# Kirchhoff equations in generalized Gevrey spaces: local existence, global existence, uniqueness

Marina Ghisi

Università degli Studi di Pisa
Dipartimento di Matematica "Leonida Tonelli"
PISA (Italy)
e-mail: `ghisi@dm.unipi.it`

Massimo Gobbino

Università degli Studi di Pisa
Dipartimento di Matematica Applicata "Ulisse Dini"
e-mail: `m.gobbino@dma.unipi.it`

**Abstract**

In this note we present some recent results for Kirchhoff equations in generalized Gevrey spaces. We show that these spaces are the natural framework where classical results can be unified and extended. In particular we focus on existence and uniqueness results for initial data whose regularity depends on the continuity modulus of the nonlinear term, both in the strictly hyperbolic case, and in the degenerate hyperbolic case.

# 1   Introduction

Let $H$ be a separable real Hilbert space. For every $x$ and $y$ in $H$, $|x|$ denotes the norm of $x$, and $\langle x, y \rangle$ denotes the scalar product of $x$ and $y$. Let $A$ be a self-adjoint linear operator on $H$ with dense domain $D(A)$. We assume that $A$ is nonnegative, namely $\langle Ax, x \rangle \geq 0$ for every $x \in D(A)$, so that for every $\alpha \geq 0$ the power $A^{\alpha}x$ is defined provided that $x$ lies in a suitable domain $D(A^{\alpha})$.

Given a continuous function $m : [0, +\infty) \to [0, +\infty)$ we consider the Cauchy problem

$$u''(t) + m(|A^{1/2}u(t)|^2)Au(t) = 0 \qquad \forall t \in [0, T), \tag{1.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{1.2}$$

It is well known that (1.1), (1.2) is the abstract setting of the Cauchy-boundary value problem for the quasilinear hyperbolic integro-differential partial differential equation

$$u_{tt}(t, x) - m\left(\int_{\Omega} |\nabla u(t, x)|^2 \, dx\right)\Delta u(t, x) = 0 \qquad \forall (x, t) \in \Omega \times [0, T), \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, and $\nabla u$ and $\Delta u$ denote the gradient and the Laplacian of $u$ with respect to the space variables. Let us set

$$\mu := \inf_{\sigma \geq 0} m(\sigma).$$

Equation (1.1) or (1.3) are called *strictly hyperbolic* if $\mu > 0$, and *weakly* (or *degenerate*) *hyperbolic* if $\mu = 0$.

From the mathematical point of view, (1.3) is probably the simplest example of quasilinear hyperbolic equation. From the mechanical point of view, this Cauchy boundary value problem is a model for the small transversal vibrations of an elastic string ($n = 1$) or membrane ($n = 2$). In this context it was introduced by G. Kirchhoff in [18].

We refer to [1] for a sketch of the deduction of (1.3) from the system of (local) equations of elasticity, and to [3] for the standard arguments in functional analysis leading from (1.3) to (1.1).

This equation has generated a considerable literature after the pioneering paper by S. Bernstein [5]. For more details on previous works we refer to the introductions of the following sections. In this note we focus on the basic analytic questions for a partial differential equation, namely local existence, uniqueness, and global existence.

Section 2 is devoted to local existence. We show that a local solution of (1.1) exists provided that the initial data (1.2) are regular enough, depending on the continuity modulus of $m$. This result is an interpolation between the two extreme cases, namely the classical local existence results for analytic data and continuous $m$, or for Sobolev data and Lipschitz continuous $m$. We show that these local solutions satisfy the expected properties of propagation of regularity and continuous dependence on the data. More important, we show with some counterexamples that the spaces involved in the local existence results are optimal.

Section 3 is devoted to uniqueness issues. We present a uniqueness result in which the nonlinear term is *not* required to be Lipschitz continuous.

Section 4 is devoted to global existence. After reviewing the classical global existence results, we state our result concerning "spectral gap" initial data. This special class of initial data is small in the sense that it is not a vector space, and it does not even contain all analytic data, but it is large in the sense that every initial condition in the spaces involved in the local existence result is the sum of two initial data for which the solution is actually global. In particular, a solution can globally exist even if its initial data have only the minimal regularity required by the local existence result.

Finally, Section 5 is devoted to open problems. We recall some old and new unsolved questions which should inspire the future investigations in this challenging research field.

We conclude by pointing out that there is a considerable literature on equation (1.1) or (1.3) with additional *dissipative terms*. The interested reader is referred to the survey [14].

## 2 Local existence

**Previous works**  Local existence has been proved in the last century with two opposite sets of assumptions.

(A) The case where $m$ is Lipschitz continuous, the equation is strictly hyperbolic, and initial data have Sobolev regularity. Under these assumptions a local existence result was first proved by S. Bernstein [5], and then extended with increasing generality by many authors. The more general form was probably stated by A. Arosio and S. Panizzi in [3], where they proved that problem (1.1), (1.2) is well posed in the phase space $D(A^{3/4}) \times D(A^{1/4})$.

(B) The case where $m$ is continuous, the equation is weakly hyperbolic, and initial data are analytic. A local (and actually global, see Section 4) existence result under these assumptions was proved with increasing generality by S. Bernstein [5], S. I. Pohozaev [22], A. Arosio and S. Spagnolo [4], P. D'Ancona and S. Spagnolo [8, 9].

More recently, F. Hirosawa [16] considered equation (1.3) with $\Omega = \mathbb{R}^n$, and proved a local existence result in classes of initial data depending on the continuity modulus of $m$. The rough idea is that the less regular is the nonlinear term, the more regularity is required on initial data. This result interpolates (A) and (B).

Our contribution is twofold: in [11] we extended these intermediate results from the special concrete case $\Omega = \mathbb{R}^n$ to the general abstract setting, and we provided counterexamples in order to show their optimality. Let us introduce the functional setting needed in these statements.

**Functional spaces and continuity moduli** For the sake of simplicity we assume that $H$ admits a countable complete orthonormal system $\{e_k\}_{k \geq 1}$ made by eigenvectors of $A$. We denote the corresponding eigenvalues by $\lambda_k^2$ (with $\lambda_k \geq 0$), so that $Ae_k = \lambda_k^2 e_k$ for every $k \geq 1$. Every $u \in H$ can be written in a unique way in the form $u = \sum_{k=1}^{\infty} u_k e_k$, where $u_k = \langle u, e_k \rangle$ are the components of $u$. In other words, every $u \in H$ can be identified with the sequence $\{u_k\}$ of its components, and under this identification the operator $A$ acts component-wise by multiplication.

This simplifying assumption is not so restrictive. Indeed the spectral theorem for self-adjoint unbounded operators on a separable Hilbert space (see [24, Chapter VIII]) states that any such operator is unitary equivalent to a multiplication operator on some $L^2$ space.

More precisely, for every $H$ and $A$ there exist a measure space $(M, \mu)$, a function $a(\xi) \in L^2(M, \mu)$, and a unitary operator $H \to L^2(M, \mu)$ which associates to every $u \in H$ a function $f(\xi) \in L^2(M, \mu)$ in such a way that $Au$ corresponds to the product $a(\xi)f(\xi)$.

