The Theta-dependence Coefficient and an Almost Sure Limit Theorem for Random Iterative Models

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Abstract: We prove a weighted *Almost Sure Limit Theorem* in the setting of *Random Iterative Models*. This Theorem generalizes previous results obtained for sequences of normalized partial sums and some other classes of random sequences.

1. Introduction

Let (S_n) be the partial sums of iid real valued random variables (X_n) with mean 0 and variance 1, defined on a probability space $(\Omega, \mathcal{A}, P)$. The classical *Almost Sure Central Limit Theorem* can be stated as follows: P-almost surely,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\Big(1_A\big(\frac{S_k}{\sqrt{k}}\big) - P\big(\frac{S_k}{\sqrt{k}} \in A\big)\Big) = 0,$$

for all Borel sets $A \subseteq \mathbb{R}$ such that $\lambda(\partial A) = 0$. Here and in the sequel λ denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For the proof, by a classical principle in the theory of pointwise Central Limit Theorem, (see [6], p. 202), it is enough to show that, for any bounded 1-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, almost surely one has

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\Big(f\big(\frac{S_k}{\sqrt{k}}\big) - \mathbf{E}\Big[f\big(\frac{S_k}{\sqrt{k}}\big)\Big]\Big) = 0, \tag{1.1}$$

The proof of (1.1) relies on a suitable estimate of

$$\mathbf{Cov}\Big(f\big(\frac{S_p}{\sqrt{p}}\big), f\big(\frac{S_q}{\sqrt{q}}\big)\Big) = \mathbf{E}\Big[f\big(\frac{S_p}{\sqrt{p}}\big)\, f\big(\frac{S_q}{\sqrt{q}}\big)\Big] - \mathbf{E}\Big[f\big(\frac{S_p}{\sqrt{p}}\big)\Big]\,\mathbf{E}\Big[f\big(\frac{S_q}{\sqrt{q}}\big)\Big],$$

for $p \le q$ integers. Typically such a kind of estimate looks as

$$\Big|\mathbf{Cov}\Big(f\big(\frac{S_p}{\sqrt{p}}\big), f\big(\frac{S_q}{\sqrt{q}}\big)\Big)\Big| \le \text{const}\sqrt{\frac{p}{q}} \tag{1.2}$$

(see for instance [6], Lemma p.203), and it is easy to prove (see Lemma (6.1) of the present paper) that we can get it from an analogous one for

$$\Big|\Big|\mathbf{E}\Big[f\big(\frac{S_q}{\sqrt{q}}\big)\Big]\Big|\frac{S_p}{\sqrt{p}}\Big] - \mathbf{E}\Big[f\big(\frac{S_q}{\sqrt{q}}\big)\Big]\Big|\Big|_1,$$

where $\big|\big|\cdot\big|\big|_1$ denotes the $L^1(\Omega, \mathcal{A}, P)$- norm. Note that the two random variables $\frac{S_{n+1}}{\sqrt{n+1}}$, $\frac{S_n}{\sqrt{n}}$ are linked by the iterative equation

$$\frac{S_{n+1}}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}}\frac{S_n}{\sqrt{n}} + \frac{X_{n+1}}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}}\frac{S_n}{\sqrt{n}} + V_{n+1},$$

Date: on May 25, 2005. *AMS Subject Classification* 2000: Primary 60F15, Secondary 60G12.
Keywords: ϑ-dependence coefficient, Almost Sure Limit Theorem, Random Iterative Model, Lipschitzian Model, Autoregressive Model, Quasi Orthogonal System.
This work is partially supported by the M.I.U.R. Italy.

where $V_{n+1} = \frac{X_{n+1}}{\sqrt{n+1}}$. In this paper we consider a system of random d-dimensional vectors (Z_n) defined on a probability space $(\Omega, \mathcal{A}, P)$ by a recursive relation $Z_{n+1} = F_{n+1}(Z_n, V_{n+1})$ and, under suitable assumptions, we prove an estimate for

$$\vartheta(Z_q, Z_p) \doteq \sup_{f \in \mathcal{L}_1} \left|\left|\mathbf{E}[f(Z_q)|Z_p] - \mathbf{E}[f(Z_q)]\right|\right|_1, \tag{1.3}$$

where $\mathcal{L}_1$ denotes the set of bounded 1-Lipschitz functions $f : \mathbb{R}^d \to \mathbb{R}$. Such an estimate (which will be in terms of the sequence (F_n)) allows us to prove an *Almost Sure Limit Theorem* (*ASLT* from now on) for the sequence (Z_n); this result enlarges the classical *Almost Sure Central Limit Theorem* since it concerns "general" weights and not "logarithmic" weights only; some new particular cases are pointed out in (5.9). Previous results in this direction can be found for instance in [1] and [7]; with respect to the results of [1], our Theorem enlightens the fact that weak theorems are not necessary in order to obtain Almost Sure Limit Theorems; moreover, condition (1.7) of [1] is more difficult to be checked than our condition (2.11) (see Remark (2.12)); last, our examples (4.1) (in particular (4.1)(iii) and 4.1(iv)), (4.4), (4.5) are new (see also Example (5.9)). On the other hand, with respect to [7] our Theorem is wider in that it concerns a general "Iterative Model" (Z_n) (see definition (2.1)), and not only a sequence of normalized partial sums $(S_n/\sqrt{n})$.
We stress the fact that the setting of iterative models considered here is rather large: see section 4 for some illuminating examples. The coefficient $\vartheta(Z_q, Z_p)$ defined in (1.3) is clearly a measure of the dependence of Z_q and Z_p. It is known in the literature (see [2] for details an the references therein); we shall call it *coefficient of ϑ-dependence*; in sections 3 and 4 we show how to calculate it in some cases. Another frequently used measure of dependence between random variables is the Rosenblatt coefficient (see section 6 for its definition); in the same section 6 we present a typical situation in which the Rosenblatt coefficient can be obtained from the ϑ-coefficient.

