\mathbb{Q}_\ell/\mathcal{E}$ is trivial by Picard–Lefschetz. $\square$

**Corollary 6.26.** *In the above situation assume that the vanishing cycles are zero. We have an exact sequence*

$$0\to\bigoplus_{s\in S}\mathbb{Q}_\ell(m-n)_s \to R^n\pi_*\mathbb{Q}_\ell \to j_*j^*R^n\pi_*\mathbb{Q}_\ell\to 0$$

*where* $\mathbb{Q}_\ell(m-n)_s$ *is a skyscraper sheaf concentrated at* $s\in S$. *The sheaf* $j_*j^*R^n\pi_*\mathbb{Q}_\ell$ *is constant.*

*Proof.* Follows immediately from the second part of Corollary 6.21. $\square$

## 6.6 Back to the proof of the main estimate

After many reductions, we are in the following situation: we have a (smooth projective) variety $X/\mathbb{F}_q$ and a Lefschetz pencil

$$\begin{array}{c} X \\ \big\downarrow \\ D_0\cong\mathbb{P}^1 \end{array}$$

which by base-change induces $\overline{X}\to D := D_0\times\overline{\mathbb{F}_q}$. We denote by $d$ and $n$ the dimensions of $X$ and of the fibres of $\pi$, respectively (so $d=n+1$). We consider the cohomology groups

$$H^0\left(D,R^{n+1}\pi_*\mathbb{Q}_\ell\right),\quad H^1\left(D,R^n\pi_*\mathbb{Q}_\ell\right),\quad H^2\left(D,R^{n-1}\pi_*\mathbb{Q}_\ell\right),$$

and we need to prove that the eigenvalues of Frobenius acting on each of these spaces satisfy (3), that is,

$$q^{\frac{d}{2}-\frac{1}{2}} \le |\alpha| \le q^{\frac{d}{2}+\frac{1}{2}}.$$

Let $S$ be the locus of points over which the fibre of the Lefschetz pencil is singular, and assume that the corresponding vanishing cycles $\delta_s$ are nonzero for all $s\in S$.

We know that $R^{n-1}\pi_*\mathbb{Q}_\ell$ and, in the case that the vanishing cycles are nonzero, $R^{n+1}\pi_*\mathbb{Q}_\ell$ are locally constant (see Corollary 6.24), hence constant because $\mathbb{P}^1$ is simply connected. Cohomology of these sheaves can then be handled by the same methods we used in the case when all the fibres of the Lefschetz pencil are smooth; see Section 6.3.

In the case where the vanishing cycles are 0, we can handle the group $H^0\left(D, R^{n+1}\pi_*\mathbb{Q}_\ell\right)$ using Corollary 6.26. It is an extension of the constant sheaf associated with $H^{n+1}(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)$ by a sum of Galois modules $\mathbb{Q}_\ell(m-n)$. By our conventions $n-m = d/2$, so Frobenius has the right eigenvalues on $\mathbb{Q}_\ell(m-n)$. The constant sheaf is treated by a Lefschetz argument as in Section 6.3.

Henceforth *we assume that the vanishing cycles are nonzero.* In this case we have seen (Corollary 6.25) that the adjunction map

$$j_*j^*R^n\pi_*\mathbb{Q}_\ell \xleftarrow{\sim} R^n\pi_*\mathbb{Q}_\ell$$

is an isomorphism, and that the sheaf $j^*R^n\pi_*\mathbb{Q}_\ell$ has a filtration

$$0 \subseteq j_*\left(\mathcal{E}\cap\mathcal{E}^\perp\right) \subseteq j_*\mathcal{E} \subseteq j_*\left(R^n\pi_*\mathbb{Q}_\ell\right),$$

where $\mathcal{E}$ is the locally constant sheaf with fibre $E_{\overline{u}}$ at $\overline{u} \in U$. We have remarked that $\mathcal{E}\cap\mathcal{E}^\perp$ is now known to be 0, but this is proven as a *consequence* of the Weil conjectures (including the Riemann hypothesis), so we cannot use this fact.

## 6.7 Study of the filtration on $R^n\pi_*\mathbb{Q}_\ell$

Pushing forward the filtration on $j^*R^n\pi_*\mathbb{Q}_\ell$ by $j_*$ we get a filtration on $j_*j^*R^n\pi_*\mathbb{Q}_\ell = R^n\pi_*\mathbb{Q}_\ell$:

$$0 \to j_*(\mathcal{E}\cap\mathcal{E}^\perp) \subset j_*\mathcal{E} \subset R^n\pi_*\mathbb{Q}_\ell.$$

Denote by $E$ the fibre $\mathcal{E}_{\overline{u}}$. We now distinguish two cases:

- Case 1: $\delta_s \notin E^\perp$ for all $s \in S$. We have an exact sequence

$$0 \to E \to H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell) \to H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)/E \to 0. \tag{7}$$

  The local inertia group $I_s$ at $s$ acts trivially on $H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)/E$ (again by Picard-Lefschetz), and the inclusion

$$H_n\left(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell\right)^{I_s} \subset H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)$$

  has codimension 1 (this follows from the exact sequence (5)) . The assumption on $\delta_s$ implies that restricting to $E$ we obtain an inclusion $E^{I_s} \subset E$ which is still of codimension 1. In particular, we see that taking invariants in (7) gives another exact sequence

$$0 \to E^{I_s} \to H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)^{I_s} \to H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)/E \to 0.$$

  Exactness of this sequence follows from a purely dimensional argument: we have $\dim H^n(X_{\overline{u}}, \mathbb{Q}_\ell)^{I_s} = \dim H^n(X_{\overline{u}}, \mathbb{Q}_\ell) - 1$ and $\dim E^{I_s} = \dim E - 1$, so the quotient has dimension equal to $H^n(X_{\overline{u}}, \mathbb{Q}_\ell)/E$. At the level of sheaves, this gives an exact sequence

$$0 \to j_*\mathcal{E} \to R^n\pi_*\mathbb{Q}_\ell \to \begin{array}{c}\text{constant}\\ \text{sheaf}\end{array} \to 0.$$

The constancy of the quotient follows from the fact that the monodromy action around the singular points is trivial (for each singular point). The long exact sequence in cohomology then yields

$$H^1\left(D, j_*\mathcal{E}\right) \twoheadrightarrow H^1\left(D, R^n\pi_*\mathbb{Q}_\ell\right).$$

Now consider the other relative quotient in the filtration,

$$0 \to E \cap E^\perp \to E \to E/(E \cap E^\perp) \to 0,$$

equipped with its natural $I_s$-action. The action is trivial on $E \cap E^\perp$ by Picard-Lefschetz, and the inclusions $E^{I_s} \subseteq E$ and $\left(E/E \cap E^\perp\right)^{I_s} \subseteq E/(E \cap E^\perp)$ are of codimension 1. As above, we obtain by dimension counting an exact sequence

$$0 \to \begin{array}{c}\text{constant}\\ \text{sheaf}\end{array} \to j_*\mathcal{E} \to j_*\left(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp\right) \to 0$$

whence an injection

$$0 \to H^1\left(D, j_*\mathcal{E}\right) \hookrightarrow H^1\left(D, j_*\left(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp\right)\right).$$

It follows that it suffices to study the eigenvalues of Frobenius on $H^1(D, j_*\mathcal{E}/(\mathcal{E}\cap\mathcal{E}^\perp))$ (this cohomology group contains $H^1(D, j_*\mathcal{E})$, which in turn surjects onto the relevant cohomology group $H^1\left(D, R^n\pi_*\mathbb{Q}_\ell\right)$).

