Large deviation principles for sequences of logarithmically weighted means *

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Abstract

In this paper we consider several examples of sequences of partial sums of triangular arrays of random variables $\{X_n : n \geq 1\}$; in each case X_n converges weakly to an infinitely divisible distribution (a Poisson distribution or a centered Normal distribution). For each sequence we prove large deviation results for the logarithmically weighted means $\{\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k : n \geq 1\}$ with speed function $v_n = \log n$. We also prove a sample path large deviation principle for $\{X_n : n \geq 1\}$ defined by $X_n(\cdot) = \frac{\sum_{i=1}^n U_i(\sigma^2 \cdot)}{\sqrt{n}}$, where $\sigma^2 \in (0,\infty)$ and $\{U_n : n \geq 1\}$ is a sequence of independent standard Brownian motions.

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1 Introduction

Let $\{X_n : n \geq 1\}$ be defined by $X_n := \frac{U_1+\cdots+U_n}{\sqrt{n}}$, where $\{U_n : n \geq 1\}$ is a sequence of i.i.d. centered random variables with unit variance. The *almost sure central limit theorem* states the almost sure weak convergence to the standard Normal distribution of the sequences of random measures

$$\left\{\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}1_{\{X_k\in\cdot\}} : n \geq 1\right\} \tag{1}$$

and, of course, of

$$\left\{\frac{1}{L(n)}\sum_{k=1}^n \frac{1}{k}1_{\{X_k\in\cdot\}} : n \geq 1\right\},\ \text{where } L(n) := \sum_{k=1}^n \frac{1}{k}. \tag{2}$$

The almost sure central limit theorem was proved independently in [4], [11] and [21] under stronger moment assumptions; successive refinements appear in [12] and [17], in which only finite variance is required.

There is a wide literature in the field of large deviations, namely the asymptotic computation of small probabilities on an exponential scale (see e.g. [7] as a reference on this topic). Some results in this field concern the almost sure central limit theorem: see e.g. Theorem 1 in [19] (the expression of the rate function is provided by Theorem 3 in the same reference) for the sequence in (1) and Theorem 1.1 in [13] for the sequence in (2). Both the results are proved assuming that all the

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(common) moments of the random variables $\{U_n : n \geq 1\}$ are finite; the optimality of the moment assumptions is discussed in [18].

A generalization of the almost sure central limit theorem is proved in [2] for sequences $\{X_n : n \geq 1\}$ where $X_n := g_n(U_1, \ldots, U_n)$ for some independent random variables $\{U_n : n \geq 1\}$ and some measurable functions $\{g_n : n \geq 1\}$ which satisfy mild technical conditions. The result in [2] allows to recover some almost sure limit results in the literature, as for instance those concerning the Fisher Tippett Theorem (see e.g. Theorem 3.2.3 in [8]) in [10] and [5]; in this direction a more general result is proved in [9].

In this paper we consider several examples for the sequence $\{X_n : n \geq 1\}$ and, for each one of them, X_n converges weakly to an infinitely divisible distribution (more precisely a Poisson distribution or a centered Normal distribution). In some sense these examples are inspired by a well known generalization of the central limit theorem concerning partial sums of triangular arrays of uniformly asymptotically negligible random variables which converge weakly to an infinitely divisible distribution with finite variance (see e.g. Theorem 28.2 in [3], where one can easily overcome the null expected value hypothesis). For any choice of $\{X_n : n \geq 1\}$ we prove large deviation principles for $\{\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} X_k : n \geq 1\}$, i.e. the sequence of integrals with respect to the random measures in (1), with speed function $v_n = \log n$ (obviously we have the same results if we consider the expected values with respect to the random probability measures in (2); the details will be omitted).

All the large deviation results presented in this paper, except Theorem 3.4, concern the almost sure convergence of logarithmically weighted means guaranteed by a suitable condition on the covariances $\mathrm{Cov}(X_h, X_k)$, for all $1 \leq j < k$ (see condition **(C1)** below). In Theorem 3.4 we have again $\{\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} X_k : n \geq 1\}$ and the sequence $\{X_n : n \geq 1\}$ is defined by $X_n(\cdot) = \frac{\sum_{i=1}^n U_i(\sigma^2 \cdot)}{\sqrt{n}}$, where $\sigma^2 \in (0, \infty)$ and $\{U_n : n \geq 1\}$ is a sequence of independent standard Brownian motions.

One of our examples concerns the framework of the almost sure central limit theorem as in [19] (and in [13]) but, as we shall see (see subsection 4.5 below for more details), the large deviation principle for the sequence of integrals with respect to the random measures cannot be derived from the one for the sequence of the random measures using standard large deviation tools. Similarly, in subsection 4.3, we show that the results in this paper cannot be derived by the main result in [15], which concerns weighted sums of independent identically distributed random variables.

The outline of the paper is as follows. We start with some preliminaries in section 2. In section 3 we give the statements of the large deviation results together with brief sketches of their proofs. Minor results and remarks are presented in section 4. We conclude with sections 5, 6 and 7 in which we present some details of the proofs of Theorems 3.1-3.2, 3.3 and 3.4, respectively.

We conclude with some notation and symbols used throughout the paper. We write $x_n \sim y_n$ (as $n \to \infty$) to mean $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$ and we adopt the convention $\sum_{i=a}^b x_i = 0$ if $a > b$. Furthermore we use the symbol $\mathcal{P}(\lambda)$ for the Poisson distribution with mean $\lambda \geq 0$ (one can also allow $\lambda = 0$ referring to the distribution of the constant random variable equal to zero) and the symbol $\mathcal{N}(\mu, \sigma^2)$ be the Normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

2 Preliminaries

2.1 Large deviations

We refer to [7] (pages 4-5). Let $\mathcal{X}$ be a topological space equipped with its completed Borel σ-field. A sequence of $\mathcal{X}$-valued random variables $\{Z_n : n \geq 1\}$ satisfies the large deviation principle (LDP for short) with speed function v_n and rate function I if: $\lim_{n\to\infty} v_n = \infty$; the function $I : \mathbb{R} \to [0, \infty]$ is lower semi-continuous;

$$\limsup_{n\to\infty} \frac{1}{v_n} \log P(Z_n \in F) \leq - \inf_{x\in F} I(x) \text{ for all closed sets } F;$$

$$\liminf_{n\to\infty} \frac{1}{v_n} \log P(Z_n \in G) \geq -\inf_{x\in G} I(x) \text{ for all open sets } G.$$

A rate function I is said to be *good* if its level sets $\{\{x \in \mathcal{X} : I(x) \leq \eta\} : \eta \geq 0\}$ are compact.

In what follows we prove LDPs with $\mathcal{X} = \mathbb{R}$ and we use the Gärtner Ellis Theorem (see e.g. Theorem 2.3.6 in [7]); the application of this theorem consists in checking the existence of the function $\Lambda : \mathbb{R} \to (-\infty, \infty]$ defined by

$$\Lambda(\theta) := \lim_{n\to\infty} \frac{1}{v_n} \log \mathbb{E}[e^{v_n \theta Z_n}] \tag{3}$$

and, if Λ is essentially smooth (see e.g. Definition 2.3.5 in [7]) and lower semi-continuous, the LDP holds with good rate function $\Lambda^* : \mathbb{R} \to [0, \infty]$ defined by

$$\Lambda^*(x) := \sup_{\theta\in\mathbb{R}} \{\theta x - \Lambda(\theta)\}. \tag{4}$$

We also prove a sample path LDP, i.e. a LDP with $\mathcal{X} = C[0, T]$, i.e. the family of all continuous functions on $[0, T]$ (for some $T \in (0, \infty)$) equipped with the topology of uniform convergence. In such a case we consider an abstract version of the Gärtner Ellis Theorem, i.e. Theorem 4.5.20 in [7] (Baldi's Theorem); note that in such a case the dual space $\mathcal{X}^*$, i.e. the space of all continuous linear functionals on $\mathcal{X}$, is the family of all signed Borel measures with bounded variation on $[0, T]$.

2.2 Strong laws of large numbers for logarithmically weighted means

We start with some strong laws of large numbers for logarithmically weighted means which can be proved using the following standard argument (see e.g. Lemma 1 in [9]; see also Theorem 1 in [20] cited therein): if $\{X_n : n \geq 1\}$ is a sequence of random variables with finite variances such that **(C1)**: *there exists* $C, \rho \in (0, \infty)$ *such that* $|\mathrm{Cov}(X_h, X_k)| \leq C \left(\frac{h}{k}\right)^\rho$ *for all* $1 \leq j < k$,
then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}(X_k - \mathbb{E}[X_k]) = 0 \text{ almost surely.} \tag{5}$$

Proposition 2.1 (Binomial laws converging to the Poisson law) *Let* $\{U_n : n \geq 1\}$ *be a sequence of independent random variables uniformly distributed on* $[0, 1]$ *and let* $\{p_n : n \geq 1\}$ *be a sequence of numbers in* $[0, 1]$ *such that* $\lim_{n\to\infty} np_n = \lambda$ *for some* $\lambda \in (0, \infty)$*. Let* $\{X_n : n \geq 1\}$ *be defined by* $X_n = \sum_{i=1}^n 1_{\{U_i \leq p_n\}}$*. Then* $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} X_k = \lambda$ *almost surely.*

Proof. Firstly $\{np_n : n \geq 1\}$ is bounded and let $B \in (0, \infty)$ a constant such that $np_n \leq B$ for all $n \geq 1$. For $1 \leq h < k$ we have $\mathrm{Cov}(X_h, X_k) = h\mathrm{Cov}(1_{\{U_1 \leq p_h\}}, 1_{\{U_1 \leq p_k\}}) = h\{p_h \wedge p_k - p_h p_k\}$ and therefore $\mathrm{Cov}(X_h, X_k) \in [0, hp_k]$. Then $|\mathrm{Cov}(X_h, X_k)| \leq kp_k \left(\frac{h}{k}\right) \leq B\frac{h}{k}$ and condition **(C1)** holds with $C = B$ and $\rho = 1$. Thus (5) holds. We complete the proof noting that $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k}\mathbb{E}[X_k] = \lambda$ as a consequence of $\mathbb{E}[X_k] = kp_k$ (for all $k \geq 1$) and of $\lim_{n\to\infty} np_n = \lambda$. □

Proposition 2.2 (Poisson laws converging to the Poisson law) *Let* $\{U_n : n \geq 1\}$ *be a sequence of independent Poisson processes with intensity 1 and let* $\{t_n : n \geq 1\}$ *be a sequence of nonnegative numbers such that* $\lim_{n\to\infty} nt_n = \lambda$ *for some* $\lambda \in (0, \infty)$*. Let* $\{X_n : n \geq 1\}$ *be defined by* $X_n = \sum_{i=1}^n U_i(t_n)$*. Then* $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} X_k = \lambda$ *almost surely.*

Proof. Firstly $\{nt_n : n \geq 1\}$ is bounded and let $B \in (0, \infty)$ a constant such that $nt_n \leq B$ for all $n \geq 1$. For $1 \leq h < k$ we have $\mathrm{Cov}(X_h, X_k) = h\mathrm{Cov}(U_1(t_h), U_1(t_k)) = h\mathrm{Var}[U_1(t_h \wedge t_k)]$ and therefore $\mathrm{Cov}(X_h, X_k) \in [0, ht_k]$. Then $|\mathrm{Cov}(X_h, X_k)| \leq kt_k \left(\frac{h}{k}\right) \leq B\frac{h}{k}$ and condition **(C1)** holds with $C = B$ and $\rho = 1$. Thus (5) holds. We complete the proof noting that $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k}\mathbb{E}[X_k] = \lambda$ as a consequence of $\mathbb{E}[X_k] = kt_k$ (for all $k \geq 1$) and of $\lim_{n\to\infty} nt_n = \lambda$. □

