Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 245 (2008) 2979–3007
Journal of Differential Equations
www.elsevier.com/locate/jde

Hyperbolic–parabolic singular perturbation for mildly degenerate Kirchhoff equations: Time-decay estimates

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Received 24 October 2007; revised 8 April 2008
Available online 4 June 2008

Abstract

We consider the second order Cauchy problem

$$\varepsilon u_\varepsilon'' + u_\varepsilon' + m\left(\left|A^{1/2}u_\varepsilon\right|^2\right)Au_\varepsilon = 0, \qquad u_\varepsilon(0) = u_0, \qquad u_\varepsilon'(0) = u_1,$$

and the first order limit problem

$$u' + m\left(\left|A^{1/2}u\right|^2\right)Au = 0, \qquad u(0) = u_0,$$

where $\varepsilon > 0$, H is a Hilbert space, A is a self-adjoint nonnegative operator on H with dense domain $D(A)$, $(u_0, u_1) \in D(A) \times D(A^{1/2})$, and $m : [0, +\infty) \to [0, +\infty)$ is a function of class C^1.

We prove decay estimates (as $t \to +\infty$) for solutions of the first order problem, and we show that analogous estimates hold true for solutions of the second order problem provided that ε is small enough. We also show that our decay rates are optimal in many cases.

The abstract results apply to parabolic and hyperbolic partial differential equations with nonlocal nonlinearities of Kirchhoff type.

MSC: 35B25; 35B40; 35L80

Keywords: Degenerate parabolic equations; Degenerate damped hyperbolic equations; Singular perturbations; Kirchhoff equations; Decay rate of solutions

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.04.017

1. Introduction

Let H be a real Hilbert space. Given x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We always assume that A is nonnegative, namely $\langle Au, u\rangle \geqslant 0$ for every $u \in D(A)$. For any such operator the power A^α is defined for every $\alpha \geqslant 0$ in a suitable domain $D(A^\alpha)$. Let $m : [0, +\infty) \to [0, +\infty)$ be a function of class C^1.

For every $\varepsilon > 0$ we consider the second order Cauchy problem

$$\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + m\big(\big|A^{1/2}u_\varepsilon(t)\big|^2\big)Au_\varepsilon(t) = 0 \quad \forall t \geqslant 0, \tag{1.1}$$

$$u_\varepsilon(0) = u_0, \qquad u_\varepsilon'(0) = u_1. \tag{1.2}$$

This problem is just an abstract setting of the initial boundary value problem for the hyperbolic partial differential equation (PDE)

$$\varepsilon u_{tt}^\varepsilon(t, x) + u_t^\varepsilon(t, x) - m\left(\int_\Omega \big|\nabla u^\varepsilon(t, x)\big|^2\,dx\right)\Delta u^\varepsilon(t, x) = 0 \tag{1.3}$$

in an open set $\Omega \subseteq \mathbb{R}^n$. This equation is a model for the damped small transversal vibrations of an elastic string ($n = 1$) or membrane ($n = 2$) with uniform density ε.

We also consider the first order Cauchy problem

$$u'(t) + m\big(\big|A^{1/2}u(t)\big|^2\big)Au(t) = 0 \quad \forall t \geqslant 0, \tag{1.4}$$

$$u(0) = u_0, \tag{1.5}$$

obtained setting formally $\varepsilon = 0$ in (1.1), and forgetting the initial condition u_1 in (1.2). In the concrete setting of (1.3) the limit problem involves a PDE of parabolic type.

The main research lines on this subject concern the behavior of $u_\varepsilon(t)$ as $t \to +\infty$ and as $\varepsilon \to 0^+$. In this paper we focus on the first issue, proving decay estimates for $u(t)$ and $u_\varepsilon(t)$ as $t \to +\infty$. The decay properties of $u(t)$ are stated in Theorem 3.2 and are proved by means of classical energy estimates for parabolic equations. We used these estimates on $u(t)$ as a benchmark when looking at the second order problem, and indeed in Theorem 3.6 we show that solutions of (1.1), (1.2) satisfy similar decay estimates provided that ε is small enough. Also the constants (and not only the decay rates) involved in our estimates for the second order problem tend (as $\varepsilon \to 0^+$) to the corresponding constants for the first order problem.