- Case 2: For all $s \in S$, the vanishing cycle $\delta_s$ is in $E^\perp$. To see that this is indeed the complementary case to case (1) above, simply recall that all the $\delta_s$ are conjugated to each other. In particular, $E \subseteq E^\perp$. Fix $s \in S$ and consider the corresponding inertia group $I_s$. As before, the action of $I_s$ is trivial on $E^\perp$ by Picard-Lefschetz, and we obtain a slightly more complicated exact sequence

$$0 \to E^\perp \to H^n\left(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell\right)^{I_s} \to H^n\left(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell\right)/E^\perp \to H^n\left(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell\right)/\langle\delta_s\rangle^\perp \to 0.$$

The inertia action on $E^\perp$ is trivial, and – as $E \subseteq E^\perp$ – so is the action on $H^n(X_{\overline{u}}, \mathbb{Q}_\ell)/E^\perp$. It follows that the whole sequence consists of $I_s$-invariant groups. Moreover,

$$H^n(X_{\overline{u}}, \mathbb{Q}_\ell)/\langle\delta_s\rangle^\perp \cong \mathbb{Q}_\ell(m-n)$$

by the exact sequence (5). We split the sheafification of the above exact sequence into two pieces,

$$0 \to \begin{array}{c}\text{constant}\\ \text{sheaf}\end{array} \to R^n\pi_*\mathbb{Q}_\ell \to \mathcal{F} \to 0$$

and

$$0 \to \mathcal{F} \to \begin{array}{c}\text{constant}\\ \text{sheaf}\end{array} \to \bigoplus_{s\in S} \mathbb{Q}_\ell(m-n) \to 0.$$

The two exact sequences give

$$\begin{tikzcd} H^1(D, R^n\pi_*\mathbb{Q}_\ell) \arrow[r, hook] & H^1(D, \mathcal{F}) \\ & H^0(D, \mathbb{Q}_\ell(m-n)) \arrow[u, two heads] \end{tikzcd}$$

and, as seen above, Frobenius acts as multiplication by $q^{n-m} = q^{\frac{n+1}{2}}$ on $H^0(D, \mathbb{Q}_\ell(m-n))$. This implies that Frobenius has the desired eigenvalues on $H^1(D, R^n\pi_*\mathbb{Q}_\ell)$, and the proof is complete in this case (which, as noted above, is a posteriori known not to occur).

### 6.8 The sheaf $\mathcal{E}/\left(\mathcal{E} \cap \mathcal{E}^\perp\right)$

From now on, let $\mathcal{F} := \mathcal{E}/\left(\mathcal{E} \cap \mathcal{E}^\perp\right)$. We have to prove that the eigenvalues of Frobenius on $H^1\left(D, j_*\mathcal{F}\right)$ satisfy

$$q^{\frac{d}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}. \tag{8}$$

**Lemma 6.27.** *It suffices to show $|\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}$.*

*Proof.* Poincaré duality. □

**Lemma 6.28.** *It is enough to prove the estimate for the eigenvalues of Frobenius on $H^1_c(U, \mathcal{F})$.*

*Proof.* There is an exact sequence

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to j_*\mathcal{F}/j_!\mathcal{F} \to 0,$$

and the sheaf $j_*\mathcal{F}/j_!\mathcal{F}$ has only finitely many nonzero stalks. This induces

$$H^1_c(U, \mathcal{F}) = H^1(D, j_!\mathcal{F}) \to H^1(D, j_*\mathcal{F}) \to 0,$$

because the $H^1$ of an étale sheaf supported at finitely many points is zero (this can be checked for example by using Čech cohomology, which coincides with étale cohomology at the level of $H^1$). □

We start with

**Lemma 6.29.** *Assume that for every closed point $x \in U$ the local Frobenius $F_{\overline{x}}^{\deg x}$ acts on $\mathcal{F}_{\overline{x}}$ with eigenvalues that are algebraic numbers and satisfy $|\alpha| \leq \left(q^{\deg x}\right)^{n/2}$. Then the eigenvalues of Frobenius on $H^1_c(\overline{U}, \mathcal{F})$ satisfy $|\alpha| \leq q^{\frac{d+1}{2}}$.*

*Proof.* By Poincaré duality we have $H^0_c(\overline{U}, \mathcal{F}) \cong H^2(\overline{U}, \mathcal{F}^\vee(-1))^\vee$, which vanishes because $\overline{U}$ is affine of dimension 1 over an algebraically closed field (Theorem 3.11). Similarly,

$$H^2_c(\overline{U}, \mathcal{F}) \cong H^0(\overline{U}, \mathcal{F}^\vee(-1))^\vee.$$

As $\mathcal{F}$ is locally constant, taking $H^0$ is the same as taking $\pi_1$-invariants, and after dualising this statement we get

$$H^2_c(\overline{U}, \mathcal{F}) \cong H^0(\overline{U}, \mathcal{F}^\vee(-1))^\vee \cong \left(E/E \cap E^\perp\right)_{\pi_1(\overline{U}, \overline{\eta})}(-1).$$