Proposition 2.3 (Central Limit Theorem) *Let $\{U_n : n \geq 1\}$ be a sequence of i.i.d. centered random variables with finite variance σ^2. Let $\{X_n : n \geq 1\}$ be defined by $X_n = \frac{\sum_{i=1}^n U_i}{\sqrt{n}}$. Then $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} X_k = 0$ almost surely.*

Proof. For $1 \leq h < k$ one can easily check that $\mathrm{Cov}(X_h, X_k) = \sigma^2 \left(\frac{h}{k}\right)^{\frac{1}{2}}$. Then condition **(C1)** holds with $C = \sigma^2$ and $\rho = \frac{1}{2}$, and (5) holds. This completes the proof since $\mathbb{E}[X_k] = 0$ for all $k \geq 1$. □

2.3 Some classical relations

We recall some classical relations which will be used in the proofs below. Firstly

$$\log(j+1) \leq \sum_{k=1}^{j} \frac{1}{k} \leq 1 + \log j \text{ and } \log \frac{j+1}{i} \leq \sum_{k=i}^{j} \frac{1}{k} \leq \log \frac{j}{i-1} \quad (j \geq i \geq 2). \tag{6}$$

Moreover, if $\alpha \in (0,1)$,

$$\frac{1}{1-\alpha}\left(j^{1-\alpha} - i^{1-\alpha}\right) \leq \sum_{k=i}^{j} \frac{1}{k^\alpha} \leq \frac{1}{1-\alpha}\left((j+1)^{1-\alpha} - (i-1)^{1-\alpha}\right) \quad (j \geq i \geq 2); \tag{7}$$

note that (7) holds if $\alpha < 0$ and $j \geq i \geq 1$. Finally, if $\alpha > 1$,

$$\frac{1}{\alpha-1}\left(i^{1-\alpha} - (j+1)^{1-\alpha}\right) \leq \sum_{k=i}^{j} \frac{1}{k^\alpha} \leq \frac{1}{\alpha-1}\left((i-1)^{1-\alpha} - j^{1-\alpha}\right) \quad (j \geq i \geq 2). \tag{8}$$

3 Large deviation results and sketches of proofs

In this section we give the statements of the LDPs proved in the present paper, together with a brief sketch of their proofs. Note that the same rate function and the same sketch of the proof pertain to Theorems 3.1 and 3.2. Moreover, in some sense, Theorem 3.4 is a sample path version of Theorem 3.3.

Theorem 3.1 (LDP for the strong law of large numbers in Proposition 2.1) *Consider the same situation as in Proposition 2.1. Assume moreover that $p_n \geq p_{n+1}$ for all $n \geq 1$. Then $\{\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} X_k : n \geq 1\}$ satisfies the LDP with speed function $v_n = \log n$ and good rate function $I_{\mathcal{P}(\lambda)}$ defined by*

$$I_{\mathcal{P}(\lambda)}(x) = \begin{cases} (\sqrt{x} - \sqrt{\lambda})^2 & if\ x \geq 0 \\ \infty & if\ x < 0. \end{cases}$$

Theorem 3.2 (LDP for the strong law of large numbers in Proposition 2.2) *Consider the same situation as in Proposition 2.2. Assume moreover that $t_n \geq t_{n+1}$ for all $n \geq 1$. Then $\{\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} X_k : n \geq 1\}$ satisfies the LDP with speed function $v_n = \log n$ and good rate function $I_{\mathcal{P}(\lambda)}$ defined by*

$$I_{\mathcal{P}(\lambda)}(x) = \begin{cases} (\sqrt{x} - \sqrt{\lambda})^2 & if\ x \geq 0 \\ \infty & if\ x < 0. \end{cases}$$

Sketch of the proof of Theorems 3.1-3.2. In section 5 we prove (3) with $Z_n = \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} X_k$, $v_n = \log n$ and

$$\Lambda(\theta) = \begin{cases} \frac{\lambda\theta}{1-\theta} & \text{if } \theta < 1 \\ \infty & \text{if } \theta \geq 1. \end{cases}$$

Then the Gärtner Ellis Theorem can be applied and the LDP holds with the good rate function $I_{\mathcal{P}(\lambda)}$ given by Λ^* in (4) (if $x > 0$ the supremum $\Lambda^*(x)$ is attained at $\theta = 1 - \sqrt{\frac{\lambda}{x}}$; if $x \leq 0$ the supremum $\Lambda^*(x)$ is attained by taking the limit as $\theta \to -\infty$). □

Theorem 3.3 (LDP for the strong law of large numbers in Proposition 2.3) *Consider the same situation as in Proposition 2.3, with $\sigma^2 > 0$. Assume moreover that $\mathbb{E}[e^{\theta U_1}] < \infty$ for all $\theta \in \mathbb{R}$; hence $m_j := \mathbb{E}[U_1^j] < \infty$ for all $j \geq 1$. For $\alpha_j := \sum_{h=3}^{j-3} \frac{m_h}{h!}\frac{m_{j-h}}{(j-h)!}$ for all $j \geq 6$, we assume the following condition:*
(C2) : *there exists $M \in (0,\infty)$ such that $C_0 := \sup_{j\geq 6} \frac{|\alpha_j|}{M^j} < \infty$.*
Then $\{\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k : n \geq 1\}$ satisfies the LDP with speed function $v_n = \log n$ and good rate function $I_{\mathcal{N}(0,\sigma^2)}$ defined by $I_{\mathcal{N}(0,\sigma^2)}(x) = \frac{x^2}{8\sigma^2}$.

Sketch of the proof of Theorem 3.3. In section 6 we prove (3) with $Z_n = \frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k$, $v_n = \log n$ and $\Lambda(\theta) = 2\sigma^2\theta^2$. Then the Gärtner Ellis Theorem can be applied and the LDP holds with the good rate function $I_{\mathcal{N}(0,\sigma^2)}$ given by Λ^* in (4) (for any $x \in \mathbb{R}$ the supremum $\Lambda^*(x)$ is attained at $\theta = \frac{x}{4\sigma^2}$). □

Theorem 3.4 (Sample path LDP) *Let $\{U_n : n \geq 1\}$ be a sequence of independent standard (real valued) Brownian motions. Let $\{X_n : n \geq 1\}$ be the sequence of continuous processes on $[0,T]$ defined by $X_n(\cdot) = \frac{\sum_{i=1}^n U_i(\sigma^2\cdot)}{\sqrt{n}}$ for some $\sigma^2 \in (0,\infty)$. Then $\{\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k : n \geq 1\}$ satisfies the LDP with speed function $v_n = \log n$ and good rate function $I_{B(\sigma^2\cdot)}$ defined by*

$$I_{B(\sigma^2\cdot)}(x) = \begin{cases} \int_0^T I_{\mathcal{N}(0,\sigma^2)}(\dot{x}(t))dt & \text{if } x \in \mathcal{A} \\ \infty & \text{otherwise,} \end{cases}$$

where $I_{\mathcal{N}(0,\sigma^2)}$ is as in Theorem 3.3 and $\mathcal{A}$ is the family of all absolutely continuous functions x on $[0,T]$ such that $x(0) = 0$.

Sketch of the proof of Theorem 3.4. We illustrate how to apply Theorem 4.5.20 in [7]; the details will be shown in section 7. Let $\mathcal{X}^*$ be the dual space of $\mathcal{X} = C[0,T]$. In subsection 7.1 we check the existence of the function $\Lambda : \mathcal{X}^* \to (-\infty,\infty]$ (actually we have $\Lambda(\theta) < \infty$ for all $\theta \in \mathcal{X}^*$) defined by

$$\Lambda(\theta) := \lim_{n\to\infty} \frac{1}{v_n} \log \mathbb{E}[e^{v_n \int_0^T Z_n(t)d\theta(t)}] \tag{9}$$

with $Z_n = \frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k$, $v_n = \log n$ and $\Lambda(\theta) = 2\sigma^2 \int_0^T \theta^2((r,T])dr$. In subsection 7.2 it is proved that $\{Z_n : n \geq 1\}$ is an exponentially tight sequence. Then the function $\Lambda^* : \mathcal{X} \to [0,\infty]$ defined by

$$\Lambda^*(x) := \sup_{\theta\in\mathcal{X}^*} \left\{ \int_0^T x(t)d\theta(t) - \Lambda(\theta) \right\}$$

coincides with $I_{B(\sigma^2\cdot)}$ in the statement; this is a consequence of a more general result Lévy processes taking values on a Banach space (see section 3 in [6] where $T = 1$; the result can be easily extended to any $T \in (0,\infty)$). We complete the proof showing that the set of exposed points $\mathcal{F}$ coincides with $\{x \in \mathcal{X} : I_{B(\sigma^2\cdot)}(x) < \infty\}$; this will be done in subsection 7.3. □

4 Minor results and remarks

Firstly we remark that the rate functions $I_{\mathcal{P}(\lambda)}$ and $I_{\mathcal{N}(0,\sigma^2)}$ presented above can be expressed in terms of the Hellinger distance between two suitable probability measures. Furthermore we present the LDPs for sums of two independent sequences of logarithmically weighted means as in the theorems of the previous section. We also show that we cannot recover any LDP in this paper as a consequence of the LDP in [15]. Finally we concentrate our attention on Theorem 3.3: we present some examples for which the condition **(C2)** holds and we illustrate a connection with the LDPs for two sequences of logarithmically weighted empirical measures in the literature (see e.g. [13] and [19]). In this section we refer to another well known large deviation result, i.e. the contraction principle (see e.g. Theorem 4.2.1 in [7]).

4.1 Rate functions and Hellinger distance

It is known that the sequences $\{X_n : n \geq 1\}$ in the theorems of the previous section converge weakly (as $n \to \infty$): the weak limit is $\mathcal{P}(\lambda)$ in Theorems 3.1-3.2 and $\mathcal{N}(0, \sigma^2)$ in Theorem 3.3. In this subsection we illustrate how the rate functions can be expressed in terms of the weak limits of the sequences $\{X_n : n \geq 1\}$. In view of what follows we introduce the *Hellinger distance* between two probability measures P_1 and P_2 on the same measurable space Ω (see e.g. section 3.2 in [16]; see also section 14.5 in [22]), which is $H^2[P_1, P_2]$ defined by

$$H^2[P_1, P_2] := \frac{1}{2}\int_\Omega \left(\sqrt{\frac{dP_1}{d\mu}} - \sqrt{\frac{dP_2}{d\mu}}\right)^2 d\mu, \quad \text{for any measure } \mu \text{ such that } P_1 \text{ and } P_2 \text{ are absolutely continuous w.r.t. } \mu.$$

Note that we also have $H^2[P_1, P_2] = 1 - A[P_1, P_2]$ where $A[P_1, P_2] := \int_\Omega \sqrt{\frac{dP_1}{d\mu}\frac{dP_2}{d\mu}}d\mu$ is the *Hellinger affinity*. We always have a choice for μ, i.e. $\mu = P_1 + P_2$. In what follows we rewrite the rate functions $I_{\mathcal{P}(\lambda)}$ and $I_{\mathcal{N}(0,\sigma^2)}$ in terms of the Hellinger distance (or affinity) between two suitable probability measures on $\mathbb{R}$.