Most of our estimates are independent on ε. For this reason we plan to apply them in a future paper in order to provide global-in-time estimates for $|u_\varepsilon - u|$ as $\varepsilon \to 0^+$ (see also [8,9]).

Our proofs involve comparison principles for ordinary differential equations (see Lemmas 4.1 and 4.2) together with estimates of suitable *first order* energies (see Proposition 3.10). Our methods are quite general and do *not* require any special assumption on the nonlinearity m. Nevertheless we obtain decay rates for $|A^{1/2}u_\varepsilon|^2$, $|Au_\varepsilon|^2$, $|u_\varepsilon'|^2$ which are optimal and often better than those stated in the literature.

As a byproduct of our energy inequalities we get also decay estimates for $\varepsilon|A^{1/2}u_\varepsilon'|^2$. We state them because in some cases they improve the existing literature, but we suspect they are not optimal (we can indeed prove better estimates both for more regular data, and for special choices of m).

This paper is organized as follows. Section 2 contains a reasonably short summary of the literature and a comparison with the estimates obtained in this paper. Our results are formally stated in Section 3 and proved in Section 4.

2. Survey of existence results and decay estimates

Let us recall some terminology.

- The operator A is called *coercive* if there exists a constant $\nu > 0$ such that $\langle Au, u\rangle \geqslant \nu|u|^2$ for every $u \in D(A)$.
- Eq. (1.1) or (1.4) is called *nondegenerate* if there exists a constant $\mu > 0$ such that $m(\sigma) \geqslant \mu$ for every $\sigma \geqslant 0$.
- Problem (1.1), (1.2) or (1.4), (1.5) is called *mildly degenerate* if the initial condition u_0 belongs to $D(A^{1/2})$ and satisfies the nondegeneracy condition

$$m\big(\big|A^{1/2}u_0\big|^2\big) > 0. \tag{2.1}$$

This means that m may vanish, but not at the initial time. In many statements we also assume that u_0 satisfies the stronger nondegeneracy condition

$$\big|A^{1/2}u_0\big|^2 m\big(\big|A^{1/2}u_0\big|^2\big) > 0. \tag{2.2}$$

Note that (2.2) is equivalent to (2.1) if $m(0) = 0$.

2.1. Existence results

Existence of a global solution for problem (1.4), (1.5) can be established under very general assumptions on m, A, u_0. In particular one can prove the following result (see [2,7,13]).

Theorem 2.1. *Let $m : [0, +\infty) \to [0, +\infty)$ be a locally Lipschitz continuous function. Let us assume that $u_0 \in D(A)$ satisfies the nondegeneracy condition* (2.2)*.*

Then problem (1.4)*,* (1.5) *has a unique solution*

$$u \in C^1\big([0, +\infty); H\big) \cap C^0\big([0, +\infty); D(A)\big).$$

Moreover $A^{1/2}u(t) \neq 0$ for every $t \geqslant 0$, and $u \in C^1((0, +\infty); D(A^\alpha))$ for every $\alpha \geqslant 0$.

The standard result concerning problem (1.1), (1.2) is the existence of a unique global solution provided that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy (2.1) and ε is small enough. This was proved by E. de Brito [3], Y. Yamada [24], and K. Nishihara [17] in the nondegenerate case, then by K. Nishihara and Y. Yamada [18] in the mildly degenerate case with $m(\sigma) = \sigma^\gamma$ ($\gamma \geqslant 1$), and finally by the authors [6] with a general locally Lipschitz continuous nonlinearity $m(\sigma) \geqslant 0$.

The following theorem is a straightforward consequence of Theorem 2.2 of [6].

Theorem 2.2. *Let $m : [0, +\infty) \to [0, +\infty)$ be a locally Lipschitz continuous function. Let us assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the nondegeneracy condition* (2.1).

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (1.1), (1.2) *has a unique global solution*

$$u_\varepsilon \in C^2\big([0, +\infty); H\big) \cap C^1\big([0, +\infty); D\big(A^{1/2}\big)\big) \cap C^0\big([0, +\infty); D(A)\big).$$

We recall also that there is a wide literature on the nondissipative case: the interested reader is referred to the surveys [1] and [23], or to the most recent papers [10,12].