Acknowledgement. The authors wish to thank Prof. L. Pratelli for helpful suggestions and an unknown referee for having pointed out an error in the proof of Theorem (2.9).

2. Main results

On the probability space $(\Omega, \mathcal{A}, P)$ we are given a filtration $(\mathcal{A}_n)$. Following [3], p. 183 we introduce the concept of *Iterative Lipschitzian Model*:

(2.1) DEFINITION. *An* **Iterative Lipschitzian Model** *adapted to $(\mathcal{A}_n)$ is a sequence (Z_n) of random d-dimensional vectors such that, for every $n \in \mathbb{N}$, Z_n is $\mathcal{A}_n$-measurable and*

$$Z_{n+1} = F_{n+1}(Z_n, V_{n+1}) \tag{2.2}$$

where,

(i) for every $n \in \mathbb{N}$, F_n is a measurable function from $\mathbb{R}^d \times \Gamma$ to $\mathbb{R}^d$ (where $(\Gamma, \mathcal{G})$ is a measurable space), α_n- Lipschitzian in its first argument, independent on the second, i. e., for each $z_1, z_2 \in \mathbb{R}^d$, $v \in \Gamma$

$$|F_n(z_1, v) - F_n(z_2, v)| \le \alpha_n |z_1 - z_2|.$$

Here and in the sequel $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$.

(ii) (V_n) is a sequence of random variables with values in $(\Gamma, \mathcal{G})$, and V_{n+1} is independent of $\mathcal{A}_n$ for every $n \in \mathbb{N}$.

Our first result is the following.

(2.3) THEOREM. *Assume that* $\sup_n \mathbf{E}[|Z_n|] = C < +\infty$. *Then, for $p < q$ integers, we have*

$$\vartheta(Z_q, Z_p) \le 2\, C \alpha_{p+1} \times \cdots \times \alpha_q. \tag{2.4}$$

Write

$$g(k) = \prod_{h=1}^{k} \frac{1}{\alpha_h}. \tag{2.5}$$

Then (2.4) can be written as

$$\vartheta(Z_q, Z_p) \leq 2C\, \frac{g(p)}{g(q)}. \tag{2.6}$$

The proof of Theorem (2.3) is in Section 3. Before stating our second result (the *ASLT* for the sequence (Z_n)) we must recall some preliminary notions and make some remarks.

(i) Let $(T, \mathcal{C}, \tau)$ be some probability space and consider a sequence (f_n) of elements of $L^2(\tau)$. Let $a_{h,k} = \int f_h f_k d\tau$. A system of functions (f_n) such that the quadratic form defined on ℓ_2 by $(x_n) \mapsto \sum_{h,k} a_{h,k} x_h x_k$ is bounded, is said *quasi orthogonal.* Say also that a sequence $c = (c_n) \in \ell_2$ is *universal* when the series $\sum_k c_k \psi_k$ converges almost everywhere for every orthonormal system of functions (ψ_n). According to Schur's Theorem ([8], pag. 56), if c is universal, then the series $\sum_k c_k f_k$ converges almost everywhere for any quasi-orthogonal system of functions (f_n).

(ii) Assume that (Z_n) is an iterative Lipschitzian model, such that

$$\alpha_n < 1 \qquad \forall\, n \geq 2, \qquad \sum_{k=2}^{\infty} \log \alpha_k = -\infty \tag{2.7}$$

$$\liminf_{n\to\infty} \alpha_n > 0. \tag{2.8}$$

Condition (2.7) amounts clearly to assuming that g, defined in (2.5), is strictly increasing to $+\infty$, so that we can suppose that it defined on $[1, +\infty)$ and strictly increasing to $+\infty$. On the other hand, condition (2.8) is equivalent to $\limsup\limits_{x\to\infty} g(x+1)/g(x) < +\infty$. We can now state our result.