By Kazhdan-Margulis (Theorem 6.23) we have

$$\left(E/E \cap E^\perp\right)_{\pi_1(\overline{U}, \overline{\eta})} = \left(E/E \cap E^\perp\right)_{\mathrm{Sp}\left(E/E \cap E^\perp\right)},$$

and it is a standard fact in invariant theory that

$$\operatorname{Hom}(V,K)^{\operatorname{Sp}(V)} = (0)$$

for any finite-dimensional $K$-vector space $V$. In our setting, this implies that $H^2_c(\overline{U},\mathcal{F})$ is trivial. As a consequence, the zeta function

$$Z(U,\mathcal{F},T) = \prod_{x\in U_{(0)}} \det\left(1 - F_{\overline{x}}^{\deg x} T^{\deg x} \mid \mathcal{F}_{\overline{x}}\right)^{-1} \tag{9}$$

is equal (by the Lefschetz trace formula) to

$$\det\left(1 - FT \mid H^1_c(\overline{U},\mathcal{F})\right).$$

We regard the right hand side of (9) as a power series in $\mathbb{C}[[T]]$[2] and show that its radius of convergence is at least $\frac{1}{q^{\frac{d+1}{2}}} = \frac{1}{q^{\frac{n}{2}+1}}$. Let $\alpha_{x,i}$, for $1 \le i \le r = \operatorname{rk}\mathcal{F}_{\overline{x}}$, be the eigenvalues of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$. Then

$$Z(U,\mathcal{F},T)^{-1} = \prod_{x,i}(1 - \alpha_{x,i}T^{\deg(x)}),$$

and it is enough to show that

$$\sum_{x,i} |\alpha_{x,i}T^{\deg(x)}| < \infty$$

for $|T| < \frac{1}{q^{\frac{n}{2}+1}}$. We have

$$\sum_{x,i} |\alpha_{x,i}T^{\deg x}| \le r\sum_{x}(q^{\deg x})^{\frac{n}{2}}|T|^{\deg x} \le r\sum_{m}(q^m)^{\frac{n}{2}}(q^m+1)|T|^m,$$

where we have used the trivial estimate

$$\#\{x\in U \text{ closed point} : \deg x = m\} \le \#\mathbb{P}^1(\mathbb{F}_{q^m}) = q^m + 1.$$

Finally, the sum $r\sum_m (q^m)^{\frac{n}{2}}(q^m+1)|T^m|$ converges for $|T| < \frac{1}{q^{\frac{n}{2}+1+\varepsilon}}$, as desired. □

We are thus reduced to the following:

**Lemma 6.30** (Main lemma)**.** *Let $\mathcal{F}$ be a locally constant sheaf of finite-dimensional $\mathbb{Q}_\ell$-vector spaces on $U$ such that:*

*(a) for all closed points $x\in U$, the inverse characteristic polynomial of the local Frobenius action of $F_{\overline{x}}^{\deg(x)}$ on $\mathcal{F}_{\overline{x}}$ is in $\mathbb{Q}[T]$.*

*(b) there exists a $\pi_1(U,\overline{\eta})$-invariant,* $\operatorname{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)$*-equivariant, alternating, non-degenerate form*

$$\mathcal{F}_{\overline{x}} \times \mathcal{F}_{\overline{x}} \to \mathbb{Q}_\ell(-n)$$

*for all geometric points $\overline{x}\to U$.*

[2] It is known that $\mathbb{Q}_\ell$ embeds in $\mathbb{C}$. But, for those who wish to avoid this fact, we shall soon prove that the coefficients of this power series are in fact rational, so there is no problem with viewing it in $\mathbb{C}[[T]]$.

*(c) the image of $\pi_1(\overline{U}, \overline{u})$ in $\mathrm{Sp}(\mathcal{F}_{\overline{x}})$ is Zariski dense.*

*Then the eigenvalues of $F_{\overline{x}}^{\deg(x)}$ on $\mathcal{F}_{\overline{x}}$ satisfy $|\alpha| \leq \left(q^{\deg x}\right)^{n/2}$.*

We now show that the main lemma implies the main estimate (8):

*Proof.* We apply the lemma to $\mathcal{F} := \mathcal{E}/\mathcal{E} \cap \mathcal{E}^{\perp}$. We will verify assumption (a) in Theorem 6.32 below; (b) is simply Poincaré duality on fibres, and (c) is Kazhdan-Margulis (Theorem 6.23). We then get the inequality stated in the main lemma, which implies the main estimate (8) by Lemma 6.29. □

## 6.9 Proof of the Main Lemma

**Reduction 1.** Rankin's trick again: it is enough to prove that the eigenvalues of Frobenius $F_{\overline{x}}^{\deg(x)}$ on $\mathcal{F}_{\overline{x}}^{\otimes 2k}$ satisfy $|\beta| \leq (q^{\deg x})^{kn+1}$ for every $k \geq 0$. Indeed, $\alpha^{2k}$ is among the eigenvalues $\beta$ of $F_{\overline{x}}^{\deg(x)}$ acting on $\mathcal{F}_{\overline{x}}^{\otimes 2k}$, and we get the conclusion by taking $2k$-th roots and passing to the limit $k \to \infty$.

**Reduction 2.** It is enough to prove that the eigenvalues of Frobenius on $H_c^2\left(\overline{U}, \mathcal{F}^{\otimes 2k}\right)$ have absolute value at most $q^{kn+1}$. The proof of this statement is very similar to that of Proposition 5.9, but we repeat the argument. Recall the elementary identity (Lemma 1.13)

$$\det\left(1 - F_{\overline{x}}^{\deg(x)} T^{\deg x} \mid \mathcal{F}_{\overline{x}}^{\otimes 2k}\right)^{-1} = \exp\left\{\sum_{m=1}^{\infty} \mathrm{Tr}\left(F_{\overline{x}}^{\deg(x)\, m} \mid \mathcal{F}_{\overline{x}}^{\otimes 2k}\right) \frac{T^{m \deg(x)}}{m}\right\}.$$

By assumption (a) of the Main Lemma, for $k = \frac{1}{2}$ on the left hand side we have a polynomial $\det\left(1 - F_{\overline{x}}^{\deg(x)} T^{\deg x} \mid \mathcal{F}_{\overline{x}}\right)$ with rational coefficients (recall that the set of its coefficients equals that of the inverse characteristic polynomial, plus perhaps 0), hence on the right hand side $\mathrm{Tr}\left(F_{\overline{x}}^{\deg(x)\, m} \mid \mathcal{F}_{\overline{x}}\right)$ is a rational number. But for any $k \geq 0$ we have

$$\mathrm{Tr}\left(F_{\overline{x}}^{\deg(x)\, m} \mid \mathcal{F}_{\overline{x}}^{\otimes 2k}\right) = \mathrm{Tr}\left(F_{\overline{x}}^{\deg(x)\, m} \mid \mathcal{F}_{\overline{x}}\right)^{2k},$$

so if we write

$$Z_{\overline{x}}(\mathcal{F}^{\otimes 2k}, T) := \det\left(1 - F_{\overline{x}}^{\deg(x)} T^{\deg x} \mid \mathcal{F}_{\overline{x}}^{\otimes 2k}\right)^{-1}$$

we see that for all integer $k \geq 0$ this is a power series with *positive* coefficients.