2.2. The hyperbolic problem: Decay estimates

A lot of papers have been devoted to decay estimates for dissipative Kirchhoff equations. Comparing such results is a hard task because of the different settings (abstract or concrete equation, with or without forcing terms), of the different approaches (either $\varepsilon = 1$ and small data, or fixed data and small ε), of the different quantities considered (u_ε, $A^{1/2}u_\varepsilon$, Au_ε, u'_ε, $A^{1/2}u'_\varepsilon$, u''_ε), and of the different assumptions on m (degenerate or nondegenerate), A (coercive or noncoercive), u_0, u_1 (more or less regular). For this reason in this section we do not quote the results exactly as they are stated in the appropriate papers, but we always rephrase them in the setting of Theorem 2.2.

We also neglect decay estimates on u_ε because in the coercive case they can be easily deduced from estimates on $A^{1/2}u_\varepsilon$, while in the noncoercive case there is no reason for $u_\varepsilon(t)$ to tend to 0, even for a linear equation (when m is a positive constant).

2.2.1. Decay estimates for coercive operators

The nondegenerate case was considered by M. Hosoya and Y. Yamada [11] (see also [4,16]).

The degenerate case with $m(\sigma) = \sigma^\gamma$ ($\gamma \geqslant 1$) was considered by K. Nishihara and Y. Yamada [18].

Later on, better estimates have been obtained by T. Mizumachi [14] and K. Ono [19] in the special case $\gamma = 1$. Indeed their decay rates for $A^{1/2}u_\varepsilon$ and u'_ε improve those obtained by setting $\gamma = 1$ in the corresponding estimates of [18].

All these results are summed up in the left column of Table 1.

The case of a general nonlinearity $m(\sigma) \geqslant 0$ was considered by the authors in [6]. When $m(\sigma) > 0$ for every $\sigma > 0$ they proved that $|A^{1/2}u_\varepsilon| \to 0$ and $|u'_\varepsilon| \to 0$, without estimates of the decay rate.

In this paper we provide such estimates in terms of m. Our results, when applied to the particular choices of m considered in the literature, improve most of the known estimates. In particular we always obtain lower bounds for $|A^{1/2}u_\varepsilon|^2$ and $|Au_\varepsilon|^2$, our estimates on $|u'_\varepsilon|^2$ are ε-independent, and we have better exponents in the case $m(\sigma) = \sigma^\gamma$ (note that our estimates for $m(\sigma) = \sigma$ are just the case $\gamma = 1$ in our estimates for $m(\sigma) = \sigma^\gamma$).

2.2.2. Decay estimates for noncoercive operators

The nondegenerate case was considered by Y. Yamada [24], then by K. Ono [21], and finally in the recent paper by H. Hashimoto and T. Yamazaki [9], where the ε-independent estimate on u'_ε is proved.

The case $m(\sigma) = \sigma^\gamma$ was considered by K. Ono [22]. Finally, better estimates were obtained by T. Mizumachi [15] and K. Ono [20] in the special case $\gamma = 1$.