(2.9) Theorem. (**ASLT for iterative models**). *Let (Z_n) be an iterative Lipschitzian model such that $\sup_n \mathbf{E}[|Z_n|] = C < +\infty$. Let $\varphi : [1, +\infty) \to \mathbb{R}^+$ be a strictly increasing function with*

$$\lim_{x\to+\infty} \varphi(x) = +\infty$$

and for which there exists a constant $\beta > 0$ such that

$$\varphi(x+1) \leq \varphi(x) + \beta \qquad \forall x \in \mathbb{R}^+. \tag{2.10}$$

Assume moreover that the composed function $G = g \circ \varphi^{-1}$ verifies the condition

$$\sup_n \Big(\sum_{k\leq n} \frac{G(k)}{G(n)} + \sum_{k>n} \frac{G(n)}{G(k)} \Big) < +\infty. \tag{2.11}$$

Let $\mathcal{E} = \{A \in \mathcal{B}(\mathbb{R}^d), \lambda(\partial A) = 0\}$. Then, for every $A \in \mathcal{E}$

(a) *for every decreasing sequence (c_n) satisfying the condition $\liminf_{n\to\infty}(c_{n+1}/c_n) > 0$ and such that $\sum_n c_n^2 (\log n)^2 < \infty$ we have, almost surely,*

$$\lim_{n\to\infty} c_{[\varphi(n)]} \sum_{k=1}^{n} (\varphi(k+1) - \varphi(k))(1_A(Z_k) - P(Z_k \in A)) = 0;$$

(b) *almost surely we have*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} (\varphi(k+1) - \varphi(k))(1_A(Z_k) - P(Z_k \in A))}{\varphi(n)} = 0.$$

(2.12) Remark. If the iterative model (Z_n) satisfies (2.7) and (2.8), a function φ verifying the two conditions (2.10) and (2.11) is easily found: in fact we can take, for instance

$$\varphi(x) = \log g(x).$$

The proofs of Theorem (2.9) and Remark (2.12) will be given in Section 5.

3. The coefficient of ϑ-dependence and proof of Theorem (2.3)

Theorem (2.3) follows easily from a general result (Proposition (3.2)), which we state and prove in this section. Let T and S be two d-dimensional random vectors defined on a probabilty space $(\Omega, \mathcal{A}, P)$ and assume that $\mathbf{E}[|T|] < +\infty$; let $\mathcal{L}_1$ be the set of bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ which are Lipschitzian of constant 1.

(3.1) Definition. *For T and S as above, we define the ϑ-**coefficient of dependence** as*

$$\vartheta(T, S) \doteq \sup_{f \in \mathcal{L}_1} \big|\big|\mathbf{E}[f(T)|S] - \mathbf{E}[f(T)]\big|\big|_1,$$

where $\big|\big| \cdot \big|\big|_1$ denotes the norm in $L^1(\Omega, \mathcal{A}, P)$.

We are interested in the following situation: assume that $T = \phi(S, V)$, where V is a random variable defined on $(\Omega, \mathcal{A}, P)$ with values in a measurable space $(\Gamma, \mathcal{G})$, independent on S, and $\phi : \mathbb{R}^d \times \Gamma \to \mathbb{R}^d$ is a measurable function β-Lipschitzian in its first argument, independent on the second, i. e. for every $s_1, s_2 \in \mathbb{R}^d$ and $v \in \Gamma$ we have

$$|\phi(s_1, v) - \phi(s_2, v)| \leq \beta|s_1 - s_2|.$$

We prove our general result:

(3.2) Proposition. *Assume that S is integrable. Then*

$$\vartheta(T, S) \leq 2\beta\mathbf{E}[|S|].$$

Proof. It is easy to see that, for every $f \in \mathcal{L}_1$ we have

$$\mathbf{E}[f(T)|S] = \int f \circ \phi(S, v) d\mu_V(v),$$

where μ_V denotes the law of V. Hence

$$\vartheta(T, S) = \sup_{f \in \mathcal{L}_1} \big|\big| \int f \circ \phi(S, v) d\mu_V(v) - \mathbf{E}[\int f \circ \phi(S, v) d\mu_V(v)]\big|\big|_1.$$

The relation

$$\big| \int f \circ \phi(s_1, v) d\mu_V(v) - \int f \circ \phi(s_2, v) d\mu_V(v)\big| \leq \int |\phi(s_1, v) - \phi(s_2, v)| d\mu_V(v) \leq \beta|s_1 - s_2|,$$

shows that the function $s \mapsto \int f \circ \phi(s, v) d\mu_V(v)$ is β-Lipschitzian, hence

$$\vartheta(T, S) \leq \beta \sup_{g \in \mathcal{L}_1} \big|\big|g(S) - \mathbf{E}[g(S)]\big|\big|_1 = \beta \sup \big\{ \int |g| d\mu_S, g \in \mathcal{L}_1, \int g d\mu_S = 0\big\},$$

where μ_S is the law of S. The statement of the proposition follows from a simple lemma.

(3.3) Lemma. *Let μ be a probability measure on $\mathbb{R}^d$, with $\int |x| d\mu(x) < \infty$. Then*

$$\sup \big\{ \int |g| d\mu, g \in \mathcal{L}_1, \int g d\mu = 0\big\} \leq 2 \int |x| d\mu(x).$$

Proof. Let $g \in \mathcal{L}_1$ with $\int g d\mu = 0$. Then

$$|g(0)| = |\int (g(x) - g(0)) d\mu(x)| \leq \int |x| d\mu(x), \tag{3.4}$$

hence $\int |g(x)| d\mu(x) \leq \int |g(x) - g(0)| d\mu(x) + |g(0)| \leq 2 \int |x| d\mu(x)$, and the lemma is proved.