We denote by $\rho$ the radius of convergence of a complex power series. We obtain

$$\rho\left(Z_{\overline{x}}\left(\mathcal{F}^{\otimes 2k}, T\right)\right) \geq \rho\left(\prod_{x \in U} Z_{\overline{x}}(\mathcal{F}^{\otimes 2k}, T)\right) = \rho\left(Z(U, \mathcal{F}^{\otimes 2k}, T)\right), \tag{10}$$

because the coefficients of the series on the right are larger than those on the left. On the other hand,

$$Z(U, \mathcal{F}^{\otimes 2k}, T) = \frac{\det\left(1 - FT \mid H_c^1(\overline{U}, \mathcal{F}^{\otimes 2k}\right)}{\det\left(1 - FT \mid H_c^2(\overline{U}, \mathcal{F}^{\otimes 2k})\right)}, \tag{11}$$

where we may omit the term corresponding to $H_c^0$ because $H_c^0$ is dual to the $H^2$ of an affine curve, which vanishes. The assumption that all eigenvalues of Frobenius acting on

$H^2_c(\overline{U}, \mathcal{F}^{\otimes 2k})$ have absolute value at most $q^{kn+1}$ implies that the right hand side of (11) converges for $|T| < q^{-1-kn}$. Inequality (10) implies that $Z_{\overline{x}}\left(\mathcal{F}^{\otimes 2k}, T\right)$ also converges for $|T| < q^{-1-kn}$. On the other hand, let $\beta$ be an eigenvalue of $F_{\overline{x}}^{\deg(x)}$ acting on $\mathcal{F}_{\overline{x}}^{\otimes 2k}$. Then $Z_{\overline{x}}(\mathcal{F}^{\otimes 2k}, T)$ has a pole in $\beta^{-1/\deg(x)}$. It follows that $\left|\beta^{-1/\deg(x)}\right| \geq q^{-1-kn}$, or equivalently $|\beta| \leq q^{(1+kn)\deg(x)}$ as desired.

**Main proof.** We have to consider

$$H^2_c\left(\overline{U}, \mathcal{F}^{\otimes 2k}\right) \cong H^0\left(\overline{U}, \mathcal{F}^{\otimes 2k}(-1)\right)^{\vee} = \operatorname{Hom}\left(\mathcal{F}_{\overline{x}}^{\otimes 2k}, \mathbb{Q}_\ell\right)^{\pi_1(\overline{U})}(-1).$$

By assumption (c), we have

$$\operatorname{Hom}\left(\mathcal{F}_{\overline{x}}^{\otimes 2k}, \mathbb{Q}_\ell\right)^{\pi_1(\overline{U})} = \operatorname{Hom}\left(\mathcal{F}_{\overline{x}}^{\otimes 2k}, \mathbb{Q}_\ell\right)^{\mathrm{Sp}(\mathcal{F}_{\overline{x}})}.$$

Recall the following general fact (e.g. from [FH91, Appendix F]):

**Theorem 6.31.** *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional symplectic space over a field $K$. For every partition $\mathcal{P}$ of $\{1, 2, \ldots, 2k\}$ in pairs $\{a_i, b_i\}$, an $\mathrm{Sp}(V)$-invariant function on $V^{\otimes 2k}$ is given by*

$$\begin{array}{rccc} f_{\mathcal{P}}: & V^{\otimes 2k} & \to & K \\ & v_1 \otimes \cdots \otimes v_{2k} & \mapsto & \prod_i \langle v_{a_i}, v_{b_i}\rangle. \end{array}$$

*Furthermore, as $\mathcal{P}$ varies, these functions give a basis for the vector space of all $\mathrm{Sp}(V)$-invariant functions $V^{\otimes 2k} \to K$.*

We obtain

$$H^2_c\left(\overline{U}, \mathcal{F}^{\otimes 2k}\right) \cong \operatorname{Hom}\left(\mathcal{F}_{\overline{x}}^{\otimes 2k}, \mathbb{Q}_\ell\right)^{\mathrm{Sp}(\mathcal{F}_x)}(-1) \cong \mathbb{Q}_\ell(-kn)^{\oplus|\mathcal{P}|}(-1).$$

Notice that the weight $-kn$ comes from the fact that the bilinear form $\langle\cdot,\cdot\rangle$ has weight $-n$, and we are taking the product of $k$ such forms. The eigenvalues of Frobenius on $\mathbb{Q}_\ell(-kn-1)$ are $q^{kn+1}$, and we are done.

## 6.10 The rationality theorem

The last result we need to conclude Deligne's proof of Conjecture 1.7 is the following rationality result. It is used to verify assumption (a) in the Main Lemma.

**Theorem 6.32** (Théorème de rationalité, Weil I)**.** *For every closed point $x \in U$ the characteristic polynomial*

$$\det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$$

*is in $\mathbb{Q}[T]$.*

The presentation below is largely influenced by Bloch's notes from a course by Katz. We will prove the theorem under the assumption $q > |S| = |X \setminus U|$ (this is no great restriction, since in the proof of the Riemann hypothesis one may enlarge the field of definition by Lemma 6.4). Let $X_x$ be the fibre of $\pi : X \to \mathbb{P}^1_{\mathbb{F}_q}$ above $x$. We already know that $Z(X_x, T)$ is a rational function of $T$ with rational coefficients.

**Lemma 6.33.** *There exist $\ell$-adic units $\alpha_i, \beta_j \in \overline{\mathbb{Q}_\ell}^\times$, independent of $x$, such that*

$$Z(X_x, T) = \frac{\prod_i (1 - \alpha_i^{\deg x} T)}{\prod_j (1 - \beta_j^{\deg x} T)} \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right).$$

Here by '$\ell$-adic units' we mean units in the valuation ring of some finite extension of $\mathbb{Q}_\ell$.

*Proof.* By the cohomological expression of $Z(X_x, T)$ we have

$$Z(X_x, T) = \prod_{i=0}^{2n} \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \left(R^i \pi_* \mathbb{Q}_\ell\right)_{\overline{x}}\right)^{(-1)^{i+1}},$$

and we have seen[3] that the sheaves $\left(R^i \pi_* \mathbb{Q}_\ell\right)_{\overline{x}}$ are constant for $i \neq n$. In particular, for $i \neq n$ the operator $F_{\overline{x}}^{\deg x}$ acts on $R^i \pi_* \mathbb{Q}_\ell$ as $A_i^{\deg x}$, where $A_i \in \mathrm{GL}_{r_i}(\mathbb{Q}_\ell)$ is a constant matrix. The eigenvalues are elements of $\overline{\mathbb{Q}_\ell}$ of valuation 0, because (by definition of an $\ell$-adic sheaf) the action of any automorphism preserves an integral structure, so all its eigenvalues are integral. It follows that

$$Z(X_x, T) = \frac{\prod_i \left(1 - (\alpha_i')^{\deg x} T\right)}{\prod_j \left(1 - (\beta_j')^{\deg x} T\right)} \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \left(R^n \pi_* \mathbb{Q}_\ell\right)_{\overline{x}}\right).$$

To deal with $R^n \pi_* \mathbb{Q}_\ell$, recall that we have a filtration

$$0 \subseteq j_*(\mathcal{E} \cap \mathcal{E}^\perp) \subseteq j_* \mathcal{E} \subseteq R^n \pi_* \mathbb{Q}_\ell$$

where the first and last relative quotients are constant. Repeating the same argument as above for these constant sheaves, we get some more powers of constant matrices, and all that is left is then the factor $\det(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}})$ as claimed. □

The idea is now that if $1 - F_{\overline{x}}^{\deg x} T$ has a zero which is not an algebraic number, then – using the fact that $Z(X_x, T)$ is a rational function with rational coefficients – this eigenvalue must cancel with some factor $1 - \beta_j^{\deg x} T$, and this *for all $x$*. The next lemma essentially shows that this cannot happen.