Table 1
Decay estimates in the coercive case

	Literature	Present paper
$m(\sigma) \geqslant \mu_1 > 0$	$\lvert A^{1/2}u_\varepsilon(t)\rvert^2 \leqslant c_2 e^{-\alpha_2 t}$	$c_1 e^{-\alpha_1 t} \leqslant \lvert A^{1/2}u_\varepsilon(t)\rvert^2 \leqslant c_2 e^{-\alpha_2 t}$
	$\lvert Au_\varepsilon(t)\rvert^2 \leqslant c_2 e^{-\alpha_2 t}$	$c_1 e^{-\alpha_1 t} \leqslant \lvert Au_\varepsilon(t)\rvert^2 \leqslant c_2 e^{-\alpha_2 t}$
	$\varepsilon\lvert u'_\varepsilon(t)\rvert^2 \leqslant c e^{-\alpha t}$	$\lvert u'_\varepsilon(t)\rvert^2 \leqslant c e^{-\alpha t}$
$m(\sigma) = \sigma^\gamma$	$\lvert A^{1/2}u_\varepsilon\rvert^2 \leqslant \frac{c_2}{(1+t)^{1/\gamma}}$	$\frac{c_1}{(1+t)^{1/\gamma}} \leqslant \lvert A^{1/2}u_\varepsilon\rvert^2 \leqslant \frac{c_2}{(1+t)^{1/\gamma}}$
	$\lvert Au_\varepsilon\rvert^2 \leqslant c$	$\frac{c_1}{(1+t)^{1/\gamma}} \leqslant \lvert Au_\varepsilon\rvert^2 \leqslant \frac{c_2}{(1+t)^{1/\gamma}}$
	$\varepsilon\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^{1+1/\gamma}}$	$\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^{2+1/\gamma}}$
$m(\sigma) = \sigma$	$\frac{c_1}{1+t} \leqslant \lvert A^{1/2}u_\varepsilon\rvert^2 \leqslant \frac{c_2}{1+t}$	$\frac{c_1}{1+t} \leqslant \lvert A^{1/2}u_\varepsilon\rvert^2 \leqslant \frac{c_2}{1+t}$
	$\frac{c_1}{1+t} \leqslant \lvert Au_\varepsilon\rvert^2 \leqslant \frac{c_2}{1+t}$	$\frac{c_1}{1+t} \leqslant \lvert Au_\varepsilon\rvert^2 \leqslant \frac{c_2}{1+t}$
	$\varepsilon\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^3}$	$\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^3}$

Table 2
Decay estimates for noncoercive operators

	Literature	Present paper
$m(\sigma) \geqslant \mu_1 > 0$	$\lvert A^{1/2}u_\varepsilon(t)\rvert^2 \leqslant \frac{c_2}{1+t}$	$c_1 e^{-\alpha_1 t} \leqslant \lvert A^{1/2}u_\varepsilon(t)\rvert^2 \leqslant \frac{c_2}{1+t}$
	$\lvert Au_\varepsilon(t)\rvert^2 \leqslant \frac{c}{(1+t)^2}$	$\lvert Au_\varepsilon(t)\rvert^2 \leqslant \frac{c}{(1+t)^2}$
	$\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^2}$	$\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^2}$
$m(\sigma) = \sigma^\gamma$	$\lvert A^{1/2}u_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^{1/(\gamma+1)}}$	$\frac{c_1}{(1+t)^{1/\gamma}} \leqslant \lvert A^{1/2}u_\varepsilon\rvert^2 \leqslant \frac{c_2}{(1+t)^{1/(\gamma+1)}}$
	$\lvert Au_\varepsilon\rvert^2 \leqslant c$	$\lvert Au_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^{1/\gamma}}$
	$\varepsilon\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{1+t}$	$\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^{1+(\gamma^2+1)/(\gamma^2+\gamma)}}$
$m(\sigma) = \sigma$	$\frac{c_1}{(1+t)^{\alpha/\varepsilon}} \leqslant \lvert A^{1/2}u_\varepsilon\rvert^2 \leqslant \frac{c_2}{\sqrt{1+t}}$	$\frac{c_1}{1+t} \leqslant \lvert A^{1/2}u_\varepsilon\rvert^2 \leqslant \frac{c_2}{\sqrt{1+t}}$
	$\lvert Au_\varepsilon\rvert^2 \leqslant \frac{c}{1+t}$	$\lvert Au_\varepsilon\rvert^2 \leqslant \frac{c}{1+t}$
	$\varepsilon\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^2}$	$\lvert u'_\varepsilon\rvert^2 \leqslant \frac{c}{(1+t)^2}$

All these results are stated in the left column of Table 2.

In this paper, with different techniques, we obtain decay estimates in the case of a general nonlinearity $m(\sigma) \geqslant 0$. When applied with special choices of m, we re-obtain or improve the results found in the literature. In particular we always have a lower bound for $|A^{1/2}u_\varepsilon|^2$, our estimate on $|u'_\varepsilon|^2$ is ε-independent, and we get better decay rates in the case $m(\sigma)=\sigma^\gamma$.