The rate function $I_{\mathcal{P}(\lambda)}$ in Theorems 3.1-3.2. It is easy to check that $H^2[\mathcal{P}(\lambda_1), \mathcal{P}(\lambda_2)] = 1 - e^{-\frac{(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2}{2}}$ for all $\lambda_1, \lambda_2 \geq 0$. Then we have

$$I_{\mathcal{P}(\lambda)}(x) = -2\log(1 - H^2[\mathcal{P}(x), \mathcal{P}(\lambda)]) = -2\log(A[\mathcal{P}(x), \mathcal{P}(\lambda)]) \text{ (for } x \geq 0).$$

The rate function $I_{\mathcal{N}(0,\sigma^2)}$ in Theorem 3.3. It is easy to check that $H^2[\mathcal{N}(\mu_1, \sigma_1^2), \mathcal{N}(\mu_2, \sigma_2^2)] = 1 - \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2+\sigma_1^2}}e^{-\frac{(\mu_1-\mu_2)^2}{4(\sigma_1^2+\sigma_1^2)}}$ for all $(\mu_1, \sigma_1^2), (\mu_2, \sigma_2^2) \in \mathbb{R} \times (0, \infty)$. Then we have

$$I_{\mathcal{N}(0,\sigma^2)}(x) = -\log(1 - H^2[\mathcal{N}(x, \sigma^2), \mathcal{N}(0, \sigma^2)]) = -\log(A[\mathcal{N}(x, \sigma^2), \mathcal{N}(0, \sigma^2)]).$$

4.2 LDPs for sums of two independent sequences

In this subsection we consider two independent sequences $\{Z_n^{(1)} : n \geq 1\}$ and $\{Z_n^{(2)} : n \geq 1\}$ as in the theorems in the previous sections (except Theorem 3.4). More precisely, for $h \in \{1, 2\}$, we define $Z_n^{(h)} = \frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k^{(h)}$, where $\{X_n^{(1)} : n \geq 1\}$ and $\{X_n^{(2)} : n \geq 1\}$ are two independent sequences. Then we give the details of proof of the LDP of $\{Z_n^{(1)} + Z_n^{(2)} : n \geq 1\}$ (with speed function $v_n = \log n$) in several cases. In each case the proof is an immediate consequence of the application of the contraction principle for the continuous function $(x_1, x_2) \mapsto x_1 + x_2$, which gives the good rate function I_{1*2} defined by

$$I_{1*2}(x) := \inf\{I_1(x_1) + I_2(x_2) : x_1 + x_2 = x\}, \tag{10}$$

where I_1 and I_2 are the rate functions for $\{Z_n^{(1)} : n \geq 1\}$ and $\{Z_n^{(2)} : n \geq 1\}$, respectively. In each case we also give the details of the proof of the LDP as a consequence of the application of the Gärtner Ellis Theorem. We remark that the rate function can be expressed in terms of the Hellinger distance with respect to the weak limit of $X_n^{(1)}+X_n^{(2)}$ as $n\to\infty$ if the weak limits (and therefore their convolution) are of the same kind. This is what happens in all the cases except the last one.

Both the sequences as in Theorems 3.1-3.2 with $I_h=I_{\mathcal{P}(\lambda_h)}$ for $h\in\{1,2\}$. The rate function I_{1*2} in (10) coincides with $I_{\mathcal{P}(\lambda_1+\lambda_2)}$. Moreover, for $x\geq 0$, the infimum in (10) is attained at $(x_1,x_2)=(\frac{\lambda_1 x}{\lambda_1+\lambda_2},\frac{\lambda_2 x}{\lambda_1+\lambda_2})$. We have the same result by applying the Gärtner Ellis Theorem: the rate function Λ^* in (4) coincides with $I_{\mathcal{P}(\lambda_1+\lambda_2)}$ because the function Λ in (3) is

$$\Lambda(\theta)=\begin{cases}\frac{\lambda_1\theta}{1-\theta}+\frac{\lambda_2\theta}{1-\theta} & \text{if } \theta<1\\ \infty & \text{if } \theta\geq 1.\end{cases}$$

Both the sequences as in Theorem 3.3 with $I_h=I_{\mathcal{N}(0,\sigma_h^2)}$ for $h\in\{1,2\}$. The rate function I_{1*2} in (10) coincides with $I_{\mathcal{N}(0,\sigma_1^2+\sigma_2^2)}$. Moreover, for $x\in\mathbb{R}$, the infimum in (10) is attained at $(x_1,x_2)=(\frac{\sigma_1^2 x}{\sigma_1^2+\sigma_2^2},\frac{\sigma_2^2 x}{\sigma_1^2+\sigma_2^2})$. We have the same result by applying the Gärtner Ellis Theorem: the rate function Λ^* in (4) coincides with $I_{\mathcal{N}(0,\sigma_1^2+\sigma_2^2)}$ because the function Λ in (3) is $\Lambda(\theta)=2\sigma_1^2\theta^2+2\sigma_2^2\theta^2$.

A sequence as in Theorems 3.1-3.2 and the other one as in Theorem 3.3. We consider $I_1=I_{\mathcal{P}(\lambda)}$ and $I_2=I_{\mathcal{N}(0,\sigma^2)}$. We do not have an explicit formula for the rate function I_{1*2} in (10); more precisely, for $x\in\mathbb{R}$, the infimum is attained at $(x_1,x_2)=(x_1(x),x-x_1(x))$ where $x_1(x)\in(0,\infty)$ is the unique solution of the equation (in x_1) $\frac{x-x_1}{4\sigma^2}+\frac{\sqrt{\lambda}}{\sqrt{x_1}}-1=0$. The Gärtner Ellis Theorem allows to prove the LDP with a different expression of the rate function: the function Λ in (3) is

$$\Lambda(\theta)=\begin{cases}\frac{\lambda\theta}{1-\theta}+2\sigma^2\theta^2 & \text{if } \theta<1\\ \infty & \text{if } \theta\geq 1,\end{cases}$$

and Λ^* in (4) becomes $\Lambda^*(x)=\theta(x)x-\Lambda(\theta(x))$, where $\theta(x)\in(-\infty,1)$ is the unique solution of the equation (in θ) $x=\Lambda'(\theta)$, i.e. $x=\frac{\lambda}{(1-\theta)^2}+4\sigma^2\theta$. Here it seems that we cannot express the rate function in terms of the Hellinger distance with respect to the convolution between $\mathcal{P}(\lambda)$ and $\mathcal{N}(0,\sigma^2)$, which is the weak limit of $X_n^{(1)}+X_n^{(2)}$ as $n\to\infty$ (indeed $X_n^{(1)}$ and $X_n^{(2)}$ converge weakly to $\mathcal{P}(\lambda)$ and $\mathcal{N}(0,\sigma^2)$, respectively).

4.3 On the LDPs in [15] and in Theorems 3.1-3.2-3.3

In this subsection we discuss the differences between the LDPs in this paper (except the one in Theorem 3.4) and the LDP in [15]. Firstly we note that, in the framework of Theorems 3.1-3.2, we cannot have a weighted sum of i.i.d. random variables; indeed we have $\frac{1}{\log n}\sum_{k=1}^n\frac{1}{k}X_k=\sum_{i=1}^n\frac{1}{\log n}\sum_{k=i}^n\frac{1}{k}1_{\{U_i\leq p_k\}}$ in Theorem 3.1 and $\frac{1}{\log n}\sum_{k=1}^n\frac{1}{k}X_k=\sum_{i=1}^n\frac{1}{\log n}\sum_{k=i}^n\frac{1}{k}U_i(t_k)$ in Theorem 3.2. On the contrary, in Theorem 3.3, we have

$$\frac{1}{\log n}\sum_{k=1}^n\frac{1}{k}X_k=\sum_{i=1}^n a_i(n)U_i,\ \text{with } a_i(n):=\frac{1}{\log n}\sum_{k=i}^n\frac{1}{k\sqrt{k}},$$

for a sequence $\{U_n:n\geq 1\}$ of i.i.d. random variables.

In what follows we show that the LDP in [15] does not allow to recover the LDP in Theorem 3.3 with $\sigma^2=1$ (this restriction meets (2.1) in [15]). Firstly we note that we should have $\Lambda(\theta)=\sum_{h=2}^\infty\frac{a_h c_h}{h!}\theta^h$ by (2.4) in [15]; thus we have

$$\frac{a_h c_h}{h!}=\begin{cases}2 & \text{if } h=2\\ 0 & \text{if } h\geq 3.\end{cases}$$

We also have $c_2 = 1$ by the definition of the function C in [15]; then, if we do not have any restriction on the (common) distribution of the random variables $\{U_n : n \geq 1\}$, we should have $a_2 = 4$ and $a_h = 0$ for all $h \geq 3$. Moreover, if we look at (2.2)-(2.3) in [15], we should have

$$\frac{1}{(\log n)^h} \sum_{i=1}^{n} \left(\sum_{k=i}^{n} \frac{1}{k\sqrt{k}} \right)^h = \frac{a_h}{(\log n)^{h-1}} R(h,n) \text{ for all } h, n \geq 1,$$

where the error term $R(h,n)$ is close to 1 in a suitable sense. This condition cannot hold because the left hand side and the right hand side have a different behavior as $n \to \infty$.

4.4 Some examples for which condition (C2) holds

In this subsection we show that condition **(C2)** holds if the (centered) random variables $\{U_n : n \geq 1\}$ are bounded or normal distributed. A natural question is whether it is possible to characterize condition **(C2)** in terms of some features of the (common) law of the random variables of $\{U_n : n \geq 1\}$.

Bounded random variables $\{U_n : n \geq 1\}$. If $P(|U_n| \leq B) = 1$ for some $B \in (0, \infty)$, we have

$$|\alpha_j| = \left| \sum_{h=3}^{j-3} \frac{m_h}{h!} \frac{m_{j-h}}{(j-h)!} \right| \leq \sum_{h=0}^{j} \frac{B^h}{h!} \frac{B^{j-h}}{(j-h)!} = \frac{1}{j!} \sum_{h=0}^{j} \binom{j}{h} B^j = \frac{(2B)^j}{j!} \leq (2B)^j.$$

Then **(C2)** holds by taking $M \geq 2B$.

Normal distributed random variables $\{U_n : n \geq 1\}$. If $\{U_n : n \geq 1\}$ are $\mathcal{N}(0, \sigma^2)$ distributed, it is known that $m_{2k} = \sigma^{2k} \frac{(2k)!}{2^k k!}$ and $m_{2k-1} = 0$ for all $k \geq 1$. Then for all $p \geq 3$ we have $\alpha_{2p+1} = \sum_{h=3}^{2p+1-3} \frac{m_h}{h!} \frac{m_{2p+1-h}}{(2p+1-h)!} = 0$ and

$$0 \leq \alpha_{2p} = \sum_{h=3}^{2p-3} \frac{m_h}{h!} \frac{m_{2p-h}}{(2p-h)!} = \sum_{k=2}^{p-2} \frac{\sigma^{2k}}{2^k k!} \frac{\sigma^{2(p-k)}}{2^{p-k}(p-k)!} \leq \frac{\sigma^{2p}}{2^p p!} \sum_{k=0}^{p} \binom{p}{k} = \frac{\sigma^{2p}}{p!} \leq \sigma^{2p}.$$

Then **(C2)** holds by taking $M \geq \sigma$.

4.5 On the LDPs in [13]-[19] and in Theorem 3.3 (with $\sigma^2 = 1$)

Let $\mathcal{M}(\mathbb{R})$ be the space of all nonnegative Borel measures on $\mathbb{R}$ and let $\mathcal{M}_1(\mathbb{R})$ be the space of all probability measures on $\mathbb{R}$. Both $\mathcal{M}(\mathbb{R})$ and $\mathcal{M}_1(\mathbb{R})$ are equipped with the topology of weak convergence. Then, in the framework of Theorem 3.3 with $\sigma^2 = 1$, it is known that the sequences of logarithmically weighted empirical measures in (1) and (2) satisfy the LDP (see the references cited in the Introduction); in both cases we have the same good rate function J defined by

$$J(\nu) := \begin{cases} \frac{1}{2} \int_{\mathbb{R}} \left(\frac{d}{dy} \sqrt{\frac{d\nu}{\mathcal{N}(0,1)}(y)} \right)^2 \mathcal{N}(0,1)(dy) & \text{if } \nu \in \mathcal{M}_1(\mathbb{R}) \text{ and } \nu \ll \mathcal{N}(0,1) \\ \infty & \text{otherwise,} \end{cases}$$

where $\nu \ll \mathcal{N}(0,1)$ means that ν is absolutely continuous with respect to $\mathcal{N}(0,1)$ and $\frac{d\nu}{\mathcal{N}(0,1)}$ is the density.