**Lemma 6.34.** *Fix $N \geq 1$. There does not exist any $\ell$-adic unit $\lambda \in \overline{\mathbb{Q}}_\ell^\times$ such that $\lambda^{\deg x}$ is an eigenvalue of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$ for all $x \in U$ with* $(\deg x, N) = 1$.

*Proof.* Assume such a $\lambda$ exists. Let $E/\mathbb{Q}_\ell$ be a finite extension such that $\lambda \in \mathcal{O}_E^\times$. Notice that the homomorphism

$$\begin{array}{ccc} \mathbb{Z} & \to & \mathcal{O}_E^\times \\ r & \mapsto & \lambda^{-r} \end{array}$$

extends to a continuous homomorphism $\rho : \widehat{\mathbb{Z}} \to \mathcal{O}_E^\times$. To see this, let $\ell^k$ be the order of the finite residue field of $\mathcal{O}_E$, observe that $\lambda^{\ell^k - 1}$ is in $1 + \mathfrak{m}_E$, and recall that one can take arbitrary $\ell$-adic powers of elements in $1 + \mathfrak{m}_E$. Consider now the exact sequence

$$1 \to \pi(\overline{U}, \overline{u})^t \to \pi_1(U, \overline{u})^t \to \mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \to 0,$$

[3] under the assumption that the vanishing cycles are nonzero. But if they are zero, the sheaf $\mathcal{F}$ is trivial and there is nothing to prove

where $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \cong \widehat{\mathbb{Z}}$. Consider the character

$$\chi : \pi_1(U, \overline{u})^t \to \widehat{\mathbb{Z}} \to \mathcal{O}_E^\times,$$

where the second map is the *inverse* of $\rho$. As before, denote by $F_x$ the Frobenius element in $\pi_1(U, \overline{u})^t$ associated with a closed point $x$; it maps to $F_{\overline{x}}^{\deg x}$ in the quotient $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)$. Observe that $\chi(F_x) = \lambda^{-\deg x}$. It follows from the assumption that

$$\det(1 - F_x \chi(F_x)) = 0.$$

On the other hand, by Chebotarev's density theorem we know that the elements $F_x$, for all $x$ of degree prime to $N$, are dense in $\pi_1(U, \overline{u})^t$. We get that for all $g \in \pi_1(U, \overline{u})^t$ the determinant $\det(1 - g\chi(g))$ vanishes. Now we use the fact that there exists $x_0 \in U$ of degree 1; let $F_0$ be the corresponding Frobenius. For all $h_0 \in \pi_1(\overline{U}, \overline{u})^t$, the operator $F_0 h_0$ has eigenvalue $\lambda$ on $\mathcal{F}_{\overline{x}}$. Since the Poincaré duality pairing on $\mathcal{F}_{\overline{x}}$ is invariant by the action of $\pi_1(\overline{U}, \overline{u})^t$ and has a twist $n = \dim X_x$, we have

$$\langle F_0 h_0(x), F_0 h_0(y) \rangle = \langle F_0(x), F_0(y) \rangle = q^n \langle x, y \rangle,$$

hence

$$\{q^{-n/2} F_0 h_0 : h_0 \in \pi_1^t(\overline{U}, \overline{u})\} \subseteq \mathrm{Sp}(\mathcal{F}_{\overline{x}}),$$

and by Kazhdan-Margulis this implies that $q^{-n/2} F_0 h_0$ are dense in $\mathrm{Sp}(\mathcal{F}_{\overline{x}})$. This implies that $\lambda q^{-n/2}$ is a common eigenvalue for all the elements of $\mathrm{Sp}(\mathcal{F}_{\overline{x}})$, which is impossible because the elements of the symplectic group have no common eigenvalue. $\square$

**Notation 6.35.** *If $f$ is a polynomial of the form $\prod_k (1 - \gamma_k T)$, we write*

$$f^{(m)} := \prod_k (1 - \gamma_k^m T).$$

**Lemma 6.36.** *Assume that $f \in \overline{\mathbb{Q}_\ell}[T]$, with $f(0) = 1$, is such that for every closed point $x$ of $U$ the product*

$$f^{(\deg x)}(T) \cdot Z(X_x, T)$$

*belongs to $\overline{\mathbb{Q}_\ell}[T]$. Then $\prod_j (1 - \beta_j T) \mid f$, where the $\beta_j$ are as in the statement of Lemma 6.33.*

*Proof.* Write $f = \prod_k (1 - \gamma_k T)$. Then

$$\frac{\prod_k (1 - \gamma_k^{\deg x} T)(1 - \alpha_i^{\deg x} T)}{\prod (1 - \beta_j^{\deg x} T)} \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$$

is a polynomial for all closed points $x$. Let $N$ be a common multiple of all $r$ such that $\beta_j^r = \gamma_k^r$ or $\alpha_i^r = \beta_j^r$ for some $\alpha_i, \beta_j, \gamma_k$ and $r$ is minimal with this property. If some $\beta_j$ is equal to some $\alpha_i$ or some $\gamma_k$, then the corresponding monomials simplify in the above expression. If after this simplification no monomial $1 - \beta_j T$ remains we are done. On the other hand, if $\beta_j$ does remain, then $\beta_j^{-\deg x}$ is an eigenvalue of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$ for all $x$ such that $(\deg x, N) = 1$. But this is impossible by Lemma 6.34. $\square$

**Corollary 6.37.** *The following hold:*

1. $\prod_j (1 - \beta_j T)$ *is a polynomial with rational coefficients.*

2. $\prod_i (1 - \alpha_i^{\deg x} T) \det \left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$ *is a polynomial with rational coefficients.*