(3.5) Remark. Though not relevant in this context, note that in (3.3) it is possible to find a better estimate; in fact, one can replace the vector 0 (used in (3.4)) with any vector $x_0 \in \mathbb{R}^d$ and then take the infimum with respect to x_0, so getting the bound

$$2 \inf_{x_0 \in \mathbb{R}^d} \int |x - x_0| d\mu(x).$$

We can now deduce Theorem (2.3) from Proposition (3.2); in fact from relation (2.2) it is easily seen by induction that

$$Z_q = \phi_{q-p}(Z_p, V_{p+1}, \ldots, V_q)$$

where ϕ_{q-p} is some function, $(\alpha_{p+1} \times \cdots \times \alpha_q)$-Lipschitzian in the first argument and $(V_{p+1}, \ldots, V_q)$ is a random variable with values in $(\Gamma^{q-p}, \mathcal{G}^{q-p})$, independent on Z_p (of course $\phi_1 = F_{p+1}$ for every p).

(3.6) Remark. We point out the important particular case (to be encountered later, see Example (4.5)) in which $\alpha_n \leq \alpha$ (constant) for every n. In this case we find

$$\vartheta(Z_q, Z_p) \leq 2\, C\, \alpha^{q-p}.$$

4. Some examples

In this section we give some relevant examples of iterative Lipschitzian models. All the involved sequences of random variables are tacitly assumed to be defined on the basic probability space $(\Omega, \mathcal{A}, P)$.

(4.1) Example. Let (X_n) be a sequence of independent r. v.'s and (γ_n) a sequence of positive numbers. Put $S_n = \sum_{k=1}^n \gamma_k X_k$ and assume that, for every n, $S_n \in L^1(\Omega, \mathcal{A}, P)$. Suppose that there exists a sequence (a_n) of real numbers such that

$$\sup_n \mathbf{E}\big[\big|\frac{S_n}{a_n}\big|\big] = C < +\infty.$$

For every integer n define $Z_n = \frac{S_n}{a_n}$ and consider the maps $F_n(z, v) = \frac{a_{n-1}}{a_n} z + v$. Observe that

$$Z_{n+1} = F_{n+1}(Z_n, V_{n+1}), \qquad \text{with} \qquad V_{n+1} = \frac{\gamma_{n+1} X_{n+1}}{a_n}$$

Theorem (2.3) gives, for $p < q$,

$$\vartheta(Z_q, Z_p) \leq 2\, C\, \frac{a_p}{a_q}. \tag{4.2}$$

We are in the above setting if for instance

(*i*) $\sup_n E[|X_n|] < +\infty$.

In this case we can take $a_n = \sum_{k=1}^n \gamma_k$, as it is easily seen.

(*ii*) If $\sigma_n^2 = \mathbf{E}[X_n^2] < +\infty$ for every n, another suitable choice for (a_n) is $a_n = \Big(\sum_{k=1}^n \sigma_k^2 \gamma_k^2\Big)^{1/2}$.

(*iii*) $\gamma_n = 1$ for every n, (X_n) are independent identically distributed and their common distribution belongs to the domain of attraction of a stable distribution Φ with exponent $\alpha \in (1, 2]$. This means that there exist two sequences of numbers (a_n) and (b_n) such that the distribution of $\frac{S_n}{a_n} - b_n = Z_n - b_n$ tends to Φ. In this case it is known (see [5], lemma 2.3) that $\mathbf{E}[|S_n|] \leq C a_n$ for a suitable constant C.

(*iv*) Let $p > 1$, and consider the class $\mathcal{F}_p$ of distribution functions verifying

$$\big(F(-x) \vee (1 - F(x)\big) = \mathcal{O}\big(x^{-p}\big) \qquad x \to +\infty.$$

Let again $\gamma_n = 1$ for every n, (X_n) be independent identically distributed with their common law belonging to $\mathcal{F}_p$. According to part b) of Lemma 2.2 in [5], if $F \in \mathcal{F}_p$, $1 < p < 2$, then $\mathbf{E}\big|S_n\big| \leq C_p n^{1/p}$. And if $p = 2$ and $\mathbf{E}X^2 = \infty$, then $\mathbf{E}\big|S_n\big| \leq C\big(n\mathbf{E}X^2 \mathbf{1}_{\{|X| \leq n^{1/2}\}}\big)^{1/2}$, which gives other examples of the same kind as above.

(4.3) Example. Let (X_n) be a sequence of independent random variables with $\sup_n \mathbf{E}[|X_n|] = C < +\infty$. Put $M_n = \max(X_1, \dots, X_n), Z_n = \frac{M_n}{n}$. Then $\mathbf{E}[|Z_n|] \leq \frac{\sum_{k=1}^n \mathbf{E}[|X_k|]}{n} \leq C < +\infty$.
Put $F_n(z, v) = \max\left(\frac{n-1}{n} z, \frac{v}{n}\right)$. Then F_n is $((n-1)/n)$-Lipschitzian and we have $Z_{n+1} = F_{n+1}(Z_n, V_{n+1})$, with $V_{n+1} = \frac{X_{n+1}}{n+1}$. Theorem (2.3) applies and we get