If the map $\nu \mapsto \int_{\mathbb{R}} y\nu(dy)$ were continuous on $\mathcal{M}_1(\mathbb{R})$, we could prove Theorem 3.3 by an application of the contraction principle and the good rate function $I_{\mathcal{N}(0,1)}$ would be

$$I_{\mathcal{N}(0,1)}(x) = \inf \left\{ J(\nu) : \int_{\mathbb{R}} y\nu(dy) = x \right\} \text{ for all } x \in \mathbb{R}. \tag{11}$$

Unfortunately $\nu \mapsto \int_{\mathbb{R}} y\nu(dy)$ is not continuous; nevertheless (11) holds. In fact, for any fixed $x \in \mathbb{R}$, let $\nu \in \mathcal{M}_1(\mathbb{R})$ be such that $\nu \ll \mathcal{N}(0,1)$ (otherwise we have $J(\nu) = \infty$) and $\int_{\mathbb{R}} y\nu(dy) = x$. Then we have

$$\left(\frac{d}{dy}\sqrt{\frac{d\nu}{\mathcal{N}(0,1)}(y)}\right)^2 = \left(\frac{\frac{d}{dy}\frac{d\nu}{\mathcal{N}(0,1)}(y)}{2\sqrt{\frac{d\nu}{\mathcal{N}(0,1)}(y)}}\right)^2 = \frac{1}{4}\left(\frac{\frac{d}{dy}\frac{d\nu}{\mathcal{N}(0,1)}(y)}{\frac{d\nu}{\mathcal{N}(0,1)}(y)}\right)^2 \frac{d\nu}{\mathcal{N}(0,1)}(y)$$
$$= \frac{1}{4}\left(\frac{d}{dy}\log\frac{d\nu}{\mathcal{N}(0,1)}(y)\right)^2 \frac{d\nu}{\mathcal{N}(0,1)}(y),$$

whence we obtain

$$J(\nu) = \frac{1}{2}\int_{\mathbb{R}}\left(\frac{d}{dy}\sqrt{\frac{d\nu}{\mathcal{N}(0,1)}(y)}\right)^2 \mathcal{N}(0,1)(dy) = \frac{1}{8}\int_{\mathbb{R}}\left(\frac{d}{dy}\log\frac{d\nu}{\mathcal{N}(0,1)}(y)\right)^2 \nu(dy).$$

Thus, by the Jensen inequality, we have

$$J(\nu) \geq \frac{1}{8}\left(\int_{\mathbb{R}}\frac{d}{dy}\log\frac{d\nu}{\mathcal{N}(0,1)}(y)\nu(dy)\right)^2,$$

and the lower bound is attained if and only if $\frac{d}{dy}\log\frac{d\nu}{\mathcal{N}(0,1)}(y)$ is a constant function, i.e. $\log\frac{d\nu}{\mathcal{N}(0,1)}(y)$ is a linear function. Thus we have $\frac{d\nu}{\mathcal{N}(0,1)}(y) = e^{\theta y - \frac{\theta^2}{2}}$ for some $\theta \in \mathbb{R}$ and, by taking into account the constraint $\int_{\mathbb{R}} y\nu(dy) = x$, we have to choose $\theta = x$. In conclusion this choice of ν gives $\frac{1}{8}\left(\int_{\mathbb{R}}\frac{d}{dy}\log\frac{d\nu}{\mathcal{N}(0,1)}(y)\nu(dy)\right)^2 = \frac{x^2}{8} = I_{\mathcal{N}(0,1)}(x)$ and this proves (11).

5 The proof of (3) for Theorems 3.1-3.2

In this section we give the details of the proofs of Theorem 3.1 and 3.2 which lead to the application of the Gärtner Ellis Theorem. In the framework of the two theorems we have to check that

$$\lim_{n\to\infty}\frac{\log\mathbb{E}[e^{\theta\sum_{k=1}^n\frac{1}{k}X_k}]}{\log n} = \begin{cases} \frac{\lambda\theta}{1-\theta} & \text{if } \theta < 1 \\ \infty & \text{if } \theta \geq 1 \end{cases} \quad (\text{for all } \theta \in \mathbb{R}). \tag{12}$$

Note that, in both the situations, the function $\theta \mapsto \frac{\log\mathbb{E}[e^{\theta\sum_{k=1}^n\frac{1}{k}X_k}]}{\log n}$ is non-decreasing because $\{\sum_{k=1}^n\frac{1}{k}X_k : n \geq 1\}$ are non-negative random variables. Thus, assuming that (12) holds for $\theta < 1$, we can easily obtain (12) for $\theta \geq 1$ as follows: for each $\eta < 1$ (and for $\theta \geq 1$) we have

$$\frac{\log\mathbb{E}[e^{\eta\sum_{k=1}^n\frac{1}{k}X_k}]}{\log n} \leq \frac{\log\mathbb{E}[e^{\theta\sum_{k=1}^n\frac{1}{k}X_k}]}{\log n} \quad (\text{for all } n \geq 1),$$

whence

$$\frac{\lambda\eta}{1-\eta} = \liminf_{n\to\infty}\frac{\log\mathbb{E}[e^{\eta\sum_{k=1}^n\frac{1}{k}X_k}]}{\log n} \leq \liminf_{n\to\infty}\frac{\log\mathbb{E}[e^{\theta\sum_{k=1}^n\frac{1}{k}X_k}]}{\log n}$$

and we conclude letting $\eta \uparrow 1$.

Thus we only have to prove (12) for $\theta < 1$. The two theorems deserve different proofs.

5.1 The proof of (12) (with $\theta < 1$) for Theorem 3.1

We start with a useful expression for $\log \mathbb{E}[e^{\theta \sum_{k=1}^n \frac{1}{k} X_k}]$ provided by the next Lemma 5.1. This expression is given in terms of the following quantities:

$$\begin{cases} b_j^{(i)} := e^{\theta \sum_{k=i}^j \frac{1}{k}} \ (i, j \in \{1, \ldots, n\},\ i \leq j); \\ \alpha_i := p_i(b_i^{(i)} - 1) \text{ and } \beta_i^{(n)} := \sum_{j=i}^{n-1} p_{j+1}(b_{j+1}^{(i)} - b_j^{(i)}) \ (i \in \{1, \ldots, n-1\}) \end{cases}$$

Lemma 5.1 *We have* $\log \mathbb{E}[e^{\theta \sum_{k=1}^n \frac{1}{k} X_k}] = \sum_{i=1}^{n-1} \log(1 + \alpha_i + \beta_i^{(n)}) + \log(1 + \alpha_n)$ *for all* $n \geq 1$.

Proof of Lemma 5.1. Firstly, since the random variables $\{U_n : n \geq 1\}$ are i.i.d. and $\sum_{k=1}^n \frac{1}{k} X_k = \sum_{k=1}^n \frac{1}{k} \sum_{i=1}^k 1_{\{U_i \leq p_k\}} = \sum_{i=1}^n \sum_{k=i}^n \frac{1}{k} 1_{\{U_i \leq p_k\}}$, we have

$$\log \mathbb{E}[e^{\theta \sum_{k=1}^n \frac{1}{k} X_k}] = \sum_{i=1}^n \log \mathbb{E}[e^{\theta \sum_{k=i}^n \frac{1}{k} 1_{\{U_i \leq p_k\}}}].$$

By the monotonicity of the sequence $\{p_n : n \geq 1\}$, the expected values in each summand at the right hand side can be written as follows:

$$\begin{aligned} \mathbb{E}[e^{\theta \sum_{k=i}^n \frac{1}{k} 1_{\{U_i \leq p_k\}}}] &= \int_0^1 e^{\theta \sum_{k=i}^n \frac{1}{k} 1_{[0,p_k]}(x)} dx \\ &= \int_0^{p_n} e^{\theta \sum_{k=i}^n \frac{1}{k} 1_{[0,p_k]}(x)} dx + \sum_{j=i}^{n-1} \int_{p_{j+1}}^{p_j} e^{\theta \sum_{k=i}^n \frac{1}{k} 1_{[0,p_k]}(x)} dx + \int_{p_i}^1 e^{\theta \sum_{k=i}^n \frac{1}{k} 1_{[0,p_k]}(x)} dx \\ &= p_n e^{\theta \sum_{k=i}^n \frac{1}{k}} + \sum_{j=i}^{n-1} (p_j - p_{j+1}) e^{\theta \sum_{k=i}^j \frac{1}{k}} + (1 - p_i). \end{aligned}$$

Then we have to prove that

$$p_n e^{\theta \sum_{k=i}^n \frac{1}{k}} + \sum_{j=i}^{n-1} (p_j - p_{j+1}) e^{\theta \sum_{k=i}^j \frac{1}{k}} + (1 - p_i) = \begin{cases} 1 + \alpha_i + \beta_i^{(n)} & \text{if } i \in \{1, \ldots, n-1\} \\ 1 + \alpha_n & \text{if } i = n. \end{cases} \tag{13}$$

We start with the left hand side in (13). For $i = n$ it is equal to

$$p_n e^{\frac{\theta}{n}} + (1 - p_n) = 1 + p_n(e^{\frac{\theta}{n}} - 1) = 1 + \alpha_n.$$

For $i \in \{1, \ldots, n-1\}$ it is equal to

$$1 - p_i + \sum_{j=i}^{n-1} p_j e^{\theta \sum_{k=i}^j \frac{1}{k}} - \sum_{j=i}^{n-1} p_{j+1} e^{\theta \sum_{k=i}^j \frac{1}{k}} + p_n e^{\theta \sum_{k=i}^n \frac{1}{k}} =$$

$$1 - p_i + p_i e^{\frac{\theta}{i}} + \sum_{j=i+1}^{n} p_j e^{\theta \sum_{k=i}^j \frac{1}{k}} - \sum_{j=i}^{n-1} p_{j+1} e^{\theta \sum_{k=i}^j \frac{1}{k}} =$$

$$1 + p_i(e^{\frac{\theta}{i}} - 1) + \sum_{j=i}^{n-1} p_{j+1}(e^{\theta \sum_{k=i}^{j+1} \frac{1}{k}} - e^{\theta \sum_{k=i}^j \frac{1}{k}}) = 1 + \alpha_i + \beta_i^{(n)}. \ \square$$

Note that, since $\lim_{n\to\infty} np_n = \lambda$,

$$\alpha_n = p_n(b_n^{(n)} - 1) = p_n(e^{\frac{\theta}{n}} - 1) \sim \frac{\lambda\theta}{n^2} \to 0 \text{ as } n \to \infty; \tag{14}$$

thus $\lim_{n\to\infty}\frac{\log(1+\alpha_n)}{\log n}=0$ and, by Lemma 5.1, (12) will be proved for $\theta<1$ if we show that

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n-1}\log(1+\alpha_i+\beta_i^{(n)})}{\log n}=\frac{\lambda\theta}{1-\theta}. \tag{15}$$