As a consequence, all the spaces we define in terms of $u_k$ and $\lambda_k$ can be defined in the general case by replacing the sequence of components $\{u_k\}$ of $u$ with the function $f(\xi)$ corresponding to $u$, the sequence $\{\lambda_k\}$ of eigenvalues of $A$ with the function $a(\xi)$, and summations over $k$ with integrals over $M$ in the variable $\xi$ with respect to the measure $\mu$. Similarly, there is no loss of generality in proving our existence and uniqueness results (Theorems 2.1, 3.1, 4.3) using components, as we did in [11, 12, 13]. On the contrary, existence of countably many *eigenvalues* is essential in the construction of our counterexamples, as stated in Theorem 2.5 and Theorem 2.6.

Coming back to functional spaces, using components we have that

$$D(A^\alpha) := \left\{ u \in H : \sum_{k=1}^{\infty} \lambda_k^{4\alpha} u_k^2 < +\infty \right\}, \qquad D(A^\infty) := \bigcap_{\alpha \geq 0} D(A^\alpha).$$

Let now $\varphi : [0, +\infty) \to [1, +\infty)$ be any function. Then for every $\alpha \geq 0$ and $r > 0$ one can set

$$\|u\|_{\varphi,r,\alpha}^2 := \sum_{k=1}^{\infty} \lambda_k^{4\alpha} u_k^2 \exp\left( r\varphi(\lambda_k) \right), \tag{2.1}$$

and then define the *generalized Gevrey spaces* as

$$\mathcal{G}_{\varphi,r,\alpha}(A) := \{ u \in H : \|u\|_{\varphi,r,\alpha} < +\infty \}. \tag{2.2}$$

These spaces can also be seen as the domain of the operator $A^\alpha \exp\left( r\varphi(A) \right)$, and they are a generalization of the usual spaces of Sobolev, Gevrey or analytic functions. They are Hilbert spaces with norm $(|u|^2 + \|u\|_{\varphi,r,\alpha}^2)^{1/2}$, and they form a scale of Hilbert spaces with respect to the parameter $r$.

A *continuity modulus* is a continuous increasing function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$, and $\omega(a + b) \leq \omega(a) + \omega(b)$ for every $a \geq 0$ and $b \geq 0$.

The function $m$ is said to be $\omega$-continuous if there exists a constant $L \in \mathbb{R}$ such that

$$|m(a) - m(b)| \leq L\,\omega(|a - b|) \qquad \forall a \geq 0,\ \forall b \geq 0. \tag{2.3}$$

We point out that the set of $\omega$-continuous functions depends only on the values of $\omega(\sigma)$ for $\sigma$ in a neighborhood of 0, while when $\varphi$ is continuous the space $\mathcal{G}_{\varphi,r,\alpha}(A)$ depends only on the values of $\varphi(\sigma)$ for large $\sigma$.

**Our local existence results** The following statement sums up the state of the art concerning existence of local solutions.

**Theorem 2.1 (Local existence)** *Let $H$ be a separable Hilbert space, and let $A$ be a nonnegative self-adjoint (unbounded) operator on $H$ with dense domain. Let $\omega$ be a continuity modulus, let $m : [0, +\infty) \to [0, +\infty)$ be an $\omega$-continuous function, and let $\varphi : [0, +\infty) \to [1, +\infty)$.*
  *Let us assume that there exists a constant $\Lambda$ such that either*

$$\sigma\omega\left(\frac{1}{\sigma}\right) \leq \Lambda\varphi(\sigma) \qquad \forall\sigma > 0 \tag{2.4}$$

*in the strictly hyperbolic case, or*

$$\sigma \leq \Lambda\varphi\left(\frac{\sigma}{\sqrt{\omega(1/\sigma)}}\right) \qquad \forall\sigma > 0 \tag{2.5}$$

*in the weakly hyperbolic case. Let us finally assume that*

$$(u_0, u_1) \in \mathcal{G}_{\varphi,r_0,\alpha+1/2}(A) \times \mathcal{G}_{\varphi,r_0,\alpha}(A) \tag{2.6}$$

*for some $r_0 > 0$, and some $\alpha \geq 1/4$.*
  *Then there exist $T > 0$, and $R > 0$ with $RT < r_0$ such that problem (1.1), (1.2) admits at least one local solution $u(t)$ in the space*

$$C^1\left([0, T]; \mathcal{G}_{\varphi,r_0-Rt,\alpha}(A)\right) \cap C^0\left([0, T]; \mathcal{G}_{\varphi,r_0-Rt,\alpha+1/2}(A)\right). \tag{2.7}$$

Condition (2.7), with the range space depending on time, simply means that

$$u \in C^1\left([0, \tau]; \mathcal{G}_{\varphi,r_0-R\tau,\alpha}(A)\right) \cap C^0\left([0, \tau]; \mathcal{G}_{\varphi,r_0-R\tau,\alpha+1/2}(A)\right)$$

for every $\tau \in (0, T]$. This amounts to say that scales of Hilbert spaces are the natural setting for this problem.

Admittedly assumptions (2.4) and (2.5) do not lend themselves to a simple interpretation. The basic idea is that in the strictly hyperbolic case the best choice for $\varphi$, namely the choice giving the largest space of initial data, is always $\varphi(\sigma) = \sigma\omega(1/\sigma)$. In the weakly hyperbolic case things are more complex because condition (2.5) is stated

4

| $\omega(\sigma) =$ | $m$ is ... | $\varphi(\sigma) =$ | Local existence with data in ... |
|:---:|:---:|:---:|:---:|
| any | continuous | $\sigma$ | analytic functions (never optimal) |
| any | continuous | $\sigma\omega(1/\sigma)$ | space larger than analytic functions |
| $\sigma^\beta$ | $\beta$-Hölder cont. | $\sigma^{1-\beta}$ | Gevrey space of order $(1-\beta)^{-1}$ |
| $\sigma\lvert\log\sigma\rvert$ | Log-Lipschitz cont. | $\log\sigma$ | $D(A^{\alpha+1/2}) \times D(A^\alpha)$ with $\alpha > 1/4$ |
| $\sigma$ | Lipschitz cont. | $1$ | $D(A^{3/4}) \times D(A^{1/4})$ |

Table 1: Examples of relations between the regularity of $m$ and the regularity of initial data for local existence in the *strictly* hyperbolic case

| $\omega(\sigma) =$ | regularity of $m$ | $\varphi(\sigma) =$ | Local existence with data in ... |
|:---:|:---:|:---:|:---:|
| any | continuous | $\sigma$ | analytic functions (never optimal) |
| any | continuous | $o(\sigma)$ | space larger than analytic functions |
| $\sigma^\beta$ | $\beta$-Hölder cont. | $\sigma^{2/(\beta+2)}$ | Gevrey space of order $1 + \beta/2$ |
| $\sigma$ | Lipschitz cont. | $\sigma^{2/3}$ | Gevrey space of order $3/2$ |

Table 2: Examples of relations between the regularity of $m$ and the regularity of initial data for local existence in the *weakly* hyperbolic case

in an implicit form. In this case the best choice for $\varphi$ is the inverse of the function $\sigma \to \sigma/\sqrt{\omega(1/\sigma)}$. Note that this inverse function is always $o(\sigma)$ as $\sigma \to +\infty$. Tables 1 and 2 provide examples of pairs of functions $m$, $\varphi$ satisfying (2.4) and (2.5).

As one could easily expect, assumption (2.5) is always stronger than assumption (2.4). We remark that, since we are interested in local solutions, assumption (2.4) is the relevant one also when the equation is degenerate but the initial condition $u_0$ satisfies the nondegeneracy condition

$$m(|A^{1/2}u_0|^2) \neq 0.$$

In this case problem (1.1), (1.2) is called *mildly degenerate*. As observed in [2], in this situation it is enough to solve the problem with a different nonlinearity which is strictly hyperbolic and coincides with the given $m$ in a neighborhood of $|A^{1/2}u_0|^2$. The solution of the modified problem is thus a solution of the original problem for $t$ small enough.