*Proof.* The second part is a consequence of the first: this is obvious for $\deg x = 1$, using that $Z(X_x, T)$ has rational coefficients. Moreover, if $\prod(1 - \beta_j T)$ has rational coefficients, so does $\prod(1 - \beta_j^r T)$ by Galois theory, which proves it for $\deg x = r$. For part (1), let

$$\mathcal{S} := \{f \in \overline{\mathbb{Q}_\ell}[T] : f(0) = 1, f^{(\deg x)}(T) \cdot Z(X_x, T) \in \overline{\mathbb{Q}_\ell}[T]\}.$$

By Lemma 6.36 the greatest common divisor of all the polynomials in $\mathcal{S}$ is $\prod(1 - \beta_j T)$, and on the other hand $\mathcal{S}$ is stable under the action of $\mathrm{Aut}(\overline{\mathbb{Q}_\ell})$. Hence in particular the coefficients of $\prod(1 - \beta_j T)$ are in $\overline{\mathbb{Q}_\ell}^{\mathrm{Aut}(\overline{\mathbb{Q}_\ell})} = \mathbb{Q}$ as desired. □

**Remark 6.38.** To see why $\overline{\mathbb{Q}_\ell}^{\mathrm{Aut}(\overline{\mathbb{Q}_\ell})} = \mathbb{Q}$, notice that every automorphism of $\overline{\mathbb{Q}_\ell}$ preserves $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}_\ell}$. Given an element $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, we can find an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ with $\sigma(\alpha) \neq \alpha$. Now extend $\sigma$ to an automorphism of $\overline{\mathbb{Q}_\ell}$ using field theory. Similarly, for $\alpha \in \overline{\mathbb{Q}_\ell} \setminus \overline{\mathbb{Q}}$ (necessarily a transcendental element) we may extend the identity map of $\overline{\mathbb{Q}}$ to an automorphism of $\overline{\mathbb{Q}_\ell}$ moving $\alpha$.

The corollary implies in particular that all the roots of $\det \left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$ are algebraic numbers. It remains to show that the set of such roots is stable under the Galois action.

*Proof of Theorem 6.32.* We already know that

$$\Phi_x := \prod_i (1 - \alpha_i^{\deg x} T) \det \left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$$

is a polynomial with rational coefficients, thus it suffices to show that $\prod_i (1 - \alpha_i^{\deg x} T)$ has rational coefficients for all closed points $x$.

It will suffice to show that the set of the $\alpha_i$ (considered with multiplicities) is stable under the Galois action, for then the same is true for the $\alpha_i^{\deg x}$ for all $x$, and we are done. Assume this is not the case. Then for some index $i_0$ there is a Galois conjugate $\sigma(\alpha_{i_0})$ of $\alpha_{i_0}$ that does not appear among the $\alpha_i$. Since $1 - \alpha_{i_0}^{\deg x} T$ divides $\Phi_x \in \mathbb{Q}[T]$, so does $1 - \sigma(\alpha_{i_0})^{\deg x} T$. As in the proof Lemma 6.36 we may find some $N$ such that for $\deg x$ prime to $N$ all conjugates of the $\alpha_i^{\deg x}$ are distinct. For all such $\deg x$ the factor $1 - \sigma(\alpha_{i_0})^{\deg x} T$ is prime to $\prod_i (1 - \alpha_i^{\deg x} T)$, hence $\sigma(\alpha_{i_0})^{-\deg x}$ is an eigenvalue of $F_{\overline{x}}$ on $\mathcal{F}_{\overline{x}}$. As before, this contradicts Lemma 6.34. □

# 7 A brief overview of Weil II

To finish this course we give an outline of some of the main results of Deligne's paper [Del80].

### 7.1 $\overline{\mathbb{Q}_\ell}$-sheaves

Let $E/\mathbb{Q}_\ell$ be a finite extension. The ring $\mathcal{O}_E$ is a finite-dimensional $\mathbb{Z}_\ell$-algebra, and one can generalise the construction of $\mathbb{Z}_\ell$-sheaves to this case to obtain an $\mathcal{O}_E$-sheaf. Tensoring with $E$ (over $\mathcal{O}_E$) we get the notion of an $E$-sheaf, and the category of $\overline{\mathbb{Q}_\ell}$-sheaves is the direct limit of the categories of $E$-sheaves for $E \subset \overline{\mathbb{Q}_\ell}$.

**Remark 7.1.** In particular, a $\mathbb{Q}_\ell$-sheaf is also a $\overline{\mathbb{Q}_\ell}$-sheaf.

### 7.2 Purity

**Definition 7.2.** Let $X$ be a scheme of finite type over $\mathbb{F}_q$ and let $\mathcal{F}$ be a $\overline{\mathbb{Q}_\ell}$-sheaf on $X$. We say that $\mathcal{F}$ is **(punctually) pure** if there exists $w \in \mathbb{Z}$ such that the eigenvalues of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$ satisfy

$$|\tau(\alpha)| = \left(q^{\deg x}\right)^{w/2}$$

for all $\tau : \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$ and for all closed points $x$ of $X$. If we want to specify $w$, we say that $\mathcal{F}$ is **pure of weight** $w$**.** We say that $\mathcal{F}$ is **mixed** if it has a finite filtration such that all successive quotients are pure. We say that it is **mixed of weights** $\leq n$ if (for some filtration) the successive quotients are pure, and weights of such quotients are all $\leq n$.

**Remark 7.3.** There exist more general notions of being $\tau$-pure and $\tau$-mixed for a fixed embedding $\tau$.

**Theorem 7.4** (Main theorem of Weil II)**.** *Let $f : X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. If $\mathcal{F}$ is a constructible $\overline{\mathbb{Q}_\ell}$-sheaf on $X$ that is mixed of weights $\leq n$, then for every integer $i \geq 0$ the sheaf $R^i f_! \mathcal{F}$ is also mixed of weights $\leq n + i$.*

**Corollary 7.5.** *In the case $Y = \operatorname{Spec} \mathbb{F}_q$, we get that $H^i_c(\overline{X}, \mathcal{F})$ is mixed of weights $\leq n+i$.*

**Corollary 7.6.** *Assume moreover that $X$ is smooth. If $\mathcal{F}$ is mixed of weights $\geq n$, then $H^i(\overline{X}, \mathcal{F})$ is mixed of weights $\geq n + i$.*

*Proof.* Poincaré duality. □

**Corollary 7.7.** *If $X$ is smooth and $\mathcal{F}$ is pure of weight $n$, then*

$$\operatorname{Im}\left(H^i_c(\overline{X}, \mathcal{F}) \to H^i(\overline{X}, \mathcal{F})\right)$$

*is pure of weight $n + i$. In particular, if $X$ is proper and smooth, then $H^i(\overline{X}, \mathcal{F})$ is pure of weight $n + i$.*

**Remark 7.8.** This immediately implies Weil I (which is the case $\mathcal{F} = \mathbb{Q}_\ell$, of weight 0). Also notice that here we assume that $X$ is only proper, and not necessarily projective.