Moreover, since $\beta_i^{(n)}=\sum_{j=i}^{n-1}p_{j+1}e^{\theta\sum_{k=i}^{j}\frac{1}{k}}(e^{\frac{\theta}{j+1}}-1)$,

$$|\beta_i^{(n)}|\le\sum_{j=i}^{n-1}p_{j+1}e^{\theta\sum_{k=i}^{j}\frac{1}{k}}|e^{\frac{\theta}{j+1}}-1|\le C\sum_{j=i}^{n-1}\frac{e^{\theta\sum_{k=i}^{j}\frac{1}{k}}}{(j+1)^2},$$

where $C:=\sup_{n\ge1}np_n\sup_{x\in(0,1]}|\frac{e^{\theta x}-1}{x}|\in(0,\infty)$. Thus, in order to check that

$$\lim_{n\ge i\to\infty}\beta_i^{(n)}=0, \tag{16}$$

we can use (6) with $i\ge2$ and the second inequality in (8) with $\alpha=2-\theta$ as follows: for $0\le\theta<1$

$$\sum_{j=i}^{n-1}\frac{e^{\theta\sum_{k=i}^{j}\frac{1}{k}}}{(j+1)^2}\le\frac{1}{(i-1)^\theta}\sum_{j=i}^{n-1}\frac{1}{(j+1)^{2-\theta}}\le\left(\frac{i}{i-1}\right)^\theta\frac{1}{i}\frac{1}{1-\theta}\to0;$$

for $\theta<0$

$$\sum_{j=i}^{n-1}\frac{e^{\theta\sum_{k=i}^{j}\frac{1}{k}}}{(j+1)^2}\le\frac{1}{i^\theta}\sum_{j=i}^{n-1}\frac{1}{(j+1)^{2-\theta}}\le\frac{1}{i(1-\theta)}\to0.$$

By (14) and (16), for every integer $i_0\ge1$,

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{i_0}\alpha_i}{\log n}=0;\ \lim_{n\to\infty}\frac{\sum_{i=1}^{i_0}\beta_i^{(n)}}{\log n}=0;\ \lim_{n\to\infty}\frac{\sum_{i=1}^{i_0}\alpha_i^2}{\log n}=0;\ \lim_{n\to\infty}\frac{\sum_{i=1}^{i_0}\{\beta_i^{(n)}\}^2}{\log n}=0.$$

Note that there exists $m>0$ such that $|\log(1+x)-x|\le mx^2$ for $|x|<\frac{1}{2}$. Then, by (14) and (16) there exists an integer i_0 such that, for any integer n and i such that $n\ge i\ge i_0$, we have $|\alpha_i+\beta_i^{(n)}|<\frac{1}{2}$ and therefore

$$\begin{aligned}\left|\sum_{i=i_0}^{n}\{\log(1+\alpha_i+\beta_i^{(n)})-(\alpha_i+\beta_i^{(n)})\}\right|&\le\sum_{i=i_0}^{n}|\log(1+\alpha_i+\beta_i^{(n)})-(\alpha_i+\beta_i^{(n)})|\\&\le m\sum_{i=i_0}^{n}(\alpha_i+\beta_i^{(n)})^2\le2m\sum_{i=i_0}^{n}(\alpha_i^2+\{\beta_i^{(n)}\}^2).\end{aligned}$$

In conclusion, for $\theta<1$, the proof of (15) (and therefore of (12)) will be a consequence of the following relations:

$$\begin{aligned}&(i):\ \lim_{n\to\infty}\frac{\sum_{i=i_0}^{n}\alpha_i}{\log n}=0;\ (ii):\ \lim_{n\to\infty}\frac{\sum_{i=i_0}^{n}\alpha_i^2}{\log n}=0;\\&(iii):\ \lim_{n\to\infty}\frac{\sum_{i=i_0}^{n}\beta_i^{(n)}}{\log n}=\frac{\lambda\theta}{1-\theta};\ (iv):\ \lim_{n\to\infty}\frac{\sum_{i=i_0}^{n}\{\beta_i^{(n)}\}^2}{\log n}=0.\end{aligned}$$

Proofs of (i)-(ii). By (14) and the Cesaro theorem we have $\lim_{n\to\infty}\frac{\sum_{i=i_0}^{n}\alpha_i}{\log n}=\lim_{n\to\infty}n\alpha_n=0$ and $\lim_{n\to\infty}\frac{\sum_{i=i_0}^{n}\alpha_i^2}{\log n}=\lim_{n\to\infty}n\alpha_n^2=0$.

Proof of (iii). Consider the quantity $\gamma_j:=\sum_{i=1}^{j}p_{j+1}(b_{j+1}^{(i)}-b_j^{(i)})$, and the following equalities hold:

$$\sum_{i=1}^{n-1}\beta_i^{(n)}=\sum_{i=1}^{n-1}\sum_{j=i}^{n-1}p_{j+1}(b_{j+1}^{(i)}-b_j^{(i)})=\sum_{j=1}^{n-1}\gamma_j.$$

Thus we have to prove that $\lim_{n\to\infty} \frac{\sum_{j=1}^{n-1}\gamma_j}{\log n} = \frac{\lambda\theta}{1-\theta}$, which is equivalent to $\lim_{n\to\infty} n\gamma_n = \frac{\lambda\theta}{1-\theta}$ by the Cesaro theorem. Since

$$n\gamma_n = np_{n+1}\sum_{i=1}^{n}(b_{n+1}^{(i)} - b_n^{(i)}) = np_{n+1}(e^{\frac{\theta}{n+1}} - 1)\sum_{i=1}^{n} b_n^{(i)} \sim \frac{\lambda\theta}{n}\sum_{i=1}^{n} b_n^{(i)} \text{ as } n\to\infty,$$

we only have to prove that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} b_n^{(i)}}{n} = \frac{1}{1-\theta}. \tag{17}$$

We prove (17) for $0 \le \theta < 1$; the proof (17) for $\theta < 0$ is similar (the inequalities must be reversed but lead to the same conclusion) and therefore omitted. By (6)

$$(n+1)^\theta \le b_n^{(1)} \le e^\theta n^\theta \text{ and } \left(\frac{n+1}{i}\right)^\theta \le b_n^{(i)} \le \left(\frac{n}{i-1}\right)^\theta \text{ (for } i\in\{2,\dots,n\});$$

hence, summing over $i \in \{1,\dots,n\}$, by (7) with $\alpha = \theta$ we obtain

$$\begin{cases} \frac{\sum_{i=1}^{n} b_n^{(i)}}{n} \le \frac{n^\theta}{n}\left(e^\theta + \sum_{i=2}^{n}\frac{1}{(i-1)^\theta}\right) \sim n^{\theta-1}\left(e^\theta + \frac{n^{1-\theta}}{1-\theta}\right) \to \frac{1}{1-\theta} \\ \frac{\sum_{i=1}^{n} b_n^{(i)}}{n} \ge \frac{(n+1)^\theta}{n}\sum_{i=1}^{n}\frac{1}{i^\theta} \sim \frac{(n+1)^\theta}{n}\frac{(n+1)^{1-\theta}}{1-\theta} \to \frac{1}{1-\theta} \end{cases} \text{ as } n\to\infty.$$

Proof of (iv). Firstly let us consider the following quantities:

$$\begin{cases} A_{i,n} := \sum_{j=i}^{n-1} p_{j+1}^2(b_{j+1}^{(i)} - b_j^{(i)})^2 \\ B_{i,n} := \sum_{j>k=i}^{n-1} p_{j+1}p_{k+1}(b_{j+1}^{(i)} - b_j^{(i)})(b_{k+1}^{(i)} - b_k^{(i)}) = \sum_{j=i+1}^{n-1}\sum_{k=i}^{j-1} p_{j+1}p_{k+1}(b_{j+1}^{(i)} - b_j^{(i)})(b_{k+1}^{(i)} - b_k^{(i)}); \end{cases}$$

then we can write

$$\{\beta_n^{(i)}\}^2 = \left\{\sum_{j=i}^{n-1} p_{j+1}(b_{j+1}^{(i)} - b_j^{(i)})\right\}^2 = A_{i,n} + 2B_{i,n},$$

and we prove (iv) showing that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} A_{i,n}}{\log n} = 0 \tag{18}$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} B_{i,n}}{\log n} = 0. \tag{19}$$

Proof of (18). Consider

$$\rho_j := \sum_{i=1}^{j} p_{j+1}^2(b_{j+1}^{(i)} - b_j^{(i)})^2.$$

Hence $\sum_{i=1}^{n-1} A_{i,n} = \sum_{i=1}^{n-1}\sum_{j=i}^{n-1} p_{j+1}^2(b_{j+1}^{(i)} - b_j^{(i)})^2 = \sum_{j=1}^{n-1}\rho_j$; then (18) is equivalent to $\lim_{n\to\infty}\frac{\sum_{j=1}^{n-1}\rho_j}{\log n} = 0$ and, by the Cesaro theorem, it is also equivalent to $\lim_{n\to\infty} n\rho_n = 0$. We note that

$$n\rho_n = np_{n+1}^2\sum_{i=1}^{n}(b_{n+1}^{(i)} - b_n^{(i)})^2 = np_{n+1}^2(e^{\frac{\theta}{n+1}} - 1)^2\sum_{i=1}^{n}\{b_n^{(i)}\}^2 \sim \frac{\lambda^2\theta^2}{(n+1)^3}\sum_{i=1}^{n}\{b_n^{(i)}\}^2 \text{ as } n\to\infty.$$

We start with the case $0 \le \theta < 1$. From (6) we get

$$(n+1)^{2\theta} \le \{b_n^{(1)}\}^2 \le e^{2\theta}n^{2\theta} \text{ and } \left(\frac{n+1}{i}\right)^{2\theta} \le \{b_n^{(i)}\}^2 \le \left(\frac{n}{i-1}\right)^{2\theta} \text{ (for } i\in\{2,\dots,n\});$$

hence

$$\begin{cases} \frac{\lambda^2\theta^2}{(n+1)^3}\sum_{i=1}^n\{b_n^{(i)}\}^2 \leq \frac{\lambda^2\theta^2 n^{2\theta}}{(n+1)^3}\left(e^{2\theta}+\sum_{i=2}^n\frac{1}{(i-1)^{2\theta}}\right) \sim \lambda^2\theta^2 n^{2\theta-3}\sum_{i=1}^n\frac{1}{i^{2\theta}} \\ \frac{\lambda^2\theta^2}{(n+1)^3}\sum_{i=1}^n\{b_n^{(i)}\}^2 \geq \frac{\lambda^2\theta^2 (n+1)^{2\theta}}{(n+1)^3}\sum_{i=1}^n\frac{1}{i^{2\theta}} \sim \lambda^2\theta^2 n^{2\theta-3}\sum_{i=1}^n\frac{1}{i^{2\theta}} \end{cases} \text{as } n\to\infty.$$

As for the proof of (iii) we have reverse inequalities for the case $\theta < 0$ and we obtain similar estimates. In conclusion we prove (18) distinguishing the following three cases:

- for $\theta < \frac{1}{2}$ we have $n^{2\theta-3}\sum_{i=1}^n\frac{1}{i^{2\theta}} \sim n^{2\theta-3}\frac{n^{1-2\theta}}{1-2\theta} = \frac{1}{n^2(1-2\theta)} \to 0$ as $n\to\infty$ by (7) with $\alpha = 2\theta$;
- for $\theta = \frac{1}{2}$ we have $n^{2\theta-3}\sum_{i=1}^n\frac{1}{i^{2\theta}} = n^{-2}\sum_{i=1}^n\frac{1}{i} \sim \frac{\log n}{n^2} \to 0$ as $n\to\infty$ by (6);
- for $\frac{1}{2} < \theta < 1$ we have $0 \leq n^{2\theta-3}\sum_{i=1}^n\frac{1}{i^{2\theta}} \leq Cn^{2\theta-3} \to 0$ as $n\to\infty$ for $C = \sum_{i=1}^\infty\frac{1}{i^{2\theta}} \in (0,\infty)$ since $2\theta - 3 < 0$ if $\frac{1}{2} < \theta < 1$.