Assumption (2.5) is therefore the relevant one only when $m(|A^{1/2}u_0|^2) = 0$. This is usually called the *really degenerate* case.

The proof of Theorem 2.1 relies on standard techniques. The first step is remarking that (1.1) admits a first-order conserved energy, namely the Hamiltonian

$$\mathcal{H}(t) := |u'(t)|^2 + M(|A^{1/2}u(t)|^2), \tag{2.8}$$

5

where $M(\sigma)$ is any function such that $M'(\sigma) = m(\sigma)$ for every $\sigma \geq 0$. This is the reason why $D(A^{1/2}) \times H$ is called the *energy space*.

The second step is to consider the linearization of (1.1), namely equation

$$u''(t) + c(t)Au(t) = 0, \tag{2.9}$$

where now $c(t) := m(|A^{1/2}u(t)|^2)$ is thought as a given coefficient. The theory of such linear hyperbolic equations with time-dependent coefficients was developed by F. Colombini, E. De Giorgi and S. Spagnolo [6] in the strictly hyperbolic case, and by F. Colombini, E. Jannelli and S. Spagnolo [7] in the weakly hyperbolic case. The result is that a local solution exists provided that the regularity of the initial data is related to the continuity modulus of $c(t)$ as in Theorem 2.1.

Unfortunately the boundedness of the Hamiltonian (2.8) is not enough to control the oscillations of $c(t)$. The main point is therefore to prove an a priori estimate for

$$\frac{d}{dt}|A^{1/2}u(t)|^2 = 2\langle A^{1/2}u(t), A^{1/2}u'(t)\rangle = 2\langle A^{3/4}u(t), A^{1/4}u'(t)\rangle,$$

which in turn is achieved through an a priori estimate for the higher order energy

$$|A^{1/4}u'(t)|^2 + |A^{3/4}u(t)|^2.$$

This a priori estimate provides an a priori control on the continuity modulus of $c(t)$. One can therefore apply the linear theory and obtain, for example, the so called *propagation of regularity*, namely the fact that solutions belong to the same space (or more precisely to the same scale of spaces) of the initial condition. The precise statement is the following.

**Theorem 2.2 (A priori estimate, Propagation of regularity)** *Let* $H$, $A$, $\omega$, $m$, $\varphi$, $\Lambda$, $u_0$, $u_1$, $r_0$, $\alpha$ *be as in Theorem 2.1.*

*Then there exist positive real numbers* $T$, $K$, $R$, *with* $RT < r_0$, *such that every solution*

$$u \in C^1\left([0,T]; D(A^{1/4})\right) \cap C^0\left([0,T]; D(A^{3/4})\right) \tag{2.10}$$

*of problem (1.1), (1.2) satisfies*

$$|A^{1/4}u'(t)|^2 + |A^{3/4}u(t)|^2 \leq K \qquad \forall t \in [0,T],$$

*and actually* $u$ *belongs to the space (2.7).*

The constants $T$, $K$, $R$ depend only on $\omega$, $m$, and on the norms of $u_0$ and $u_1$ in the spaces $\mathcal{G}_{\varphi,r_0,\alpha+1/2}(A)$ and $\mathcal{G}_{\varphi,r_0,\alpha}(A)$, respectively.

In Theorem 2.2, as in every a priori estimate, we assumed the existence of a solution. So the final step in the proof of Theorem 2.1 is proving that a solution exists. Thanks to the a priori estimate this can be done in several standard ways.

A first possibility is to use Galerkin approximations. In this case one approximates $(u_0, u_1)$ with a sequence of data $(u_{0n}, u_{1n})$ belonging to $A$-invariant subspaces of $H$ where the restriction of $A$ is a *bounded* operator. For such data solutions exist, and thanks to the a priori estimate the corresponding coefficients $c_n(t)$ are relatively compact in $C^0([0, T])$. The conclusion follows from the fact that solutions of the linear problem depend continuously on the initial data and on the coefficient $c(t)$ (see [6] and [7]).

A second possibility is to apply Schauder's fixed point theorem in the space of coefficients. In this case one defines $X_{K,T}$ as the space of functions $a : [0, T] \to \mathbb{R}$ such that $a(0) = |A^{1/2}u_0|^2$, and with Lipschitz constant less than or equal to $K$. For every $a \in X_{K,T}$, one considers the solution $u(t)$ of the linear problem (2.9) with $c(t) = m(a(t))$, and initial data (1.2). Finally one sets $[\Phi(a)](t) := |A^{1/2}(t)|^2$. For suitable values of $K$ and $T$ (those given by Theorem 2.2), $\Phi$ defines a continuous map from $X_{K,T}$ into itself which has a fixed point due to Schauder's theorem. This fixed point corresponds to a solution of (1.1), (1.2).

The same techniques (a priori estimate + compactness + results for the linear equation) lead to the continuity of the map

$$(\text{initial data}, m) \to \text{solution}.$$

Since the solution is not necessarily unique, this has to be intended in the sense that "the limit of solutions is again a solution". The precise statement is the following.

**Theorem 2.3 (Continuous dependence on initial data)** *Let $H$, $A$, $\omega$, $m$, $\varphi$, $\Lambda$, $u_0$, $u_1$, $r_0$, $\alpha$, $T$, $R$ be as in Theorem 2.2.*

*Let $\{m_n\}$ be a sequence of $\omega$-continuous functions $m_n : [0, +\infty) \to [0, +\infty)$ satisfying (2.3) with the same constant $L$, and such that $m_n \to m$ uniformly on compact sets. Let $\{u_{0n}, u_{1n}\} \subseteq \mathcal{G}_{\varphi, r_0, \alpha+1/2}(A) \times \mathcal{G}_{\varphi, r_0, \alpha}(A)$ be a sequence converging to $(u_0, u_1)$ in the same space.*

*Let finally $T_1 \in (0, T)$ and $R_1 > R$ be real numbers such that $R_1 T_1 < r_0$.*

*Then we have the following conclusions.*

*(1) For every $n \in \mathbb{N}$ large enough the Cauchy problem (1.1), (1.2) (with of course $m_n$, $u_{0n}$, $u_{1n}$ instead of $m$, $u_0$, $u_1$) has at least one solution $u_n(t)$ in the space*

$$C^1\left([0, T_1]; \mathcal{G}_{\varphi, r_0 - R_1 t, \alpha}(A)\right) \cap C^0\left([0, T_1]; \mathcal{G}_{\varphi, r_0 - R_1 t, \alpha+1/2}(A)\right). \qquad (2.11)$$

*(2) The sequence $\{u_n(t)\}$ is relatively compact in the space (2.11).*

*(3) Any limit point of $\{u_n(t)\}$ is a solution of the limit problem.*

With minimal technicalities the theory can be extended in order to allow time-dependent right-hand sides $f_n(t)$ with suitable regularity assumptions, for example in $L^2\left([0, T], \mathcal{G}_{\varphi, r_0, 1/4}(A)\right)$. We spare the reader from the details.

7

**Derivative loss** Our goal is now to prove the optimality of the spaces involved in the local existence result. To this end we show that solutions with less regular data can exhibit an instantaneous derivative loss. Let us introduce the precise notion.