$$\vartheta(Z_q, Z_p) \leq C\,\frac{p}{q}.$$

(4.4) Example. Let X_n be a sequence of independent identically distributed complex-valued random variables and let $f_1, f_2, \dots$ be complex-valued functions defined on some metric space $(\mathbf{T}, \mathbf{d})$, and form the quantities

$$Z_n(t) = \frac{\sum_{k=1}^{N_n} \Re(X_k \overline{f_k(t)})}{B_n(t)}, \qquad B_n(t) = \|\sum_{k=1}^{N_n} \Re(X_k \overline{f_k(t)})\|_{2,P}.$$

where (N_n) is some given sequence of integers. Here t is *fixed* and we write more simply $Z_n := Z_n(t)$, $B_n := B_n(t)$. It is clear that

$$Z_{n+1} = \frac{B_n}{B_{n+1}} Z_n + V_{n+1}, \qquad V_{n+1} = \frac{\sum_{k=N_n+1}^{N_{n+1}} \Re(X_k \overline{f_k(t)})}{B_{n+1}}$$

Define $F_{n+1}(x, y) = \frac{B_n}{B_{n+1}} x + y$. Then we have $Z_{n+1} = F_{n+1}(Z_n, V_{n+1})$. According to Definition (2.1), (Z_n) is an iterative Lipschitzian model. Now by contruction $\sup_n \mathbf{E}|Z_n| = C < \infty$. Indeed, by Cauchy-Schwarz inequality, $\mathbf{E}[|Z_n(t)|] = \mathbf{E}\big[\big|\sum_{k=1}^{N_n} \Re(X_k \overline{f_k(t)})\big|\big]/B_n(t) \leq 1$ so that Theorem (2.3) applies in force with $C = 1$, and we get for $p \leq q$:

$$\vartheta(Z_p, Z_q) \leq 2\frac{B_p}{B_q}.$$

(4.5) Example. Let (M_n) be a sequence of square $d \times d$ matrices with elements in $\mathbb{R}$, and let $||M_n||$ be the sequence of their norms. We assume that $\rho \doteq \sup_n ||M_n|| < 1$. We consider the autoregressive model

$$Z_{n+1} = F_{n+1}(Z_n, \varepsilon_{n+1}) = M_n Z_n + \varepsilon_{n+1},$$

with initial state Z_0, where $\varepsilon = (\varepsilon_n)$ is a noise independent on Z_0. We assume that $|Z_0|$ is integrable. Moreover the (ε_n) are independent and $\sup_n \mathbf{E}[|\varepsilon_n|] = C < +\infty$. By induction one sees easily that Z_n and ε_{n+1} are independent. Put

$$B_j = \begin{cases} I_d & \text{for } j = 0; \\ \prod_{k=n-j+1}^n M_k & \text{for } 1 \leq j \leq n. \end{cases}$$

Then Z_n can be written in the closed form $Z_n = B_n Z_0 + \sum_{k=1}^n B_{n-k} \varepsilon_k$. We have

$$\sup_n \mathbf{E}[|X_n|] \leq \rho^n \mathbf{E}[|Z_0|] + C\sum_{k=1}^n \rho^{n-k} \leq \mathbf{E}[|Z_0|] + C\frac{1}{1-\rho} = C_1.$$

Since F_n is $||M_n||$-Lipschitzian, we deduce from Theorem (2.3) that

$$\vartheta(Z_q, Z_p) \leq C_1 \rho^{q-p}, \qquad q \geq p.$$

5. Proof of Theorem (2.9)

As we saw in the Introduction, in order to prove (2.9) it is enough to prove an analogous result by substituting 1_A with any bounded function $f \in \mathcal{L}_1$. So, fix $f \in \mathcal{L}_1$. Throughout the present section we put, for every integer n

$$Y_n = f(Z_n) - \mathbf{E}[f(Z_n)].$$

Assertion (b) of Theorem (2.9) easily follows easily from assertion (a): take the sequence (c_n) defined by $c_n = \frac{1}{\sqrt{n}\log^2 n}$, and observe that (c_n) is universal by Rademacher-Menchov Theorem, (asserting that a sequence (c_n) is universal if $\sum_n c_n^2 \log^2 n < +\infty$). Now we have

$$\frac{\left|\sum_{k=1}^n (\varphi(k+1)-\varphi(k))Y_k\right|}{\varphi(n)} \leq \frac{\log^2 \varphi(n)}{\sqrt{\varphi(n)}} \left| c_{[\varphi(n)]} \sum_{k=1}^n (\varphi(k+1)-\varphi(k))Y_k \right|,$$

and the second term of the above inequality tends to 0 by assertion (a); hence we prove assertion (a). With no loss of generality we can assume $\beta = 1$ in (2.10). This is plain if $\beta \leq 1$: if $\beta > 1$, we prove first the result for the function $\tilde{\varphi} \doteq \beta^{-1}\varphi$; in order to get the desired conclusion for the function φ also, we need only to observe that

$$c_{[\varphi(n)]} \leq c_{[\varphi(n)\beta^{-1}]}.$$

We now observe that the relation

$$\varphi(x+1) \leq \varphi(x) + 1 \qquad \forall x \in \mathbb{R}^+$$

implies (in fact, is equivalent) to

$$\varphi^{-1}(x) + 1 \leq \varphi^{-1}(x+1) \qquad \forall x \in \mathbb{R}^+. \tag{5.1}$$