Proof of (19). Consider

$$\xi_j := \sum_{i=1}^{j-1}\sum_{k=i}^{j-1} p_{j+1}p_{k+1}(b_{j+1}^{(i)} - b_j^{(i)})(b_{k+1}^{(i)} - b_k^{(i)}).$$

Hence we have $\sum_{i=1}^{n-1} B_{i,n} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n-1}\sum_{k=i}^{j-1} p_{j+1}p_{k+1}(b_{j+1}^{(i)} - b_j^{(i)})(b_{k+1}^{(i)} - b_k^{(i)}) = \sum_{j=2}^{n-1}\xi_j$; then (19) is equivalent to $\lim_{n\to\infty}\frac{\sum_{j=2}^{n-1}\xi_j}{\log n} = 0$ and, by the Cesaro theorem, it is also equivalent to $\lim_{n\to\infty} n\xi_n = 0$. We note that

$$\begin{aligned} n\xi_n =& n\sum_{i=1}^{n-1}\sum_{k=i}^{n-1} p_{n+1}p_{k+1}(b_{n+1}^{(i)} - b_n^{(i)})(b_{k+1}^{(i)} - b_k^{(i)}) = np_{n+1}(e^{\frac{\theta}{n+1}} - 1)\sum_{k=1}^{n-1}\sum_{i=1}^{k} p_{k+1}(e^{\frac{\theta}{k+1}} - 1)b_n^{(i)}b_k^{(i)} \\ \sim& \frac{\lambda\theta}{n}\sum_{k=1}^{n-1}\sum_{i=1}^{k} p_{k+1}(e^{\frac{\theta}{k+1}} - 1)b_n^{(i)}b_k^{(i)} \text{ as } n\to\infty; \end{aligned}$$

thus, if $\Phi(n) := \sum_{k=1}^{n-1}\sum_{i=1}^{k} p_{k+1}(e^{\frac{\theta}{k+1}} - 1)b_n^{(i)}b_k^{(i)}$, (19) will be proved if we show that

$$\lim_{n\to\infty}\frac{\Phi(n)}{n} = 0. \tag{20}$$

We start with the case $0 \leq \theta < 1$. From (6) we get

$$\begin{aligned} 0 \leq \Phi(n) \sim& \lambda\theta\sum_{k=1}^{n-1}\frac{1}{(k+1)^2}\left(e^\theta n^\theta e^\theta k^\theta + \sum_{i=2}^k\left(\frac{n}{i-1}\right)^\theta\left(\frac{k}{i-1}\right)^\theta\right) \text{ as } n\to\infty \\ \leq& \lambda\theta n^\theta\sum_{k=1}^{n-1}\frac{1}{k^{2-\theta}}\left(e^{2\theta} + \sum_{i=2}^k\frac{1}{(i-1)^{2\theta}}\right). \end{aligned}$$

Then we prove (20) (and therefore (19)) for $0 \leq \theta < 1$ distinguishing the following three cases:

- for $0 \leq \theta < \frac{1}{2}$ we have $\lambda\theta n^\theta\sum_{k=1}^{n-1}\frac{1}{k^{2-\theta}}\left(e^{2\theta} + \sum_{i=2}^k\frac{1}{(i-1)^{2\theta}}\right) \leq \lambda\theta n^\theta\sum_{k=1}^{n-1}\frac{1}{k^{2-\theta}}\left(e^{2\theta} + \frac{k^{1-2\theta}}{1-2\theta}\right) = \lambda\theta n^\theta e^{2\theta}\sum_{k=1}^{n-1}\frac{1}{k^{2-\theta}} + \frac{\lambda\theta n^\theta}{1-2\theta}\sum_{k=1}^{n-1}\frac{1}{k^{\theta+1}} \leq C_\theta n^\theta$ for a suitable constant $C_\theta \in (0,\infty)$ by (7) with $\alpha = 2\theta$ and noting that $\sum_{k=1}^\infty\frac{1}{k^{2-\theta}}, \sum_{k=1}^\infty\frac{1}{k^{\theta+1}} \in (0,\infty)$;
- for $\theta = \frac{1}{2}$ we have $\frac{\lambda}{2}n^{\frac{1}{2}}\sum_{k=1}^{n-1}\frac{1}{k^{\frac{3}{2}}}\left(e + \sum_{i=2}^k\frac{1}{i-1}\right) \leq \frac{\lambda}{2}n^{\frac{1}{2}}\sum_{k=1}^{n-1}\frac{1}{k^{\frac{3}{2}}}(e + 1 + \log(k-1)) \leq C\sqrt{n}$ (for a suitable constant $C \in (0,\infty)$) by (6);

- for $\frac{1}{2} < \theta < 1$ we have $\lambda\theta n^\theta \sum_{k=1}^{n-1} \frac{1}{k^{2-\theta}} \left(e^{2\theta} + \sum_{i=2}^{k} \frac{1}{(i-1)^{2\theta}}\right) \leq C_\theta n^\theta$ where $C_\theta = \left(e^{2\theta} + \sum_{i=2}^{\infty} \frac{1}{(i-1)^{2\theta}}\right) \sum_{k=1}^{\infty} \frac{1}{k^{2-\theta}} \in (0,\infty)$.

We conclude with the case $\theta < 0$. Firstly note that $C_\theta := \sum_{k=1}^{\infty} \frac{1}{k^{2-\theta}} \in (0,\infty)$ and $C := \sup_{n\geq 1} np_n \inf_{x\in(0,1]} \frac{e^{\theta x}-1}{x} \in (-\infty, 0)$. Then, in order to prove (20) (and therefore (19)), we use (6), (7) with $\alpha = 2\theta$ and (7) with $\alpha = 1+\theta$ as follows:

$$\begin{aligned}
0 \geq \frac{\Phi(n)}{n} &\geq \frac{C}{n} \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{i=1}^{k} b_n^{(i)} b_k^{(i)} \\
&\geq \frac{C}{n} \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \left(e^\theta n^\theta e^\theta k^\theta + \sum_{i=2}^{k} \left(\frac{n}{i-1}\right)^\theta \left(\frac{k}{i-1}\right)^\theta\right) \\
&\geq \frac{C}{n} n^\theta \sum_{k=1}^{n-1} \frac{1}{k^{2-\theta}} \left(e^{2\theta} + \sum_{i=2}^{k} \frac{1}{(i-1)^{2\theta}}\right) \geq \frac{C}{n} e^{2\theta} C_\theta n^\theta + \frac{Cn^\theta}{n(1-2\theta)} \sum_{k=1}^{n-1} \frac{1}{k^{1+\theta}} \\
&\sim \frac{C}{n} e^{2\theta} C_\theta n^\theta + \frac{Cn^\theta}{n(1-2\theta)} \frac{n^{1-(1+\theta)}}{1-(1+\theta)} = \frac{C}{n} e^{2\theta} C_\theta n^\theta + \frac{C}{n(1-2\theta)|\theta|} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

5.2 The proof of (12) (with $\theta < 1$) for Theorem 3.2

Firstly we have

$$\begin{aligned}
\log \mathbb{E}[e^{\theta \sum_{k=1}^{n} \frac{1}{k} X_k}] &= \log \mathbb{E}[e^{\theta \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} U_i(t_k)}] \\
&= \log \mathbb{E}[e^{\theta \sum_{i=1}^{n} \sum_{k=i}^{n} \frac{1}{k} U_i(t_k)}] = \sum_{i=1}^{n} \log \mathbb{E}[e^{\theta \sum_{k=i}^{n} \frac{1}{k} U_1(t_k)}],
\end{aligned}$$

since $U_1 \ldots, U_n$ are i.i.d. processes. By the monotonicity of the sequence $\{t_n : n \geq 1\}$, for any $k \in \{i, \ldots, n\}$ we have $U_1(t_k) = U_1(t_n) + \sum_{h=k}^{n-1} \{U_1(t_h) - U_1(t_{h+1})\}$; hence $U_1(t_k)$ is the sum of independent Poisson distributed random variables $U_1(t_n), U_1(t_{n-1}) - U_1(t_n), \ldots, U_1(t_k) - U_1(t_{k+1})$ with means $t_n, t_{n-1} - t_n, \ldots, t_k - t_{k+1}$, respectively, and we have

$$\begin{aligned}
\log \mathbb{E}[e^{\theta \sum_{k=1}^{n} \frac{1}{k} X_k}] &= \sum_{i=1}^{n} \log \mathbb{E}[e^{\theta \sum_{k=i}^{n} \frac{1}{k} U_1(t_n) + \theta \sum_{k=i}^{n} \frac{1}{k} \sum_{h=k}^{n-1} \{U_1(t_h) - U_1(t_{h+1})\}}] \\
&= \sum_{i=1}^{n} \log \mathbb{E}[e^{\theta \sum_{k=i}^{n} \frac{1}{k} U_1(t_n) + \theta \sum_{h=i}^{n-1} \{U_1(t_h) - U_1(t_{h+1})\} \sum_{k=i}^{h} \frac{1}{k}}] \\
&= \sum_{i=1}^{n} \left\{ t_n (e^{\theta \sum_{k=i}^{n} \frac{1}{k}} - 1) + \sum_{h=i}^{n-1} (t_h - t_{h+1})(e^{\theta \sum_{k=i}^{h} \frac{1}{k}} - 1) \right\}.
\end{aligned}$$

We obtain the following expression handling the latter sum as follows:

$$\begin{aligned}
\log \mathbb{E}[e^{\theta \sum_{k=1}^{n} \frac{1}{k} X_k}] &= \sum_{i=1}^{n} \left\{ t_n e^{\theta \sum_{k=i}^{n} \frac{1}{k}} + \sum_{h=i}^{n-1} (t_h - t_{h+1}) e^{\theta \sum_{k=i}^{h} \frac{1}{k}} - \left(t_n + \sum_{h=i}^{n-1} (t_h - t_{h+1}) \right) \right\} \\
&= \sum_{i=1}^{n} \left\{ t_n e^{\theta \sum_{k=i}^{n} \frac{1}{k}} + \sum_{h=i}^{n-1} (t_h - t_{h+1}) e^{\theta \sum_{k=i}^{h} \frac{1}{k}} - t_i \right\} \\
&= \sum_{i=1}^{n} \left\{ \sum_{h=i}^{n} t_h e^{\theta \sum_{k=i}^{h} \frac{1}{k}} - \left(t_i + \sum_{h=i}^{n-1} t_{h+1} e^{\theta \sum_{k=i}^{h} \frac{1}{k}} \right) \right\} \\
&= \sum_{i=1}^{n} \left\{ \sum_{h=i}^{n} t_h e^{\theta \sum_{k=i}^{h} \frac{1}{k}} - \sum_{h=i}^{n} t_h e^{\theta \sum_{k=i}^{h-1} \frac{1}{k}} \right\} = \sum_{i=1}^{n} \sum_{h=i}^{n} t_h e^{\theta \sum_{k=i}^{h-1} \frac{1}{k}} (e^{\frac{\theta}{h}} - 1) \\
&= \sum_{h=1}^{n} t_h (e^{\frac{\theta}{h}} - 1) \sum_{i=1}^{h} e^{\theta \sum_{k=i}^{h-1} \frac{1}{k}}.
\end{aligned}$$

In conclusion we have

$$\begin{aligned}
\frac{\log \mathbb{E}[e^{\theta \sum_{k=1}^{n} \frac{1}{k} X_k}]}{\log n} &= \frac{\sum_{h=1}^{n} t_h (e^{\frac{\theta}{h}} - 1) \sum_{i=1}^{h} e^{\theta \sum_{k=i}^{h-1} \frac{1}{k}}}{\log n} \\
&\sim n t_n (e^{\frac{\theta}{n}} - 1) \sum_{i=1}^{n} e^{\theta \sum_{k=i}^{n-1} \frac{1}{k}} \sim \frac{\lambda \theta}{n} \sum_{i=1}^{n} e^{\theta \sum_{k=i}^{n-1} \frac{1}{k}} \text{ as } n \to \infty
\end{aligned}$$

by the Cesaro Theorem and $\lim_{n \to \infty} n t_n = \lambda$, and we complete the proof (12) for $\theta < 1$ noting that

$$\frac{\log \mathbb{E}[e^{\theta \sum_{k=1}^{n} \frac{1}{k} X_k}]}{\log n} \sim \frac{\lambda \theta e^{\theta}}{n^{1-\theta}} + \frac{\lambda \theta}{n^{1-\theta}} \frac{n^{1-\theta}}{1-\theta} \text{ as } n \to \infty$$

by (6) and (7) (for the cases $i = 1$ and $i \in \{2, \ldots, n\}$, respectively).