**Definition 2.4** Let $H$ and $A$ be as in Theorem 2.1. Let $\varphi : [0, +\infty) \to [1, +\infty)$ be any function, and let $\alpha \geq 1/4$. We say that a solution $u$ of problem (1.1), (1.2) has *instantaneous strong derivative loss* of type

$$\mathcal{G}_{\varphi,\infty,3/4}(A) \times \mathcal{G}_{\varphi,\infty,1/4}(A) \to D(A^{\alpha+1/2}) \times D(A^\alpha) \tag{2.12}$$

if the following conditions are fulfilled.

(1) *Regularity of the solution.* There exists $T_0 > 0$ such that

$$u \in C^1\left([0, T_0]; D(A^{1/4})\right) \cap C^0\left([0, T_0]; D(A^{3/4})\right). \tag{2.13}$$

(2) *High regularity at $t = 0$.* We have that

$$(u_0, u_1) \in \mathcal{G}_{\varphi,\infty,3/4}(A) \times \mathcal{G}_{\varphi,\infty,1/4}(A),$$

(3S) *Low regularity for subsequent times.* We have that

$$(u(t), u'(t)) \notin D(A^{\alpha+1/2+\varepsilon}) \times D(A^{\alpha+\varepsilon}) \qquad \forall \varepsilon > 0, \ \forall t \in (0, T_0].$$

We say that the same solution has *instantaneous weak derivative loss* of type (2.12) if it satisfies (1), (2), and

(3W) *Unboundedness as $t \to 0^+$.* There exists a sequence $\tau_k \to 0^+$ such that

$$\left|A^{\alpha+1/2+\varepsilon}u(\tau_k)\right| \to +\infty \qquad \forall \varepsilon > 0.$$

The second notion is weaker in the sense that what is actually lost is the control on the norm of $(u(t), u'(t))$ in $D(A^{\alpha+1/2+\varepsilon}) \times D(A^{\alpha+\varepsilon})$ as $t \to 0^+$. We are now ready to state our counterexamples, the first one in the strictly hyperbolic case, the second one in the weakly hyperbolic case.

**Theorem 2.5 (Derivative loss: strictly hyperbolic case)** *Let $A$ be a self-adjoint linear operator on a Hilbert space $H$. Let us assume that there exist a countable (not necessarily complete) orthonormal system $\{e_k\}_{k\geq 1}$ in $H$, and an increasing unbounded sequence $\{\lambda_k\}_{k\geq 1}$ of positive real numbers such that $Ae_k = \lambda_k^2 e_k$ for every $k \geq 1$.*

*Let $\omega : [0, +\infty) \to [0, +\infty)$ be a continuity modulus such that $\sigma \to \sigma/\omega(\sigma)$ is a nondecreasing function.*

Let $\varphi : [0, +\infty) \to [1, +\infty)$ be a function such that

$$\lim_{k \to +\infty} \frac{\lambda_k}{\varphi(\lambda_k)} \omega\left(\frac{1}{\lambda_k}\right) = +\infty. \tag{2.14}$$

Then there exist an $\omega$-continuous function $m : [0, +\infty) \to [1/2, 3/2]$, and a solution $u$ of the corresponding problem (1.1) with instantaneous strong derivative loss of type

$$\mathcal{G}_{\varphi,\infty,3/4}(A) \times \mathcal{G}_{\varphi,\infty,1/4}(A) \to D(A^{3/4}) \times D(A^{1/4}).$$

**Theorem 2.6 (Derivative loss: weakly hyperbolic case)** *Let $H$, $A$, $\{e_k\}$, $\{\lambda_k\}$, $\omega$ be as in Theorem 2.5. Let $\varphi : [0, +\infty) \to [1, +\infty)$ be a function such that*

$$\lim_{k \to +\infty} \lambda_k \left[ \varphi\left(\frac{\lambda_k}{\sqrt{\omega(1/\lambda_k)}}\right) \right]^{-1} = +\infty. \tag{2.15}$$

*Then there exist an $\omega$-continuous function $m : [0, +\infty) \to [0, 3/2]$, and a solution $u$ of the corresponding equation (1.1) with instantaneous strong derivative loss of type*

$$\mathcal{G}_{\varphi,\infty,3/4}(A) \times \mathcal{G}_{\varphi,\infty,1/4}(A) \to D(A) \times D(A^{1/2}),$$

*and instantaneous weak derivative loss of type*

$$\mathcal{G}_{\varphi,\infty,3/4}(A) \times \mathcal{G}_{\varphi,\infty,1/4}(A) \to D(A^{3/4}) \times D(A^{1/4}).$$

Note that assumptions (2.14) and (2.15) are the counterpart of (2.4) and (2.5), respectively. Table 3 and Table 4 below present examples of functions $\omega$ and $\varphi$ satisfying the assumptions of Theorem 2.5 and Theorem 2.6 above. They are the counterpart of Table 1 and Table 2, respectively.

These examples are based on the construction introduced in [6] and [7] in the linear context. In those papers the authors gave examples of coefficients $c(t)$ and initial data $(u_0, u_1)$ in such a way that the "solution" of the linear problem has instantaneous derivative loss

$$\mathcal{G}_{\varphi,\infty,1/2}(A) \times \mathcal{G}_{\varphi,\infty,0}(A) \to \text{hyperdistributions}.$$

This means that the solution is quite regular at time $t = 0$, but it is even outside the space of distributions for $t > 0$. This is usually presented as a nonexistence result in the space of distributions, but it is proved by showing that the solution exists and is unique (since the equation is linear) in a space of hyperdistributions (which in our notations is a space of the form $\mathcal{G}_{\varphi,r,0}$ with $r < 0$), but exhibits an instantaneous derivative loss up to hyperdistributions. In the quoted papers the derivative loss is always intended in the weak sense, but those examples are quite flexible and can be modified in order to obtain the derivative loss even in the strong sense.

Our strategy is similar. In the first step we modify the parameters in those examples in order to stop the derivative loss up to the $D(A^{3/4}) \times D(A^{1/4})$ level. In the second step we find a function $m$ in such a way that the coefficient $c(t)$ is actually equal to $m(|A^{1/2}u(t)|^2)$. This can be easily done as soon as the function $t \to |A^{1/2}u(t)|^2$ is invertible in a neighborhood of $t = 0$, and this can be obtained by modifying just one dominant component of $u(t)$. We refer to [11] for the details.

| $m(\sigma) =$ | $m$ is ... | $\varphi(\sigma) =$ | Derivative loss for data in ... |
|---|---|---|---|
| $\dfrac{1}{\|\log \sigma\|^{1/2}}$ | just continuous | $\dfrac{\sigma}{\log \sigma}$ | quasi-analytic functions |
| $\sigma^{\beta}$ | $\beta$-Hölder cont. | $\dfrac{\sigma^{1-\beta}}{\log \sigma}$ | Gevr. sp. of order $> (1-\beta)^{-1}$ |
| $\sigma\|\log \sigma\|^{3}$ | $\beta$-Höld. cont. $\forall \beta \in (0,1)$ | $\log^{2}\sigma$ | $D(A^{\infty})$ |

Table 3: Pairs of functions $m$, $\varphi$ for which a derivative loss example can be found in the strictly hyperbolic case

| $m(\sigma) =$ | $m$ is ... | $\varphi(\sigma) =$ | Derivative loss for data in ... |
|---|---|---|---|
| $\sigma^{\beta}$ | $\beta$-Hölder cont. | $\dfrac{\sigma^{2/(\beta+2)}}{\log \sigma}$ | Gevrey sp. of order $> 1 + \beta/2$ |
| $\sigma$ | Lipschitz cont. | $\dfrac{\sigma^{2/3}}{\log \sigma}$ | Gevrey sp. of order $> 3/2$ |