**Theorem 7.9** (Semisimplicity Theorem)**.** *If $X$ is smooth and $\mathcal{F}$ is a lisse and pure $\overline{\mathbb{Q}_\ell}$-sheaf, then $\mathcal{F}$ is semisimple, that is, a direct sum of irreducible subsheaves.*

**Corollary 7.10.** *If $f : X \to Y$ is proper and smooth, the sheaves $R^i f_* \mathbb{Q}_\ell$ are semisimple.*

Indeed, they are pure by Weil I.

**Remark 7.11.** Let $f : X \to Y$ be a proper and smooth morphism of schemes over $\mathbb{C}$. It is again true that the sheaves $R^i f_* \mathbb{C}$ are semisimple, and this can be deduced from the finite field case (using an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$).

*Proof of the Semisimplicity Theorem.* Let $\overline{\mathcal{F}'} \subseteq \overline{\mathcal{F}}$ on $\overline{X}$ be the largest semisimple lisse subsheaf (in other words, the – automatically direct – sum of all the irreducible subsheaves). By construction, $\overline{\mathcal{F}'}$ is stable by Frobenius, and therefore it comes from a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ over $X$. Let $\mathcal{F}'' := \mathcal{F}/\mathcal{F}'$. We have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{12}$$

that gives rise to a class in $\operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$. We can map this class to $\operatorname{Ext}^1_{\overline{X}}\left(\overline{\mathcal{F}''}, \overline{\mathcal{F}'}\right)^F$, where the superscript $F$ denotes the subset of elements fixed by Frobenius. Hence we get

$$\operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}') \to \operatorname{Ext}^1_{\overline{X}}(\mathcal{F}'', \mathcal{F}')^F = H^1\left(\overline{X}, \underline{\operatorname{Hom}}_{\overline{X}}(\mathcal{F}'', \mathcal{F}')\right)^F.$$

To justify the last equality, note that there is an exact sequence

$$0 \to H^1\left(\overline{X}, \underline{\operatorname{Hom}}_{\overline{X}}(\mathcal{F}'', \mathcal{F}')\right) \to \operatorname{Ext}^1_{\overline{X}}(\mathcal{F}'', \mathcal{F}') \to H^0\left(\overline{X}, \underline{\operatorname{Ext}}^1_{\overline{X}}(\mathcal{F}'', \mathcal{F}')\right)$$

coming from the spectral sequence of Ext's, and $\underline{\operatorname{Ext}}^1_{\overline{X}}(\mathcal{F}'', \mathcal{F}') = 0$ because $\mathcal{F}''$, $\mathcal{F}'$ are both locally constant $\overline{\mathbb{Q}_\ell}$-sheaves.

Since $\mathcal{F}', \mathcal{F}''$ are both pure of weight $w$, the sheaf $\underline{\operatorname{Hom}}(\mathcal{F}'', \mathcal{F}')$ is pure of weight 0, hence $H^1(\overline{X}, \underline{\operatorname{Hom}}(\mathcal{F}'', \mathcal{F}'))$ is mixed of weights $\geq 1$. In particular it has no weight-0 part, hence it contains no nontrivial Frobenius-invariant element. This proves that the pullback of sequence (12) to the algebraic closure splits (because the class defining the extension vanishes over the algebraic closure). But this is a contradiction, because one can then enlarge $\overline{\mathcal{F}'}$ (simply add to it a simple subsheaf of $\mathcal{F}''$), contradicting its maximality. □

**Remark 7.12.** Using Theorem 7.4 it is possible to define, on every lisse mixed sheaf $\mathcal{F}$, an increasing **weight filtration** by lisse subsheaves $W_i\mathcal{F}$ such that each graded quotient $\operatorname{gr}^W_i \mathcal{F}$ is pure of weight $i$. This filtration is functorial in $\mathcal{F}$. Moreover, morphisms $\mathcal{F} \to \mathcal{G}$ are even *strictly* compatible with the filtration $W$: the pullback of the weight filtration on $\mathcal{G}$ gives the weight filtration on $\mathcal{F}$.

## 7.3 Some reductions in the proof of Theorem 7.4

The key case of Theorem 7.4 is the following:

**Theorem 7.13.** *Let $X$ be a smooth projective curve over $\mathbb{F}_q$, $j : U \hookrightarrow X$ a dense open subscheme, and $\mathcal{F}$ is lisse and pure of weight $w$ on $U$. Then $H^i\left(\overline{X}, j_*\mathcal{F}\right)$ is pure of weight $w + i$ for $i = 0, 1, 2$.*

Deligne proved this using a considerable amplification of the methods of Weil I (use of L-series and monodromy computations). As mentioned in the introduction, there is a second proof by Laumon, using Fourier transforms for $\ell$-adic sheaves. We'll not discuss any of these proofs but give a very rough sketch of how to deduce Theorem 7.4 from this key case.

### 7.4 Dévissages

The following reductions are more or less easy to check.

1. if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and the conclusion of the theorem holds for $\mathcal{F}', \mathcal{F}''$, then it also holds for $\mathcal{F}$. One can therefore assume that $\mathcal{F}$ is pure.

2. let $U \xrightarrow{i} X \xleftarrow{j} Z = X \setminus U$ with $U$ open. Then if the theorem holds for $i^*\mathcal{F}, j^*\mathcal{F}$ it also holds for $\mathcal{F}$.

3. it is slightly harder to check that the same holds on $Y$: if $V$ is an open subset of $Y$, $T$ is its closed complement, and the statement holds upon restriction to $V$ and $T$, then it holds on all of $Y$.

4. if $f = g \circ h$, then the theorem for $g$ and $h$ implies the theorem for $f$ (by the spectral sequence for composite functors)

5. one may assume that $Y$ is reduced, and also replace it with a purely inseparable cover if necessary.

6. the case when $\mathcal{F}$ is pure and $f$ is of relative dimension 0 is easy (one reduces to finite morphisms).

### 7.5 Structure of the main proof

The proof is by Noetherian induction on the relative dimension. After the above dévissages and after fibering $X$ successively by curves, we arrive at the following situation:

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & D \arrow[l, hook', "i"'] \arrow[dl] \\ & Y & \end{tikzcd}$$

Here $\overline{X}$ is a smooth projective relative curve over $Y$, and $D$ is an étale divisor. We have

$$R^i f_! \mathcal{F} = R^i \overline{f}_* j_! \mathcal{F},$$

and the key case says that $R^i\overline{f}_* j_* \mathcal{F}$ is pure of weight $w+i$ (if $\mathcal{F}$ is pure of weight $i$). There is an exact sequence

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to i_! i^* j_* \mathcal{F} \to 0,$$

which (by passing to the corresponding long exact sequence) shows that it is enough to prove that $i^* j_* \mathcal{F}$ is mixed of weights at most $w$ (in fact, it would be enough to prove that the weights are at most $w+i$, but they turn out to be at most $w$).