6 The proof of (3) for Theorem 3.3

In this section we give the details of the proof of Theorem 3.3 which lead to the application of the Gärtner Ellis Theorem. In the framework of that theorem we have to check that

$$\lim_{n \to \infty} \frac{\log \mathbb{E}[e^{\theta \sum_{k=1}^{n} \frac{1}{k} X_k}]}{\log n} = 2\sigma^2 \theta^2 \text{ (for all } \theta \in \mathbb{R}). \tag{21}$$

In what follows we set

$$s_{i,n} := \sum_{k=i}^{n} \frac{1}{k\sqrt{k}} \text{ (for } n \geq i \geq 1). \tag{22}$$

Then, since the random variables $\{U_n : n \geq 1\}$ are i.i.d. and $\sum_{k=1}^{n} \frac{1}{k} X_k = \sum_{k=1}^{n} \frac{1}{k} \frac{\sum_{i=1}^{k} U_i}{\sqrt{k}} = \sum_{i=1}^{n} s_{i,n} U_i$, (21) becomes

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} \log \mathbb{E}[e^{\theta s_{i,n} U_1}]}{\log n} = 2\sigma^2 \theta^2 \text{ (for all } \theta \in \mathbb{R}).$$

Let i_0 be a fixed integer and let n and i be such that $i < i_0$ and $n \geq i$. Note that $P(\theta U_1 \geq 0) > 0$ since the random variables $\{U_n : n \geq 1\}$ are centered and $0 \leq s_{i,n} \leq s_{1,\infty} < \infty$ where $s_{1,\infty} := \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$. Then we have

$$\mathbb{E}[e^{\theta s_{i,n} U_1}] = \mathbb{E}[e^{\theta s_{i,n} U_1} 1_{\{\theta U_1 \geq 0\}}] + \mathbb{E}[e^{\theta s_{i,n} U_1} 1_{\{\theta U_1 < 0\}}] \leq \mathbb{E}[e^{\theta s_{1,\infty} U_1}] + 1 < \infty$$

and

$$\mathbb{E}[e^{\theta s_{i,n}U_1}] = \mathbb{E}[e^{\theta s_{i,n}U_1}1_{\{\theta U_1\geq 0\}}] + \mathbb{E}[e^{\theta s_{i,n}U_1}1_{\{\theta U_1<0\}}] \geq \mathbb{E}[e^{\theta s_{1,\infty}U_1}1_{\{\theta U_1\geq 0\}}] \geq P(\theta U_1 \geq 0) > 0,$$

whence we easily obtain

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{i_0} \log \mathbb{E}[e^{\theta s_{i,n}U_1}]}{\log n} = 0 \text{ (for all } \theta \in \mathbb{R}).$$

Thus (21) is equivalent to

$$\lim_{n\to\infty} \frac{\sum_{i=i_0}^{n} \log \mathbb{E}[e^{\theta s_{i,n}U_1}]}{\log n} = 2\sigma^2\theta^2 \text{ (for all } \theta \in \mathbb{R} \text{ and } i_0 \geq 1). \tag{23}$$

In what follows we shall choose i_0 in a suitable way.

For the function ϕ defined by $\phi(y) := \sum_{j=1}^{\infty} \frac{m_j}{j!} y^j$, we have

$$\phi(y) = \frac{\sigma^2}{2}y^2 + \sum_{j=3}^{\infty} \frac{m_j}{j!} y^j \text{ and } \mathbb{E}[e^{yU_1}] = 1 + \phi(y).$$

The function ϕ is continuous on $\mathbb{R}$ and, since $\phi(0) = 0$, there exists $\delta > 0$ such that $|\phi(y)| < \frac{1}{2}$. Hence, for all $\theta \in \mathbb{R}$, there exists an integer i_0 such that, for any integer n and i such that $n \geq i \geq i_0$, we have $|\theta s_{i,n}| < \delta$ and therefore $|\phi(\theta s_{i,n})| < \frac{1}{2}$. Introduce $A_{i,n}$ and $B_{i,n}$ defined by

$$A_{i,n} := \frac{\sigma^2}{2}\theta^2 s_{i,n}^2 \text{ and } B_{i,n} := \sum_{j=3}^{\infty} \frac{m_j}{j!}\theta^j s_{i,n}^j;$$

moreover note that there exists $m > 0$ such that $|\log(1+x) - x| \leq mx^2$ for $|x| < \frac{1}{2}$. Then, since $\phi(\theta s_{i,n}) = A_{i,n} + B_{i,n} = \mathbb{E}[e^{\theta s_{i,n}U_1}] - 1$, for $n \geq i \geq i_0$ we have $|A_{i,n} + B_{i,n}| < \frac{1}{2}$, whence we obtain $|\log \mathbb{E}[e^{\theta s_{i,n}U_1}] - (A_{i,n} + B_{i,n})| \leq m(A_{i,n} + B_{i,n})^2$, and the following inequalities:

$$\left|\sum_{i=i_0}^{n} \{\log \mathbb{E}[e^{\theta s_{i,n}U_1}] - (A_{i,n} + B_{i,n})\}\right| \leq \sum_{i=i_0}^{n} |\log \mathbb{E}[e^{\theta s_{i,n}U_1}] - (A_{i,n} + B_{i,n})|$$
$$\leq m \sum_{i=i_0}^{n} (A_{i,n} + B_{i,n})^2 \leq 2m \sum_{i=i_0}^{n} (A_{i,n}^2 + B_{i,n}^2).$$

In conclusion the proof of (23) (and therefore of (21)) will be a consequence of the following relations:

$$(i): \lim_{n\to\infty} \frac{\sum_{i=i_0}^{n} A_{i,n}}{\log n} = 2\sigma^2\theta^2; \; (ii): \lim_{n\to\infty} \frac{\sum_{i=i_0}^{n} B_{i,n}}{\log n} = 0;$$
$$(iii): \lim_{n\to\infty} \frac{\sum_{i=i_0}^{n} A_{i,n}^2}{\log n} = 0; \; (iv): \lim_{n\to\infty} \frac{\sum_{i=i_0}^{n} B_{i,n}^2}{\log n} = 0.$$

Proof of (i). By the definition of $A_{i,n}$, (i) is equivalent to

$$\lim_{n\to\infty} \frac{\sum_{i=i_0}^{n} s_{i,n}^2}{4 \log n} = 1. \tag{24}$$

By (8) with $\alpha = \frac{3}{2}$ we have $2\left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{n+1}}\right) \leq s_{i,n} \leq 2\left(\frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{n}}\right)$, whence

$$\frac{1}{i} + \frac{1}{n+1} - \frac{2}{\sqrt{n+1}\sqrt{i}} \leq \frac{s_{i,n}^2}{4} \leq \frac{1}{i-1} + \frac{1}{n} - \frac{2}{\sqrt{n}\sqrt{i-1}}.$$

Thus we have the bounds

$$\begin{cases} \frac{\sum_{i=i_0}^n s_{i,n}^2}{4\log n} \geq \frac{1}{\log n}\left(\sum_{i=i_0}^n \frac{1}{i} + \frac{n-i_0+1}{n+1} - \frac{2}{\sqrt{n+1}}\sum_{i=i_0}^n \frac{1}{\sqrt{i}}\right) \\ \frac{\sum_{i=i_0}^n s_{i,n}^2}{4\log n} \leq \frac{1}{\log n}\left(\sum_{i=i_0}^n \frac{1}{i-1} + \frac{n-i_0+1}{n} - \frac{2}{\sqrt{n}}\sum_{i=i_0}^n \frac{1}{\sqrt{i-1}}\right), \end{cases}$$

whence we obtain (24), noting that $\sum_{i=i_0}^n \frac{1}{i} \sim \sum_{i=i_0}^n \frac{1}{i-1} \sim \log n$ as $n \to \infty$ by (6), and $\sum_{i=i_0}^n \frac{1}{\sqrt{i}} \sim \sum_{i=i_0}^n \frac{1}{\sqrt{i-1}} \sim 2\sqrt{n}$ as $n \to \infty$ by (7) with $\alpha = \frac{1}{2}$.

Proof of (ii). Firstly, by (8) with $\alpha = \frac{j}{2}$ and $j \geq 3$ (and also $i_0 \geq 2$), we have

$$\sum_{i=i_0}^n \frac{1}{(i-1)^{\frac{j}{2}}} \leq \frac{2}{j-2}\frac{1}{(i_0-1)^{\frac{j}{2}-1}} \leq 2.$$

Then, by the second inequality in (8) with $\alpha = \frac{3}{2}$ and the latter equality, we obtain

$$\left|\frac{\sum_{i=i_0}^n B_{i,n}}{\log n}\right| \leq \frac{\sum_{i=i_0}^n \sum_{j=3}^\infty \frac{|m_j|}{j!}(2|\theta|)^j(\frac{1}{\sqrt{i-1}})^j}{\log n} = \frac{\sum_{j=3}^\infty \frac{|m_j|}{j!}(2|\theta|)^j \sum_{i=i_0}^n \frac{1}{(i-1)^{\frac{j}{2}}}}{\log n}$$
$$\leq 2\frac{\sum_{j=3}^\infty \frac{|m_j|}{j!}(2|\theta|)^j}{\log n} \to 0 \text{ (as } n \to \infty),$$

noting that $\sum_{j=3}^\infty \frac{|m_j|}{j!}(2|\theta|)^j < \infty$ because a convergent power series is also absolutely convergent.

Proof of (iii). By the definition of $A_{i,n}$, *(iii)* is equivalent to

$$\lim_{n\to\infty} \frac{\sum_{i=i_0}^n s_{i,n}^4}{\log n} = 0. \tag{25}$$

Moreover we already remarked that $s_{i,n} \leq 2\left(\frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{n}}\right)$, whence we obtain the inequality $s_{i,n}^4 \leq \frac{16}{(i-1)^2}$. Thus (25) holds noting that $0 \leq \frac{\sum_{i=i_0}^n s_{i,n}^4}{\log n} \leq 16\frac{\sum_{i=i_0}^n \frac{1}{(i-1)^2}}{\log n} \to 0$ as $n \to \infty$.