Table 4: Pairs of functions $m$, $\varphi$ for which a derivative loss example can be found in the weakly hyperbolic case

# 3 Uniqueness

**Previous works** As one can easily guess, uniqueness holds whenever $m$ is (locally) Lipschitz continuous. In the strictly hyperbolic case a proof of this result is contained for example in [3], of course for initial data in $D(A^{3/4}) \times D(A^{1/4})$. In the weakly hyperbolic case a proof of the same result is given in [4] for analytic initial data. Now from Theorem 2.1 we know that, when $m$ is Lipschitz continuous and the equation is degenerate, local solutions exist for all initial data in $\mathcal{G}_{\varphi,r_0,3/4}(A) \times \mathcal{G}_{\varphi,r_0,1/4}(A)$ with $\varphi(\sigma) = \sigma^{2/3}$. The uniqueness result under these assumptions has never been put into writing, but it can be easily proved by standard arguments. The main tool is indeed always the same, namely a Gronwall type lemma for the difference between two solutions.

As a general fact, uniqueness for a nonlinear evolution equation is much more difficult to establish if the nonlinear term is not locally Lipschitz continuous. Therefore it is hardly surprising that also in the case of Kirchhoff equations the non-Lipschitz case remained widely unexplored for a long time. To our knowledge indeed uniqueness issues have been previously considered only in section 4 of [4], where two results are presented.

The first one is a one-dimensional example ($H = \mathbb{R}$) where problem (1.1), (1.2) admits infinitely many local solutions. The second result is a detailed study of the case where $u_0$ and $u_1$ are *eigenvectors* of $A$ relative to the *same eigenvalue*. In this

special situation (which can be easily reduced to the two dimensional case $H = \mathbb{R}^2$) the authors proved that uniqueness of the local solution fails if and only if the following three conditions are satisfied:

(AS1) $\langle Au_0, u_1 \rangle = 0$,

(AS2) $|A^{1/2}u_1|^2 - m(|A^{1/2}u_0|^2)|Au_0|^2 = 0$,

(AS3) $m$ satisfies a suitable integrability condition in a neighborhood of $|A^{1/2}u_0|^2$.

As a consequence, the local solution is unique if at least one of the conditions above is not satisfied.

**Our uniqueness result**   Our contribution is the extension of the first two parts of the above result from the two dimensional case with equal eigenvalues to the infinite dimensional case with arbitrary eigenvalues. In other words, we prove that in the general case the solution is necessarily unique whenever either (AS1) or (AS2) are not satisfied. The precise statement is the following.

**Theorem 3.1 (Uniqueness)** *Let $H$, $A$, $\omega$, $m$, $\varphi$, $\Lambda$ be as in Theorem 2.1. Let us assume that*

$$(u_0, u_1) \in \mathcal{G}_{\varphi, r_0, 3/2}(A) \times \mathcal{G}_{\varphi, r_0, 1}(A) \tag{3.1}$$

*for some $r_0 > 0$, and*

$$|\langle Au_0, u_1 \rangle| + \left| |A^{1/2}u_1|^2 - m(|A^{1/2}u_0|^2)|Au_0|^2 \right| \neq 0. \tag{3.2}$$

*Let us assume that problem (1.1), (1.2) admits two local solutions $v_1$ and $v_2$ in*

$$C^2\left([0, T]; \mathcal{G}_{\varphi, r_1, 1/2}(A)\right) \cap C^1\left([0, T]; \mathcal{G}_{\varphi, r_1, 1}(A)\right) \cap C^0\left([0, T]; \mathcal{G}_{\varphi, r_1, 3/2}(A)\right) \tag{3.3}$$

*for some $T > 0$, and some $r_1 \in (0, r_0)$.*
*Then we have the following conclusions.*

*(1) There exists $T_1 \in (0, T]$ such that*

$$v_1(t) = v_2(t) \qquad \forall t \in [0, T_1]. \tag{3.4}$$

*(2) Let $T_*$ denote the supremum of all $T_1 \in (0, T]$ for which (3.4) holds true. Let $v(t)$ denote the common value of $v_1$ and $v_2$ in $[0, T_*]$.*
*Then either $T_* = T$ or*

$$|\langle Av(T_*), v'(T_*) \rangle| + \left| |A^{1/2}v'(T_*)|^2 - m(|A^{1/2}v(T_*)|^2)|Av(T_*)|^2 \right| = 0.$$

Let us make some comments on the assumptions. Inequality (3.2) is equivalent to say that either (AS1) or (AS2) are not satisfied. The space (3.3) is the natural one when initial data satisfy (3.1). Indeed from the propagation of regularity (see Theorem 2.2) it follows that any solution $u(t)$ satisfying (2.10) with initial data as in (3.1) lies actually in (3.3). Assumption (3.1) on the initial data is stronger than the corresponding assumption in Theorem 2.1. This is due to a technical point in the proof. However in most cases the difference is only apparent. For example if $\omega(\sigma) = \sigma^\beta$ for some $\beta \in (0, 1]$, then the following implication

$$u \in \mathcal{G}_{\varphi,r,0}(A) \implies u \in \mathcal{G}_{\varphi,r-\varepsilon,\alpha}(A)$$

holds true for every $r > 0$, $\varepsilon \in (0, r)$, $\alpha \geq 0$. Therefore in this case every solution satisfying (2.7) fulfils (3.3) with $r_1 = (r_0 - RT)/2$.

In the proof of Theorem 3.1, for which we refer to [12], we introduced a technique which seems to be new, and hopefully useful to handle also different evolution equations with non-Lipschitz terms. The main idea is to split the uniqueness problem in two steps, which we call trajectory uniqueness and parametrization uniqueness.

*Trajectory uniqueness* The first step of the proof consists in showing that the image of the curve $(A^{1/2}u(t), u'(t))$ in the phase space (for example in $D(A^{3/4}) \times D(A^{1/4})$) is unique. To this end we introduce the new variable

$$s = \psi(t) := |A^{1/2}u(t)|^2 - |A^{1/2}u_0|^2.$$

If the function $\psi$ is invertible in a right-hand neighborhood of the origin, then we can parametrize the curve using the variable $s$. If $(z(s), w(s))$ is this new parametrization, then $z$ and $w$ are solutions of the following system

$$z'(s) = \frac{A^{1/2}w(s)}{2\langle A^{1/2}z(s), w(s)\rangle}, \qquad w'(s) = -m\left(s + |A^{1/2}u_0|^2\right)\frac{A^{1/2}z(s)}{2\langle A^{1/2}z(s), w(s)\rangle}, \quad (3.5)$$

with initial data

$$z(0) = A^{1/2}u_0, \qquad w(0) = u_1. \tag{3.6}$$

What is important is that the non-Lipschitz term $m(|A^{1/2}u|^2)$ of the original equation has become the non-Lipschitz coefficient $m(s + |A^{1/2}u_0|^2)$ in the second equation of system (3.5), and it is well known that nonregular *coefficients* do not affect uniqueness. Therefore the solution of the system is unique.