Put $\psi(k) = [\varphi^{-1}(k)]$ for every integer k. From (5.1) we get also

$$\psi(k) + 1 \leq \psi(k+1) \qquad \forall k \in \mathbb{N}^*. \tag{5.2}$$

Relation (5.2) implies in turn

$$\varphi\big(\psi(k)\big) - \varphi\big(\psi(k-1)\big) \leq \varphi\big(\psi(k)\big) - \varphi\big(\psi(k-2)+1\big) \leq \varphi\big(\varphi^{-1}(k)\big) - \varphi\big(\varphi^{-1}(k-2)\big) = 2. \tag{5.3}$$

We now need two Lemmas.

(5.4) Lemma. *Let $p \leq q$ be two integers. Then, for every $g \in \mathcal{L}_1$ the following inequality holds*

$$|\mathbf{Cov}(Y_p, Y_q)| \leq \sup|g|\, \vartheta(Z_p, Z_q).$$

The proof of the above Lemma is quite similar to the proof of Lemma (6.1), and is omitted.

(5.5) Remark. Note that the function g defined by the formula $g(x) = \big(f(x) - \mathbf{E}[f(Z_n)]\big)^+$ belongs to $\mathcal{L}_1$; moreover $\sup|g| \leq 2\sup|f|$. Thus, Lemma (5.4) implies that

$$|\mathbf{Cov}(Y_p^+, Y_q^+)| \leq 2\sup|f|\, \vartheta(Z_p, Z_q). \tag{5.6}$$

By (5.6), with no loss of generality we can assume in what follows that $Y_n \geq 0$ for all n: if this is not true, it is sufficient to write $Y_n = Y_n^+ - Y_n^-$.

(5.7) Lemma. *We have $\lim_{n\to\infty} c_n \sum_{k=1}^n Y_{\psi(k)} = 0$, P-almost surely.*

Proof. For $h \leq k$ we have, by Lemma (5.4) and relations (2.6) and (5.2),

$$|\mathbf{Cov}(Y_{\psi(h)}, Y_{\psi(k)})| \leq 2\,C\,\frac{G(h)}{G(k-1)}.$$

Now condition (2.11) assures that the sequence $\big(Y_{\psi(n)}\big)$ is quasi orthogonal by lemma 7.4.3 p. 139 of [10]. The result thus follows from Kronecker's Lemma. ■

We can now pass to the proof of point (a) of (2.9). For every n put

$$U_n = c_n \sum_{k=1}^{n} Y_{\psi(k)}, \quad V_n = c_n \sum_{k=1}^{\psi(n)-1} (\varphi(k+1) - \varphi(k))Y_k, \quad T_n = c_{[\varphi(n)]} \sum_{k=1}^{n} (\varphi(k+1) - \varphi(k))Y_k.$$

It is easily verified that (V_n) converges to 0 iff (T_n) does; in fact, for $\psi(r) \le n \le \psi(r+1) - 1$ we have

$$V_r \frac{c_{r+1}}{c_r} \le T_n \le V_{r+1} \frac{c_{r-1}}{c_{r+1}}.$$

Hence, by Lemma (5.7), it is enough to prove that $(U_n - V_n)$ converges to 0 almost surely. Now

$$\begin{aligned} U_n - V_n &= c_n \Big(\sum_{k=1}^{n} Y_{\varphi^{-1}(k)} - \sum_{k=1}^{\psi(n)-1} (\varphi(k+1) - \varphi(k))Y_k \Big) \\ &= c_n \Big(\sum_{k=1}^{n} Y_{\varphi^{-1}(k)} - \sum_{k=1}^{n} \sum_{j=\psi(k-1)}^{\psi(k)-1} (\varphi(j+1) - \varphi(j))Y_j \Big) = c_n \sum_{k=1}^{n} \sum_{j=\psi(k-1)}^{\psi(k)} \delta_j Y_j \ , \end{aligned}$$

where we put

$$\delta_j = \begin{cases} \varphi(j) - \varphi(j+1) & \text{for } \psi(k-1) \le j \le \psi(k) - 1, \\ 1 & \text{for } j = \psi(k). \end{cases}$$

We have easily, by (5.3)

$$\sum_{j=\psi(k-1)}^{\psi(k)} |\delta_j| = 1 + \sum_{j=\psi(k-1)}^{\psi(k)-1} (\varphi(j+1) - \varphi(j)) = 1 + \big[\varphi\big(\psi(k)\big) - \varphi\big(\psi(k-1)\big)\big] \le 3. \tag{5.8}$$

Put now $R_k = \sum_{j=\psi(k-1)}^{\psi(k)} \delta_j Y_j$. We have to prove that $\lim_{n\to\infty} c_n \sum_{k=1}^{n} R_k = 0$ almost surely.