Proof of (iv). By the Cauchy formula for the product of two convergent series we have $B_{i,n}^2 = \sum_{j=6}^\infty \alpha_j \theta^j s_{i,n}^j$ (where α_j is as in the statement of Theorem 3.3); moreover **(C2)** and the second inequality in (8) with $\alpha = \frac{3}{2}$ yield

$$B_{i,n}^2 \leq \sum_{j=6}^\infty |\alpha_j||\theta|^j\left(\frac{2}{\sqrt{i-1}}\right)^j \leq C_0\sum_{j=6}^\infty\left(\frac{2|\theta|M}{\sqrt{i-1}}\right)^j.$$

Now define $i_1 := [(2|\theta|M)^2] + 2$. Then, for $i \geq i_1$, we have $\frac{2|\theta|M}{\sqrt{i-1}} \leq \frac{2|\theta|M}{\sqrt{i_1-1}} < 1$, whence

$$\sum_{j=6}^\infty\left(\frac{2|\theta|M}{\sqrt{i-1}}\right)^j = \frac{1}{1-\frac{2|\theta|M}{\sqrt{i-1}}}\left(\frac{2|\theta|M}{\sqrt{i-1}}\right)^6 \leq \frac{1}{1-\frac{2|\theta|M}{\sqrt{i_1-1}}}\left(\frac{2|\theta|M}{\sqrt{i-1}}\right)^6 = \frac{C}{(i-1)^3}.$$

for a suitable constant $C > 0$. Thus, for $n > i_0 \vee i_1$, we have

$$\frac{\sum_{i=i_0}^n B_{i,n}^2}{\log n} = \frac{\sum_{i=i_0}^{i_0\vee i_1} B_{i,n}^2 + \sum_{i=(i_0\vee i_1)+1}^n B_{i,n}^2}{\log n} \leq \frac{\sum_{i=i_0}^{i_0\vee i_1} B_{i,n}^2}{\log n} + \frac{CC_0\sum_{i=(i_0\vee i_1)+1}^n \frac{1}{(i-1)^3}}{\log n}$$

and we trivially have $\lim_{n\to\infty}\frac{\sum_{i=(i_0\vee i_1)+1}^n \frac{1}{(i-1)^3}}{\log n} = 0$. We also get that $\lim_{n\to\infty}\frac{\sum_{i=i_0}^{i_0\vee i_1} B_{i,n}^2}{\log n} = 0$ noting that, by the second inequality in (8) with $\alpha = \frac{3}{2}$ as before, $|B_{i,n}|$ is bounded by a positive constant: $|B_{i,n}| \leq \sum_{j=3}^\infty \frac{|m_j|}{j!}|\theta|^j\left(\frac{2}{\sqrt{i-1}}\right)^j \leq \sum_{j=3}^\infty \frac{|m_j|}{j!}(2|\theta|)^j < \infty$.

7 Some details on the proof of Theorem 3.4

We start by checking the existence of the function $\Lambda : \mathcal{X}^* \to [0, \infty]$ defined by (9), where $\mathcal{X}^*$ is the dual space of $\mathcal{X} = C[0, T]$. Later we shall prove the exponential tightness for $\left\{\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k : n \geq 1\right\}$. In the final subsection we shall study the exposed points. In view of what follows it is useful to remark that, by (22), we have

$$\sum_{k=1}^n \frac{1}{k}X_k = \sum_{k=1}^n \frac{1}{k}\frac{\sum_{i=1}^k U_i(\sigma^2\cdot)}{\sqrt{k}} = \sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot). \tag{26}$$

7.1 The proof of (9)

We have to check that

$$\lim_{n\to\infty} \frac{\log \mathbb{E}[e^{\int_0^T \sum_{k=1}^n \frac{1}{k}X_k(t)d\theta(t)}]}{\log n} = 2\sigma^2 \int_0^T \theta^2((r, T])dr \text{ (for all } \theta \in \mathcal{X}^*). \tag{27}$$

Then, by (26), remembering that $\{U_n : n \geq 1\}$ are i.i.d. processes and $U_1(\sigma^2\cdot)$ and $\sigma U_1(\cdot)$ are equally distributed, (27) becomes

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \log \mathbb{E}[e^{\sigma s_{i,n}\int_0^T U_1(t)d\theta(t)}]}{\log n} = 2\sigma^2 \int_0^T \theta^2((r, T])dr \text{ (for all } \theta \in \mathcal{X}^*).$$

Now note that

$$\int_0^T U_1(t)d\theta(t) = \int_0^T \int_0^t dU_1(r)d\theta(t) = \int_0^T \int_r^T d\theta(t)dU_1(r) = \int_0^T \theta((r, T])dU_1(r),$$

whence we obtain

$$\log \mathbb{E}[e^{\sigma s_{i,n}\int_0^T U_1(t)d\theta(t)}] = \log \mathbb{E}[e^{\sigma s_{i,n}\int_0^T \theta((r,T])dU_1(r)}] = \frac{\sigma^2}{2}s_{i,n}^2 \int_0^T \theta^2((r, T])dr,$$

and in turn (27) by (24).

7.2 The exponential tightness for $\left\{\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k : n \geq 1\right\}$

By (26) the exponential tightness condition for $\left\{\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}X_k : n \geq 1\right\}$ can be written as follows: **(ET)**: *For all $R \in (0, \infty)$ there exists a compact set $K_R \subset C[0, T]$ (with respect to the uniform topology) such that*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P\left(\left\{\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot) \notin K_R\right\}\right) \leq -R.$$

Our aim is to find the compact set K_R in **(ET)** following the procedure in [1]. In view of this let us consider the modulus of continuity of $f \in C[0, T]$, i.e.

$$w_f(\eta) := \sup\{|f(t_2) - f(t_1)| : 0 \leq t_1 \leq t_2 \leq 1, t_2 - t_1 < \eta\} \text{ for } \eta > 0.$$

Then, given a sequence $\delta := \{\delta_n : n \geq 1\}$ such that $\delta_n \downarrow 0$ as $n \uparrow \infty$, consider the sets $\{A_{\delta,k} : k \geq 1\}$ defined by $A_{\delta,k} := \left\{f \in C[0, T] : w_f(\delta_k) \leq \frac{1}{k}\right\}$, and the set $A_\delta := \cap_{k\geq 1}A_{\delta,k}$ is compact by the Ascoli-Arzelà Theorem. Our aim is to check **(ET)** choosing $K_R = A_{\delta(R)}$, i.e. choosing for any $R \in (0, \infty)$ the sequence $\delta = \delta(R)$ in a suitable way.

We trivially have $P\left(\left\{\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot)\in A_\delta^c\right\}\right)\leq\sum_{k=1}^\infty P\left(\left\{\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot)\in A_{\delta,k}^c\right\}\right)$ since $A_\delta^c=\cup_{k\geq 1}A_{\delta,k}^c$. Then it suffices to show that, for a suitable choice of $\delta=\delta(R)$, there exists a sequence of positive numbers $\{\beta_n : n\geq 1\}$ such that $\sum_{n\geq 1}\beta_n<\infty$ and

$$P\left(\left\{\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot)\in A_{\delta,k}^c\right\}\right)\leq\beta_k e^{-R\log n}.$$

Now let B_t^k ($k\geq 1$ and $t\in[0,T]$) be the set defined by

$$B_t^k:=\left\{f\in C[0,T]:\sup_{r\in[t,t+\delta_k]}|f(r)-f(t)|>\frac{1}{3k}\right\}.$$

Then, by the triangle inequality, we have $A_{\delta,k}^c=\cup_{j=0}^{T\delta_k^{-1}}B_{j\delta_k}^k$. Thus

$$\begin{aligned}
&P\left(\left\{\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot)\in A_{\delta,k}^c\right\}\right)\\
\leq&\sum_{j=0}^{T\delta_k^{-1}}P\left(\left\{\sup_{r\in[j\delta_k,(j+1)\delta_k]}\left|\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2 r)-\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2 j\delta_k)\right|>\frac{1}{3k}\right\}\right)\\
=&\sum_{j=0}^{T\delta_k^{-1}}P\left(\left\{\sup_{r\in[j\delta_k,(j+1)\delta_k]}\left|\sum_{i=1}^n s_{i,n}\{U_i(\sigma^2 r)-U_i(\sigma^2 j\delta_k)\}\right|>\frac{\log n}{3k}\right\}\right)\\
=&(1+T\delta_k^{-1})P\left(\left\{\sup_{r\in[0,\delta_k]}\left|\sum_{i=1}^n s_{i,n}U_i(\sigma^2 r)\right|>\frac{\log n}{3k}\right\}\right)\\
=&(1+T\delta_k^{-1})P\left(\left\{\sup_{r\in[0,\delta_k]}\left|U_1\left(\sum_{i=1}^n s_{i,n}^2\sigma^2 r\right)\right|>\frac{\log n}{3k}\right\}\right).
\end{aligned}$$

Then, by the Désiré André reflection principle (and noting that U_1 and $-U_1$ are equally distributed) and by a well known estimate for the tail of Gaussian random variables, we have

$$\begin{aligned}
P\left(\left\{\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot)\in A_{\delta,k}^c\right\}\right)\leq&4(1+T\delta_k^{-1})P\left(\left\{U_1\left(\sigma^2\delta_k\sum_{i=1}^n s_{i,n}^2\right)>\frac{\log n}{3k}\right\}\right)\\
=&4(1+T\delta_k^{-1})\frac{1}{\sqrt{2\pi}}\int_{\frac{\log n}{3k\sigma\sqrt{\delta_k\sum_{i=1}^n s_{i,n}^2}}}^\infty e^{-\frac{x^2}{2}}dx\\
\leq&4(1+T\delta_k^{-1})\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{\log^2 n}{18k^2\sigma^2\delta_k\sum_{i=1}^n s_{i,n}^2}}}{\frac{\log n}{3k\sigma\sqrt{\delta_k\sum_{i=1}^n s_{i,n}^2}}}.
\end{aligned}$$

Thus, setting $a_n:=\frac{\sum_{i=1}^n s_{i,n}^2}{\log n}$, we get

$$P\left(\left\{\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot)\in A_{\delta,k}^c\right\}\right)\leq\frac{4}{\sqrt{2\pi}}(1+T\delta_k^{-1})3k\sigma\sqrt{\delta_k}\frac{e^{-\frac{\log n}{18k^2\sigma^2\delta_k a_n}}}{\frac{\sqrt{\log n}}{\sqrt{a_n}}};$$

moreover, since $a_n\to 4$ as $n\to\infty$ by (24), there exist two positive constants C_1 and C_2 such that

$$P\left(\left\{\frac{1}{\log n}\sum_{i=1}^n s_{i,n}U_i(\sigma^2\cdot)\in A_{\delta,k}^c\right\}\right)\leq C_1 k\sqrt{\delta_k}e^{-\frac{\log n}{C_2k^2\delta_k}}$$

for n and k large enough.

Then a suitable choice for the sequence $\delta = \{\delta_n : n \geq 1\}$ is $\delta_n = \frac{1}{n^6}$. Indeed $\sum_{k\geq 1} k\sqrt{\delta_k} = \sum_{k\geq 1} \frac{1}{k^2} < \infty$ and $\frac{1}{C_2 k^2 \delta_k} = \frac{k^4}{C_2} \geq R$ for k large enough.

7.3 The exposed points

We recall that x is an exposed point of Λ^* if there exists an exposing hyperplane θ_x such that

$$\Lambda^*(x) + \int_0^T (z(t) - x(t))d\theta_x(t) < \Lambda^*(z), \text{ for all } z \neq x.$$

Note that, obviously, x is not an exposed point of Λ^*. Then we have to show that this condition holds for any $x \in \mathcal{X}$ such that $\Lambda^*(x) < \infty$. If $\Lambda^*(z) = \infty$ there is nothing to prove. Moreover we can say that, if $\Lambda^*(x) < \infty$, there exists a unique $\theta_x \in \mathcal{X}^*$ such that $\theta_x((r,T]) = \frac{\dot{x}(r)}{4\sigma^2}$ for all $r \in [0,T]$; thus if $\Lambda^*(z) < \infty$ we have

$$\Lambda^*(x) + \int_0^T (z(t) - x(t))d\theta_x(t) = \int_0^T z(t)d\theta_x(t) - \Lambda(\theta_x) < \int_0^T z(t)d\theta_z(t) - \Lambda(\theta_z) = \Lambda^*(z).$$

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