*Parametrization uniqueness* The second part of the proof consists in showing that the unique trajectory obtained in the previous step can be covered by solutions in a unique way. To this end, we first show that the parametrization $\psi(t)$ is a solution of the Cauchy problem

$$\psi'(t) = F(\psi(t)), \qquad \psi(0) = 0, \tag{3.7}$$

where $F(\sigma) := 2\langle A^{1/2}z(\sigma), w(\sigma)\rangle$. The function $F$ is just continuous in $\sigma = 0$, and this in not enough to conclude that the solution of (3.7) is unique. On the other hand, the differential equation in (3.7) is *autonomous*, and for autonomous equations it is well known that there is a unique solution such that $\psi(t) > 0$ for $t > 0$.

Proving that $\psi(t) > 0$ for $t > 0$, and more generally that $\psi$ is invertible in a right-hand neighborhood of $t = 0$ (as required in the first step), is the point where the quite strange assumptions (AS1) and (AS2) play their role. Indeed we have that

$$\psi'(0) = 0 \iff \text{(AS1) holds true,}$$
$$\psi''(0) = 0 \iff \text{(AS2) holds true.}$$

If (3.2) is true, then either (AS1) or (AS2) are false, hence either $\psi'(0) \neq 0$ or $\psi''(0) \neq 0$. In both cases $\psi(t)$ is invertible where needed.

We conclude by pointing out that the denominators in (3.5) are actually $\psi'(t)$, hence they can vanish for $t = 0$. Since in that case we have that $\psi''(0) \neq 0$, then for sure denominators are different from 0 for all $t > 0$ small enough, and their vanishing in $t = 0$ is of order one. This kind of singularity doesn't affect existence or uniqueness for system (3.5), but it is in some sense the limit exponent. For this reason we cannot deal with the same technique the case where $\psi'(0) = \psi''(0) = 0$ but $\psi'''(0) \neq 0$, which originates denominators with a singularity of order 2.

# 4   Global existence

**Previous works**   Global existence for Kirchhoff equations has been proved in at least five special cases.

*Analytic data*   This is the result we quoted as (B) in the history of local existence results. We recall the main assumptions: the equation is weakly hyperbolic, the nonlinearity is continuous, and initial data are analytic.

*Quasi-analytic data*   K. Nishihara [21] proved global existence for a class of initial data which strictly contains analytic functions. His assumptions are that the equation is strictly hyperbolic, the nonlinearity is Lipschitz continuous, and

$$(u_0, u_1) \in \mathcal{G}_{\varphi,r_0,1/2}(A) \times \mathcal{G}_{\varphi,r_0,0}(A),$$

where $r_0 > 0$, and $\varphi : [0, +\infty) \to [1, +\infty)$ is an increasing function satisfying suitable convexity and integrability conditions. He proves existence of a global solution

$$u \in C^1\left([0, +\infty); \mathcal{G}_{\varphi,r_0,0}(A)\right) \cap C^0\left([0, +\infty); \mathcal{G}_{\varphi,r_0,1/2}(A)\right).$$

We point out that, in contrast with our local existence results, this solution lives in a Hilbert *space*, instead of a Hilbert *scale*.

The most celebrated example of function $\varphi$ satisfying the assumptions is $\varphi(\sigma) = \sigma/\log\sigma$, in which case one has global existence in a space which contains non-analytic initial data. On the contrary, the function $\varphi(\sigma) = \sigma^\beta$ with $\beta < 1$ never satisfies the assumptions. In other words, Nishihara's spaces are intermediate classes between Gevrey and analytic functions.

It would be interesting to compare Nishihara's assumptions with

$$\int_1^{+\infty} \frac{\varphi(\sigma)}{\sigma^2}\, d\sigma = +\infty,$$

which is the usual definition of quasi-analytic classes.

*Special nonlinearities*   In a completely different direction, S. I. Pohozaev [23] considered the special case where $m(\sigma) := (a + b\sigma)^{-2}$ for some $a > 0$ and $b \in \mathbb{R}$. He proved global existence for initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfying the nondegeneracy condition $a + b|A^{1/2}u_0|^2 > 0$.

The main point is that in this case (and in a certain sense only in this case) equation (1.1) admits the second order nonnegative invariant

$$\mathcal{P}(t) := \left(a + b|A^{1/2}u(t)|^2\right)|A^{1/2}u'(t)|^2 + \frac{|Au(t)|^2}{a + b|A^{1/2}u(t)|^2} - \frac{b}{4}\langle Au(t), u'(t)\rangle^2.$$

Exploiting that $\mathcal{P}(t)$ is constant, it is not difficult to obtain a uniform bound on $\langle Au(t), u'(t)\rangle^2$, from which global existence follows in a standard way.

Recently, some new results have been obtained along this path. The interested reader is referred to [27].

*Dispersive equations*   Global existence results have been obtained for the concrete equation (1.3) in cases where *dispersion* plays a crucial role, namely when $\Omega = \mathbb{R}$ (see J. M. Greenberg and S. C. Hu [15]), $\Omega = \mathbb{R}^n$ (see P. D'Ancona and S. Spagnolo [10]), or $\Omega = $ exterior domain (see T. Yamazaki [25, 26] and the references quoted therein).

The prototype of these results is global existence provided that the equation is strictly hyperbolic, the nonlinearity is Lipschitz continuous, and initial data have Sobolev regularity and satisfy suitable *smallness assumptions* and *decay conditions* at infinity. We refer to the quoted literature for precise statements.

*Spectral gap initial data*   More recently, R. Manfrin [19] (see also [20], [17]) proved global existence in a new class of nonregular initial data. In order to describe the most astonishing aspect of his work, we need the following definition.

**Definition 4.1** Let $\mathcal{M}$ and $\mathcal{F}$ be two *subsets* of $D(A^{1/2}) \times H$. We say that $\mathcal{M}$ has the "Sum Property" in $\mathcal{F}$ if $\mathcal{M} \subseteq \mathcal{F}$ and $\mathcal{M} + \mathcal{M} \supseteq \mathcal{F}$.

In other words, for every $(u_0, u_1) \in \mathcal{F}$ there exist $(\overline{u}_0, \overline{u}_1) \in \mathcal{M}$ and $(\widehat{u}_0, \widehat{u}_1) \in \mathcal{M}$ such that $u_0 = \overline{u}_0 + \widehat{u}_0$, and $u_1 = \overline{u}_1 + \widehat{u}_1$.

Let us assume now that the equation is strictly hyperbolic, and the nonlinearity is of class $C^2$. The main result of [19] is that there exists a subset $\mathcal{M} \subseteq D(A) \times D(A^{1/2})$ such that $\mathcal{M}$ has the "Sum Property" in $D(A) \times D(A^{1/2})$, and problem (1.1), (1.2) admits a global solution for every $(u_0, u_1) \in \mathcal{M}$.

As a corollary, any initial condition $(u_0, u_1) \in D(A) \times D(A^{1/2})$ is the sum of two pairs of initial conditions for which the solution is global! Of course the set $\mathcal{M}$ is not a vector space, but just a star-shaped subset (actually a cone). We refer to the quoted papers for the definition of $\mathcal{M}$.

**Our global existence result**    In [13] we proved a result in the same spirit of Manfrin's one, but without assuming the strict hyperbolicity or the regularity of the nonlinearity.