We need a bound for $\mathbf{Cov}(R_h, R_k)$. We have

$$\mathbf{Cov}(R_h, R_k) = \sum_{i=\psi(h-1)}^{\psi(h)-1} \sum_{j=\psi(k-1)}^{\psi(k)-1} \delta_i \delta_j \mathbf{Cov}\big(Y_i, Y_j\big).$$

Now, for every i, j with $\psi(h-1) \le i \le \psi(h)$, $\quad \psi(k-1) \le j \le \psi(k)$, we have, again by (5.4) and (2.6),

$$|\mathbf{Cov}(Y_i, Y_j)| \le C_1 \frac{G(h)}{G(k-2)}.$$

Condition (2.11) and relation (5.8) assure that the sequence (R_n) is quasi orthogonal, and we can now argue as in the proof of Lemma (5.7). The Theorem is proved. ■

(5.9) Examples We give here some particular cases:

(i) We refer to Example (4.1) (i). Assume that $\gamma_n = n^\beta$, where $\beta > -1$. Then Theorem (2.9) gives

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \big(1_A\big(Z_k\big) - P\big(Z_k \in A\big)\big) = 0.$$

On the other hand, in the case $\gamma_n = n^{-1}$ we get $\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \frac{1}{k \log k} \big(1_A\big(Z_k\big) - P\big(Z_k \in A\big)\big) = 0$.

(ii) We refer here to Example (4.5). In this case it is easy to see that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \big(1_A(Z_k) - P(Z_k \in A)\big) = 0$.

We conclude this section by proving Remark (2.12). For $\varphi(x) = \log g(x)$ we have $\varphi^{-1}(x) = g^{-1}(e^x)$, hence $G(x) = e^x$. Condition (2.11) is verified since

$$\sum_{k\le n}\frac{G(k)}{G(n)} + \sum_{k>n}\frac{G(n)}{G(k)} = \sum_{k\le n}\frac{e^k}{e^n} + \sum_{k>n}\frac{1}{e^{k-n}};$$

now the first sum is equal to $\frac{e^{n+1}-1}{e^n(e-1)}$, which is bounded as $n\to\infty$, while the second sum is equal to $\sum_j e^{-j} < \infty$.

6. From the ϑ-dependence coefficient to the Rosenblatt coefficient

Let T and S be two r. v.'s defined on $(\Omega, \mathcal{A}, P)$. The coefficient of ϑ-dependence $\vartheta(T,S)$ is useful in some cases in order to estimate the Rosenblatt coefficient of dependence of S and T, defined as

$$\alpha(T,S) \doteq \sup_{A,B} |\mathbf{Cov}(1_A(S), 1_B(T))|,$$

where the sup is taken over all Borel sets in $\mathbb{R}$.

(6.1) Lemma. *Let A be any Borel set in $\mathbb{R}$. Then,*

$$\sup_{f\in\mathcal{L}_1} |\mathbf{Cov}(1_A(S), f(T))| \le \vartheta(T,S).$$

Proof. For any function $f : \mathbb{R}^d \to \mathbb{R}$ such that $f(T)$ is integrable we have

$$\begin{aligned}|\mathbf{Cov}(1_A(S), f(T))| &= |\mathbf{E}[1_A(S)f(T)] - \mathbf{E}[1_A(S)]\mathbf{E}[f(T)]| \\ &= |\mathbf{E}[1_A(S)\mathbf{E}[f(T)|S]] - \mathbf{E}[1_A(S)\mathbf{E}[f(T)]]| \\ &= |\mathbf{E}[1_A(S)\big(\mathbf{E}[f(T)|S] - \mathbf{E}[f(T)]\big)]| \le \big|\big|(\mathbf{E}[f(T)|S] - \mathbf{E}[f(T)]\big|\big|_1.\end{aligned}$$

■

(6.2) Remark. If g is L-Lipschitzian, since g/L is in $\mathcal{L}_1$ we have from (6.1)

$$|\mathbf{Cov}(1_A(S), g(T))| = L|\mathbf{Cov}(1_A(S), \frac{g}{L}(T))| \le L \sup_{f\in\mathcal{L}_1} |\mathbf{Cov}(1_A(S), g(T))| \le L\,\vartheta(T,S).$$

(6.3) Proposition. *Let $Q_T(\epsilon) = \sup_x P(x < T \le x+\epsilon)$, $\epsilon > 0$. be the concentration function of T. Then for every $x \in \mathbb{R}$,*

$$|\mathbf{Cov}(1_A(S), 1_{(-\infty,x]}(T))| \le \inf_\epsilon \Big(\frac{1}{\epsilon}\vartheta(T,S) + Q_T(\epsilon)\Big). \tag{6.4}$$

Proof. Fix $\epsilon > 0$. Put $g_\epsilon(t) = \big(1 + \frac{x-t}{\epsilon} 1_{(x,x+\epsilon]}(t)\big)$ and consider the $(1/\epsilon)$-Lipschitz function $f_\epsilon(t) = 1_{(-\infty,x]}(t) + g_\epsilon(t)$. In view of Remark (6.2), we have

$$\begin{aligned}|\mathbf{Cov}(1_A(S), 1_{(-\infty,x]}(T))| &= |\mathbf{Cov}(1_A(S), f_\epsilon(T)) - \mathbf{Cov}(1_A(S), g_\epsilon(T))| \\ &\le |\mathbf{Cov}(1_A(S), f_\epsilon(T))| + |\mathbf{Cov}(1_A(S), g_\epsilon(T))| \le \frac{1}{\epsilon}\theta(T,S) + Q_T(\epsilon).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the proof is achieved. ■