### 7.6 Local monodromy

Let $X$ be a smooth curve over $\mathbb{F}_q$, $U \subset X$ an open dense subscheme, $s \in X \setminus U$ a closed point, and $\overline{\eta}$ a geometric generic point of $U$. Let $\mathcal{F}$ be a lisse sheaf over $X$. The stalk $\mathcal{F}_{\overline{\eta}}$ has an action of the local Galois group $G_s$. The structure theory for the Galois group of local fields gives the exact sequences

$$1 \to I \to G_s \to \widehat{\mathbb{Z}} \to 0,$$

where $I$ is the inertia subgroup, and

$$1 \to P \to I \to \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to 0,$$

where $P$ is the (pro)-$p$-Sylow of $I$ and the map $I \to \mathbb{Z}_\ell(1)$ is the $\ell$-adic cyclotomic character.

**Theorem 7.14** (Grothendieck's local monodromy theorem)**.** *There exists an open subgroup $I' \subseteq I$ of finite index such that the action of $I'$ on $V = \mathcal{F}_{\overline{\eta}}$ is unipotent. If $\rho$ is the representation of $G_s$ on $V$, we may write*

$$\rho(\sigma) = \exp(Nt_\ell(\sigma)) \quad \forall \sigma \in I'$$

*for a suitable nilpotent operator $N$.*

**Remark 7.15.** The action of $\sigma \in I'$ on $V$ factors through $t_\ell$. To see this, one fixes a Galois-stable lattice $\Lambda$ in $V$ and works in $\mathrm{GL}(\Lambda)$. By passing to a finite quotient of $\mathrm{GL}(\Lambda)$, we obtain that the action of $\sigma$ is through a pro-$\ell$ group, hence it factors via $I \to \mathbb{Z}_\ell(1)$, which is the cyclotomic character. We therefore have a map $\mathbb{Z}_\ell(1) \to \mathrm{End}(V)$, and since the image consists of unipotent operators we can take the logarithm by using the defining power series.

Notice furthermore that we can consider $N$ as a map $V(1) \to V$. The following lemma is a not very difficult piece of linear algebra.

**Lemma 7.16.** *There exists a unique increasing filtration $M_\bullet$ of $V$ such that $N$ sends $M_i V(1)$ into $M_{i-2}V$ and for all $k$ the operator $N^k$ induces an isomorphism $\mathrm{gr}_k^M V(k) \xrightarrow{\sim} \mathrm{gr}_{-k}^M(V)$.*

**Remark 7.17.** The above filtration $M_\bullet$ is called the **monodromy filtration** on $V$. Its construction shows that $M_0 = \ker N$.

The following difficult theorem of Deligne usually goes under the name of 'purity of the monodromy filtration'.

**Theorem 7.18** (Deligne)**.** *In the above situation, if $\mathcal{F}$ is pure of weight $w$, then $\mathrm{gr}_k^M(V)$ is pure of weight $w + k$.*

## 7.7 Conclusion of the proof

Theorem 7.18 is applied as follows. Recall our setting: we have

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & D \arrow[l, hook', "i"'] \arrow[dl] \\ & Y & \end{tikzcd}$$

Let $V = \mathcal{F}_{\overline{\eta}}$, which gives rise to

$$\mathcal{F}_{\overline{s}} = V^{I_s} \subseteq V^{\ker N} = M_0(V).$$

This shows that only the negative parts of the monodromy filtration contribute, so – as $\mathcal{F}$ is pure of weight $w$ – the only nontrivial contributions come from $\mathrm{gr}_k^M(V)$ for $k \leq 0$, which have weights $w + k \leq w$ by Theorem 7.18.

### 7.8 Aside: the weight-monodromy conjecture

Let $K$ be any local field and $X/K$ be smooth and proper. Let $V = H^i(\overline{X}, \mathbb{Q}_\ell)$ for some $\ell$ prime to the residue characteristic. Applying Theorem 7.14 we get a monodromy filtration on $V$.

**Conjecture 7.19** (Weight-monodromy conjecture)**.** $\mathrm{gr}_k^M(V)$ *is pure of weight* $i+k$.

**Remark 7.20.** If $X$ has good reduction $X_0$, the monodromy filtration is trivial. On the other hand, by smooth proper base change one gets that $H^i(\overline{X}, \mathbb{Q}_\ell)$ is the same as $H^i(\overline{X_0}, \mathbb{Q}_\ell)$, which is pure by Weil I.

**Remark 7.21.** Scholze [Sch12] has proven the weight-monodromy conjecture for char $K = 0$ and for $X$ a smooth complete intersection in $\mathbb{P}^n$ (or, more generally, in a smooth projective toric variety).

### 7.9 A conjecture from Weil II

**Conjecture 7.22** (Deligne, Weil II)**.** *Let $X$ be a normal connected scheme over $\mathbb{F}_q$ and let $\mathcal{F}$ be a lisse irreducible $\overline{\mathbb{Q}_\ell}$-sheaf such that its determinant (seen as a representation of the fundamental group) is a character of finite order of $\pi_1(X, \overline{x})$. Then the following hold:*

1. *$\mathcal{F}$ is pure;*
2. *the characteristic polynomials of Frobenius on $\mathcal{F}_{\overline{x}}$, as $x$ varies over the closed points $x \in X$, all have coefficients in a fixed number field $E \subset \overline{\mathbb{Q}_\ell}$;*
3. *("companions") for all $\ell' \notin \{\ell, p\}$ there exists a $\overline{\mathbb{Q}_{\ell'}}$-sheaf $\mathcal{G}$ with the same Frobenius eigenvalues;*
4. *moreover, 'on espère des petits camarades cristallins', i.e. a similar statement should hold for $\ell = p$ (étale cohomology needs to be replace by crystalline cohomology in this setting).*

**Remark 7.23.** Deligne shows that, after twisting by a sheaf of the form $\overline{\mathbb{Q}_\ell}(i)$, we may always achieve the condition that $\det \mathcal{F}$ is a character of finite order.

**Remark 7.24.** Parts (1)-(2) and the dimension 1 case of (3) in Conjecture 7.22 were proven by L. Lafforgue [Laf02] as a consequence of his proof of the Langlands correspondence for $\mathrm{GL}_n$ over function fields. Drinfeld [Dri12] then showed how to reduce statement (3) for smooth schemes to the curve case. Conjecture (4) was proven by [Abe18] in the curve case using a crystalline analogue of Lafforgue's work. Work is currently in progress to reduce the general case to the dimension 1 case.

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