Let $\mathcal{L}$ denote the set of all sequences $\{\rho_n\}$ of positive real numbers such that $\rho_n \to +\infty$. Let $\varphi : [0, +\infty) \to [1, +\infty)$ be any function, and let $\alpha \geq 0$, $\beta \geq 0$. We can now define what we call *generalized Gevrey-Manfrin spaces*, namely

$$\mathcal{GM}^{(\beta)}_{\varphi,\{\rho_n\},\alpha}(A) := \left\{ u \in H : \sum_{\lambda_k > \rho_n} \lambda_k^{4\alpha} u_k^2 \exp\left(\rho_n^\beta \varphi(\lambda_k)\right) \leq \rho_n \quad \forall n \in \mathbb{N} \right\}, \qquad (4.1)$$

and

$$\mathcal{GM}^{(\beta)}_{\varphi,\alpha}(A) := \bigcup_{\{\rho_n\} \in \mathcal{L}} \mathcal{GM}^{(\beta)}_{\varphi,\{\rho_n\},\alpha}(A). \qquad (4.2)$$

Admittedly this definition has no immediate interpretation. Let us compare (4.1) with (2.2) and (2.1). In the inequalities in (4.1) the weight $\rho_n$ appears in the right-hand side, and in the left-hand side in place of $r$. Moreover $\rho_n$ appears also in the summation, which is now restricted to eigenvalues $\lambda_k > \rho_n$. The weight in the left-hand side is inside an exponential term, hence it dominates on the weight in the right-hand side. It follows that the inequalities in (4.1) are smallness assumptions on the "tails" of suitable series. More important, the smallness is not required for all tails, but only for a subsequence.

It is easy to see that the space defined in (4.1) is actually a vector space, while the space defined by (4.2) is a cone in $\mathcal{G}_{\varphi,\infty,\alpha}(A)$, because its elements may be defined starting from different sequences in $\mathcal{L}$. This fact is crucial in the proof of the "Sum Property" (see Proposition 3.2 in [13]).

**Proposition 4.2 (Sum Property)** *For every $\varphi : [0, +\infty) \to [1, +\infty)$, $\alpha \geq 0$, $\beta > 0$ we have that*

$$\mathcal{GM}^{(\beta)}_{\varphi,\alpha+1/2}(A) \times \mathcal{GM}^{(\beta)}_{\varphi,\alpha}(A)$$

*has the "Sum Property" in*

$$\mathcal{G}_{\varphi,\infty,\alpha+1/2}(A) \times \mathcal{G}_{\varphi,\infty,\alpha}(A).$$

The proof of the "Sum Property" is based on the following idea. Let us consider an increasing and divergent sequence $s_n$ of positive real numbers. Then any $u_0 \in H$

can be written as the sum of $\overline{u}_0$ and $\widehat{u}_0$, where $\overline{u}_0$ has the same components of $u_0$ with respect to eigenvectors corresponding to eigenvalues belonging to intervals of the form $[s_{2n}, s_{2n+1})$, and components equal to 0 with respect to the remaining eigenvectors, and vice versa for $\widehat{u}_0$. If the sequence $s_n$ grows fast enough, then it turns out that $\overline{u}_0$ and $\widehat{u}_0$ lie in suitable generalized Gevrey-Manfrin spaces corresponding to the sequences $s_{2n}$ and $s_{2n+1}$. Note that the spectrum of both $\overline{u}_0$ and $\widehat{u}_0$ has a sequence of "big holes", which justify the term "spectral gap" initial data.

We are now ready to state our global existence result (see Theorem 3.1 and Theorem 3.2 in [13]).

**Theorem 4.3 (Global existence)** *Let $H$, $A$, $\omega$, $m$, $\varphi$, $\Lambda$ be as in Theorem 2.1. Let $\{\rho_n\} \in \mathcal{L}$, and let*

$$(u_0, u_1) \in \mathcal{GM}^{(\beta)}_{\varphi, \{\rho_n\}, 3/4}(A) \times \mathcal{GM}^{(\beta)}_{\varphi, \{\rho_n\}, 1/4}(A),$$

*where $\beta = 2$ if the equation is strictly hyperbolic, and $\beta = 3$ if the equation is weakly hyperbolic.*

*Then problem (1.1), (1.2) admits at least one global solution $u$ with*

$$u \in C^1\left([0, +\infty); \mathcal{G}_{\varphi, r, 3/4}(A)\right) \cap C^0\left([0, +\infty); \mathcal{G}_{\varphi, r, 1/4}(A)\right)$$

*for every $r > 0$.*

Combining Theorem 4.3 and Proposition 4.2 we obtain the following statement: every pair of initial conditions satisfying (2.6) with $r_0 = \infty$ is the sum of two pairs of initial conditions for which the solution is global. We have thus extended to the general case the astonishing aspect of Manfrin's result.

The extra requirement that $r_0 = \infty$ is hardly surprising. It is indeed a necessary condition for existence of global solutions even in the theory of linear equations with nonsmooth time-dependent coefficients.

We conclude by remarking that, in the concrete case, these spaces do not contain any compactly supported function.

# 5  Open problems

The main open problem in the theory of Kirchhoff equations is for sure the existence of global solutions in $C^\infty$. In the abstract setting it can be stated as follows.

**Open problem 1** *Let us assume that equation (1.1) is strictly hyperbolic, that $m \in C^\infty(\mathbb{R})$, and $(u_0, u_1) \in D(A^\infty) \times D(A^\infty)$.*
*Does problem (1.1), (1.2) admit a global solution?*

The same problem can be restated in all situations where a local solution has been proved to exist (see Theorem 2.1). Up to now indeed we know no example of local solution, with any regularity, which is not global.

Now we would like to mention some other open questions. The first one concerns local (but of course also global) existence for initial data in $D(A^{1/2}) \times H$, which is the natural *energy space* for a second order wave equation.

**Open problem 2** *Let us assume that equation (1.1) is strictly hyperbolic, that $m \in C^\infty(\mathbb{R})$, and $(u_0, u_1) \in D(A^{\alpha+1/2}) \times D(A^\alpha)$ for some $\alpha \in [0, 1/4)$.*

*Does problem (1.1), (1.2) admit a local solution? Of course in this case we accept solutions*

$$u \in C^1([0, T_0]; H) \cap C^0([0, T_0]; D(A^{1/2})). \tag{5.1}$$

Once again we know no counterexample, even with degenerate equations or nonlinearities which are just continuous.

We stress that counterexamples are the missing element in all the theory. We proved the optimality of our local existence results by showing examples of solutions with derivative loss. These are actually counterexamples to propagation of regularity, but not counterexamples to existence. We can therefore ask the following question.

**Open problem 3** *Do there exist a nonnegative continuous function $m$, and initial data $(u_0, u_1) \in D(A^{3/4}) \times D(A^{1/4})$, such that problem (1.1), (1.2) admits no (local) solution $u$ satisfying (2.13)?*

*Do there exist a nonnegative continuous function $m$, and initial data $(u_0, u_1) \in D(A^{1/2}) \times H$, such that problem (1.1), (1.2) admits no (local) solution $u$ satisfying (5.1)?*

We conclude by mentioning three open questions related to uniqueness results. The first one concerns once again counterexamples. The motivation is that we know no example where uniqueness fails apart from those given in [4]. So we ask whether different counterexamples can be provided.

**Open problem 4** *Let $H$, $A$, $\omega$, $m$, $\varphi$, $A$, $u_0$, $u_1$ be as in Theorem 3.1, but without assumption (3.2). Let us assume that problem (1.1), (1.2) admits two local solutions.*

*Can one conclude that $u_0$ and $u_1$ are eigenvectors of $A$ relative to the same eigenvalue?*

We stress that this problem is open even in the simple case $H = \mathbb{R}^2$, where $\omega$ and $\varphi$ play non role, and no regularity is required on initial data.

