Dynamics of regular Polynomial automorphisms of $\mathbb{C}^k$.

Nessim Sibony

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Abstract. Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a polynomial automorphism. We extend it to a birational self-map of the projective space $\mathbb{P}^k$ that we still denote by f. We assume that f is not an automorphism of $\mathbb{P}^k$, otherwise the associated dynamical system is elementary.

We say that f is *regular or of Hénon-type* if the indeterminacy sets I_+ and I_- of f and of its inverse f^{-1} satisfy $I_+ \cap I_- = 0$. We refer to [5] for basic properties of this class of maps.

In dimension 2, Hénon type maps satisfy this property and any dynamically interesting polynomial automorphism is conjugated to a Hénon type map. In dimension k, there might be non-trivial dynamics for f on I_- . It could be an endomorphism of some projective space.

Consider an automorphism f as above. Then, there is an integer $1 \leq p \leq k-1$ such that $\dim I_+ = k - p - 1$ and $\dim I_- = p - 1$. Let d_+ (resp. d_-) denote the algebraic degrees of f^+ (resp. of f^-), i.e. the maximal degrees of its components which are polynomials in $\mathbb{C}^k$. It follows that $d_+^p = d_-^{k-p}$, we denote this integer by d. In [5], the author constructs for such a map an invariant measure μ with compact support in $\mathbb{C}^k$ which turns out to be the unique measure of maximal entropy $\log d$, see de Thélin [2]. The measure μ is called *the Green measure or the equilibrium measure of* f. It is obtained as the intersection of the main Green current T_+ of f and the one associated to f^{-1}.

In this lecture, I will discuss the following results which are joint work with T.C Dinh.

The first one is about uniqueness of the Green currents. It is shown, in [3], that T_+ (resp. T_-) is the unique positive closed (p, p)-current (resp. $(k - p, k - p)$-current) of mass 1 supported by the set $\mathscr{K}_+$ (resp. $\mathscr{K}_-$) of points of bounded orbit (resp. backward bounded orbit) in $\mathbb{C}^k$. They are also the unique currents having no mass at infinity which are invariant under $d^{-1}f^*$ (resp. $d^{-1}f_*$). This result uses heavily the theory of super-potentials. The results for, Hénon maps in dimension 2 are due to J.E Fornaess and N.Sibony. They imply many results on equidistribution of varieties.

The second result is very recent, it deals with equidistribution of periodic points.

In dimension 2 the equidistribution of periodic points is due to E.Bedford , M. Lyubich and J. Smillie [1]

Let P_n denote the set of periodic points of period n of f in $\mathbb{C}^k$ and SP_n the set of saddle periodic points of period n in $\mathbb{C}^k$. We have the following result.

Theorem 0.1. *Let f, d, μ, P_n and SP_n be as above. Then the saddle periodic points of f are asymptotically equidistributed with respect to μ. More precisely, if Q_n denotes P_n or SP_n we have*

$$d^{-n} \sum_{a \in Q_n} \delta_a \to \mu \qquad as \quad n \to \infty,$$

where δ_a denotes the Dirac mass at a.

We can replace Q_n with other subsets of SP_n. This gives the nature of typical periodic points. For example, given a small number $\epsilon > 0$, we can take only periodic points a of period n such that the differential Df^n at a admits p eigenvalues of modulus larger than $(\delta - \epsilon)^{n/2}$ and $k - p$ eigenvalues of modulus smaller than $(\delta - \epsilon)^{-n/2}$ with $\delta := \min(d_+, d_-)$.

The approach uses an extension of the notion of Lelong number for positive closed currents, it is the notion of density of a positive closed current along a subvariety. It permits to measure the non-generic intersections of analytic varieties or positive closed currents.

So the main tool is an appropriate intersection theory. Let Δ denote the diagonal of $\mathbb{P}^k \times \mathbb{P}^k$ and Γ_n denote the compactification of the graph of f^n in $\mathbb{P}^k \times \mathbb{P}^k$. The set P_n can be identified with the intersection of Γ_n and Δ in $\mathbb{C}^k \times \mathbb{C}^k$. The dynamical system associated to the map $F := (f, f^{-1})$ on $\mathbb{P}^k \times \mathbb{P}^k$ is similar to the one associated to Hénon-type maps on $\mathbb{P}^k$. So a property similar to the uniqueness of the main Green currents mentioned above implies that the positive closed (k,k)-current $d^{-n}[\Gamma_n]$ converges to the main Green current of F which is equal to $T_+ \otimes T_-$. Therefore, since $\mu = T_+ \wedge T_-$ can be identified with $[\Delta] \wedge (T_+ \otimes T_-)$, Theorem 0.1 is equivalent to

$$\lim_{n\to\infty} [\Delta] \wedge d^{-n}[\Gamma_n] = [\Delta] \wedge \lim_{n\to\infty} d^{-n}[\Gamma_n]$$

on $\mathbb{C}^k \times \mathbb{C}^k$. Obviously, this requires a good intersection theory.

References

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