3. Statements

3.1. Notations and preliminaries

Throughout this paper we assume that $m:[0,+\infty)\to[0,+\infty)$ is a function of class C^1. We set $\sigma_0:=|A^{1/2}u_0|^2$, and $\mu_0:=m(\sigma_0)$. Since we consider mildly degenerate equations we always have that $\mu_0\neq 0$. Let

$$\sigma_1:=\sup\bigl\{\sigma\in[0,\sigma_0]:\ \sigma\cdot m(\sigma)=0\bigr\}.$$

In a few words, σ_1 is either 0 or the largest $\sigma<\sigma_0$ such that $m(\sigma)=0$. Let us choose $\sigma_2>\sigma_0$ in such a way that $m(\sigma)>0$ for every $\sigma\in(\sigma_1,\sigma_2]$. We set

$$\mu_1:=\min_{\sigma\in[\sigma_1,\sigma_2]} m(\sigma),\qquad \mu_2:=\max_{\sigma\in[\sigma_1,\sigma_2]} m(\sigma),$$

and we denote by L the Lipschitz constant of m in $[\sigma_1,\sigma_2]$. We finally set

$$c(t):=m\bigl(\bigl|A^{1/2}u(t)\bigr|^2\bigr),\qquad c_\varepsilon(t):=m\bigl(\bigl|A^{1/2}u_\varepsilon(t)\bigr|^2\bigr).$$

The following result contains the fundamental ε-independent estimates on the solutions of (1.1), (1.2).

Proposition 3.1. *Let A, m, u_0, u_1, ε_0 be as in Theorem* 2.2.
Then there exist $\delta_1>0$ and $\varepsilon_1\in(0,\varepsilon_0]$ such that for every $\varepsilon\in(0,\varepsilon_1)$ the unique global solution of (1.1), (1.2) *satisfies the following estimates*:

$$\sigma_1\leqslant\bigl|A^{1/2}u_\varepsilon(t)\bigr|^2\quad\text{and}\quad\varepsilon\frac{|u'_\varepsilon(t)|^2}{c_\varepsilon(t)}+\bigl|A^{1/2}u_\varepsilon(t)\bigr|^2\leqslant\sigma_2\quad\forall t\geqslant 0,\tag{3.1}$$

$$\mu_1\leqslant c_\varepsilon(t)\leqslant\mu_2\quad\forall t\geqslant 0,\tag{3.2}$$

$$c_\varepsilon(t)\neq 0\quad\text{and}\quad\left|\frac{c'_\varepsilon(t)}{c_\varepsilon(t)}\right|\leqslant\delta_1\quad\forall t\geqslant 0.\tag{3.3}$$

The proof of Proposition 3.1 involves a careful examination of the main step of the proof of the existence Theorem 2.2, and heavily depends on the particular form of the nonlinearity. In Proposition 3.10 below we state more ε-independent estimates on the solutions of (1.1), (1.2), but in that case all of them hold true more generally for solutions of the linear equation obtained from (1.1) by replacing $m(|A^{1/2}u_\varepsilon(t)|^2)$ with any function $c_\varepsilon(t)$ satisfying (3.2) and (3.3).

We point out that (3.1) means in particular that we are interested in the behavior of $m(\sigma)$ only for $\sigma\in[\sigma_1,\sigma_2]$: in particular we can say that Eq. (1.1) is nondegenerate if and only if $\mu_1>0$, which in turn is true if and only if $\sigma_1=0$ and $m(0)>0$.

The function ψ. There exists a function $\psi \in C^1([\sigma_1, \sigma_2])$ such that

$$0 < \psi(\sigma) \leqslant \sigma m(\sigma) \quad \forall \sigma \in (\sigma_1, \sigma_2], \tag{3.4}$$

$$\psi(\sigma) \text{ is strictly increasing in } [\sigma_1, \sigma_2]. \tag{3.5}$$

Indeed we can set $\psi(\sigma) = \sigma m(\sigma)$ whenever $\sigma m(\sigma)$ is strictly increasing. When this is not the case $\psi(\sigma)$ is any positive (for $\sigma > \sigma_1$) strictly increasing function less or equal than $\sigma m(\sigma)$. For example, in the nondegenerate case ($\mu_1 > 0$) we can take $\psi(\sigma) = \mu_1 \sigma$, in the case $m(\sigma) = \sigma^\gamma$ we can take $\psi(\sigma) = \sigma^{\gamma+1}$.