The second open problem concerns trajectory uniqueness, namely the key step in the proof of our uniqueness result. One can indeed observe that, even in the non-uniqueness examples of [4], all the different solutions describe (a subset of) the same trajectory with a different pace. With our notations this is equivalent to say that the solution of (3.5), (3.6) is unique. We ask whether this property is true in general.

17

**Open problem 5** *Let $H$, $A$, $\omega$, $m$, $\varphi$, $A$, $u_0$, $u_1$ be as in Theorem 3.1, but without assumption (3.2). Let us consider system (3.5), with initial data (3.6).*

*Does this system admit at most one solution?*

Note that in the case where $\langle Au_0, u_1 \rangle = 0$ it is by no means clear that the system admits at least one solution, since this implicitly requires that $\langle A^{1/2}z(s), w(s) \rangle \neq 0$ for every $s \in (0, s_0]$. In any case the above question doesn't concern existence, but just uniqueness provided that a solution exists.

The last open problem concerns the regularity assumptions on initial data and solutions required in the uniqueness result. Indeed in Theorem 3.1 we proved that inequality (3.2) yields uniqueness provided that initial data satisfy (3.1) and solutions satisfy (3.3). Similar assumptions are required in the uniqueness result for Lipschitz continuous nonlinearities. On the other hand, solutions of problem (1.1), (1.2) may exist also if (3.1) is not satisfied (this is the case, for example, of our solutions with derivative loss). We ask whether uniqueness results can be proved for these solutions.

**Open problem 6** *Is it possible to prove the known uniqueness results (namely the Lipschitz case and our Theorem 3.1) with less regularity requirements on initial data or for solution in the energy space?*

Just to give an extreme example, let us consider problem (1.1), (1.2) in the strictly hyperbolic case, with an analytic nonlinearity $m$, and analytic initial data. We know that there exists a unique solution in $D(A^{3/4}) \times D(A^{1/4})$, which is actually analytic. However, as far as we know, no one can exclude that there exists a different solution in $D(A^{1/2}) \times H$ with the same initial data!

## Acknowledgments

# References

[1] A. AROSIO; Averaged evolution equations. The Kirchhoff string and its treatment in scales of Banach spaces. *Functional analytic methods in complex analysis and applications to partial differential equations (Trieste, 1993)*, 220–254, World Sci. Publ., River Edge, NJ, 1995.

[2] A. AROSIO, S. GARAVALDI; On the mildly degenerate Kirchhoff string. *Math. Methods Appl. Sci.* **14** (1991), no. 3, 177–195.

[3] A. Arosio, S. Panizzi; On the well-posedness of the Kirchhoff string. *Trans. Amer. Math. Soc.* **348** (1996), no. 1, 305–330.

[4] A. Arosio, S. Spagnolo; Global solutions to the Cauchy problem for a nonlinear hyperbolic equation. *Nonlinear partial differential equations and their applications. Collège de France seminar, Vol. VI (Paris, 1982/1983)*, 1–26, Res. Notes in Math., 109, Pitman, Boston, MA, 1984.

[5] S. Bernstein; Sur une classe d'équations fonctionnelles aux dérivées partielles, (Russian, French summary) *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **4** (1940), 17–26.

[6] F. Colombini, E. De Giorgi, S. Spagnolo; Sur le équations hyperboliques avec des coefficients qui ne dépendent que du temp. (French) *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **6** (1979), no. 3, 511–559.

[7] F. Colombini, E. Jannelli, S. Spagnolo; Well-posedness in the Gevrey classes of the Cauchy problem for a nonstrictly hyperbolic equation with coefficients depending on time. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **10** (1983), no. 2, 291–312.

[8] P. D'Ancona, S. Spagnolo; Global solvability for the degenerate Kirchhoff equation with real analytic data. *Invent. Math.* **108** (1992), no. 2, 247–262.

[9] P. D'Ancona, S. Spagnolo; On an abstract weakly hyperbolic equation modelling the nonlinear vibrating string. *Developments in partial differential equations and applications to mathematical physics (Ferrara, 1991)*, 27–32, Plenum, New York, 1992.

[10] P. D'Ancona, S. Spagnolo; A class of nonlinear hyperbolic problems with global solutions. *Arch. Rational Mech. Anal.* **124** (1993), no. 3, 201–219.

[11] M. Ghisi, M. Gobbino; Derivative loss for Kirchhoff equations with non-Lipschitz nonlinear term. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* (2009), doi:10.2422/2036-2145.2009.4.01.

[12] M. Ghisi, M. Gobbino; A uniqueness result for Kirchhoff equations with non-Lipschitz nonlinear term. *Adv. Math.* (2009), doi:10.1016/j.aim.2009.09.017.

[13] M. Ghisi, M. Gobbino; Spectral gap global solutions for degenerate Kirchhoff equations. *Nonlinear Anal.* **71** (2009), no. 9, 4115–4124.

[14] M. Ghisi, M. Gobbino; Hyperbolic-parabolic singular perturbation for Kirchhoff equations with weak dissipation. Preprint.

[15] J. M. Greenberg, S. C. Hu; The initial value problem for a stretched string. *Quart. Appl. Math.* **38** (1980/81), no. 3, 289–311.

[16] F. Hirosawa; Degenerate Kirchhoff equation in ultradifferentiable class. *Nonlinear Anal.* **48** (2002), no. 1, Ser. A: Theory Methods, 77–94.

[17] F. Hirosawa; Global solvability for Kirchhoff equation in special classes of non-analytic functions. *J. Differential Equations* **230** (2006), no. 1, 49–70.

[18] G. Kirchhoff; *Vorlesungen ober mathematische Physik: Mechanik* (section 29.7), Teubner, Leipzig, 1876.

[19] R. Manfrin; On the global solvability of Kirchhoff equation for non-analytic initial data. *J. Differential Equations* **211** (2005), no. 1, 38–60.

[20] R. Manfrin; Global solvability to the Kirchhoff equation for a new class of initial data. *Port. Math. (N.S.)* **59** (2002), no. 1, 91–109.

[21] K. Nishihara; On a global solution of some quasilinear hyperbolic equation. *Tokyo J. Math.* **7** (1984), no. 2, 437–459.

[22] S. I. Pohozaev; On a class of quasilinear hyperbolic equations. *Mat. Sb. (N.S.)* **96(138)** (1975), 152–166) (English transl.: *Math. USSR Sbornik* **25** (1975), 145–158).

[23] S. I. Pohozaev; The Kirchhoff quasilinear hyperbolic equation, *Differentsial'nye Uravneniya* **21** (1985), no. 1 (1985), 101–108 (English transl.: *Differential Equations* **21** (1985), 82–87).

[24] M. Reed, B. Simon; *Methods of Modern Mathematical Physics, I: Functional Analysis. Second edition.* Academin Press, New York, 1980.

[25] T. Yamazaki; Global solvability for the Kirchhoff equations in exterior domains of dimension three. *J. Differential Equations* **210** (2005), no. 2, 290–316.

[26] T. Yamazaki; Global solvability for the Kirchhoff equations in exterior domains of dimension larger than three. *Math. Methods Appl. Sci.* **27** (2004), no. 16, 1893–1916.

[27] B. Yordanov; Global existence in Sobolev spaces for a class of nonlinear Kirchhoff equations. Talk given during the 7th ISAAC congress (London 2009).