We now consider a case in which the infimum in (6.4) can be explicitly calculated. Assume that T is such that, for some fixed $\gamma > 0$ and for every $\epsilon > 0$,

$$Q_T(\epsilon) \le C_1\epsilon^\gamma. \tag{6.5}$$

(6.6) Proposition. *We have*

$$\sup_{A,x} |\mathbf{Cov}(1_A(S), 1_{(-\infty,x]}(T))| \le C_2(\gamma)\big(\vartheta(T,S)\big)^{\gamma/(\gamma+1)}. \tag{6.7}$$

Proof. We introduce the bound of Q_T given in (6.5) into the infimum in (6.4). Then such infimum can be found by means of elementary calculus. It is attained for $\epsilon = \big(\vartheta(T,S)/C_1\big)^{1/(\gamma+1)}$ proving (6.7). ■

(6.8) REMARK. Assumption (6.5) is simply γ-holderianity of the distribution function of T, since

$$P(x < T \le x + \epsilon) = F_T(x + \epsilon) - F_T(x).$$

If the law of T has a bounded density, then (6.5) holds with $\gamma = 1$. A uniform version of (6.5) for a sequence of random variables is considered in the paper [2] (formula (2.9) pag. 317).

We now discuss another relevant case in which (6.5) holds. Let (X_n) be a sequence of independent identically distributed random variables with distribution belonging to the domain of attraction of a stable distribution Φ with exponent $\alpha \in (1, 2]$ (see section 4, example (4.1) (iii)). Put, as in section 4, $Z_n = \frac{S_n}{a_n}$. For $p < q$ we consider $T = Z_q$ and $S = Z_p$, so that our task is to bound the concentration function of Z_q. This can be done by using the following result (see [9], pag 68 for the proof).

(6.9) LEMMA. *Let (X_n) be a sequence of independent random variables and put $S_n = X_1 + \cdots + X_n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive numbers such that $\lambda_k \le \lambda$, $k = 1, \ldots, n$. Let (X_k^s) denotes a symetrized version of (X_k). Then*

$$Q_{S_n}(\lambda) \le C_3\lambda\Big(\sum_{k=1}^{n} \lambda_k 2P\{|X_k^s| \ge (\lambda_k/2)\}\Big)^{-1/2}, \tag{6.10}$$

For each $k = 1, \ldots, q$ take $\lambda_k = \lambda = a_q\epsilon$ in (6.10). We get

$$\begin{aligned} Q_{Z_q}(\epsilon) = Q_{S_q}(a_q\epsilon) &\le C_3 a_q \epsilon\big(a_q^2\epsilon^2 q\big(1 - F(a_q(\epsilon/2)) + F(-a_q(\epsilon/2))\big)\big)^{-1/2} \\ &= C_3\big(q\big(1 - F(a_q(\epsilon/2)) + F(-a_q(\epsilon/2))\big)\big)^{-1/2}. \end{aligned}$$

By formulas (5.6) and (5.9) p. 575 of [4] we know that $\lambda^\alpha q\big(1 - F(a_q\lambda) + F(-a_q\lambda)\big) \to C_4 > 0$. Hence, for large q we obtain $Q_{Z_q}(\epsilon) \le C_5\epsilon^{\alpha/2}$. ■

In [5] the following statement is proved:

(6.11) PROPOSITION. *For large q and every $p \le q$ we have*

$$\sup_{A,x} |\mathbf{Cov}(1_A(Z_p), 1_{(-\infty,x]}(Z_q))| \le C_\alpha\Big(\frac{a_p}{a_q}\Big)^{\alpha/(\alpha+2)}.$$

It is also a consequence of Proposition (6.6) and formula (4.2) above.

References

[1] Berkes I., Csáki E. (2001) *A universal result in almost sure central limit theory,* Stoch. Proc. Appl. **94**, 105–134.

[2] Dedecker J., Doukhan P. (2003) *A new covariance inequality and applications,* Stoch. Proc. Appl. **106**, 63–80.

[3] Duflo M. (1997) *Random Iterative Models*, Springer.

[4] Feller W. (1971) *An Introduction to Probability Theory and Its Applications,* Wiley.

[5] Giuliano-Antonini R., Weber M. (2005) *Counting Occurrences in Almost Sure Limit Theorems,* Coll. Math. **102**, no. 2, 271–289 .

[6] Lacey M., Philipp W. (1990) *A note on the almost sure central limit theorem*, Stat. & Prob. Letters **9**, 201–205.

[7] Móri T. F., Székely B. (2003) *Almost sure convergence of weighted partial sums*, Acta Math. Hungar. **99** (**4**), 285–303.

[8] Olevskii A. M. (1975) *Fourier series with respect to general orthogonal systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **86**.

[9] Petrov V. V. (1995) *Limit Theorems of Probability Theory,* Oxford Science Publications.

[10] Weber M. (1998) *Entropie métrique et convergence presque partout,* Coll. "Travaux en Cours", **58**, Hermannn, Paris.

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