A Cauchy problem. We consider the Cauchy problem

$$y' = -2ym(y), \qquad y(0) = \sigma_0. \tag{3.6}$$

If $\sigma_0 m(\sigma_0) = 0$ the solution $y(t)$ is constant. If $\sigma_0 m(\sigma_0) \neq 0$, which corresponds to the strong nondegeneracy condition (2.2), there exists $t_0 > 0$ and a unique decreasing function $y : (-t_0, +\infty) \to (\sigma_1, \sigma_2)$ satisfying (3.6). Moreover $y(t) \to \sigma_1$ as $t \to +\infty$.

The heuristic reason for considering this Cauchy problem is the following. Let us assume that $H = \mathbb{R}$ and A is the identity operator, and let $u(t)$ be the solution of the first order problem (1.4), (1.5). Then $y(t) := |A^{1/2}u(t)|^2 = |u(t)|^2$ solves (3.6), and therefore in this trivial case $y(t)$ is by definition the best estimate on the decay rate of $|A^{1/2}u(t)|^2$. In statement (3) of Theorem 3.2 and Theorem 3.6 we show that $y(t)$ gives the decay rate of solution both for the first order and for the second order problem, even for general nonnegative operators.

3.2. Decay estimates for the parabolic equation

If $A^{1/2}u_0 = 0$ or $m(|A^{1/2}u_0|^2) = 0$ the solution of (1.4), (1.5) is constant. Therefore in the parabolic case we can always assume (2.2) without loss of generality.

Theorem 3.2. *Let A be a nonnegative operator, and let $m \in C^1([0, +\infty), [0, +\infty))$. Let us assume that $u_0 \in D(A)$ satisfies the strong nondegeneracy condition* (2.2).

Then we have the following estimates.

(1) *If $\psi \in C^1([\sigma_1, \sigma_2])$ satisfies* (3.4) *and* (3.5) *then*

$$t \cdot \psi\left(\left|A^{1/2}u(t)\right|^2\right) \leqslant \frac{|u_0|^2}{2} \quad \forall t \geqslant 0. \tag{3.7}$$

(2) *We have that*

$$\left|Au(t)\right|^2 \leqslant \frac{|Au_0|^2}{|A^{1/2}u_0|^2} \cdot \left|A^{1/2}u(t)\right|^2 \quad \forall t \geqslant 0. \tag{3.8}$$

(3) *Let $y : (-t_0, +\infty) \to (\sigma_1, \sigma_2)$ be the solution of the Cauchy problem* (3.6). *Then*

$$\left|A^{1/2}u(t)\right|^2 \geqslant y\left(\frac{|Au_0|^2}{|A^{1/2}u_0|^2} t\right) \quad \forall t \geqslant 0. \tag{3.9}$$

If moreover A is coercive with constant $\nu > 0$ then

$$\left|A^{1/2}u(t)\right|^2 \leqslant y(\nu t) \quad \forall t \geqslant 0. \tag{3.10}$$

(4) *If $\mu_1 > 0$ then*

$$t^2 \cdot \left|Au(t)\right|^2 \leqslant \frac{|u_0|^2}{2\mu_1^2} \quad \forall t \geqslant 0. \tag{3.11}$$

(5) *Let $\phi : [0, +\infty) \to [0, +\infty)$ be defined by*

$$\phi(t) := \int_0^t \frac{m(|A^{1/2}u(s)|^2)}{|A^{1/2}u(s)|^2}\, ds \quad \forall t \geqslant 0.$$

Then

$$\phi(t) \cdot \left|Au(t)\right|^2 \leqslant \frac{1}{2} \quad \forall t \geqslant 0. \tag{3.12}$$

Remark 3.3. Let us make a few comments on the estimates provided by Theorem 3.2.

- *Estimates on $|A^{1/2}u|$.* A lower bound is given by (3.9), and two upper bounds are given by (3.7) and (3.10). The second one is in general better, but it requires the coerciveness of the operator. In conclusion:
  - if A is coercive we have upper and lower bounds with the same decay rate given by (3.10) and (3.9);
  - if A is noncoercive we have an upper bound given by (3.7) and a (generally worse) lower bound given by (3.9).
- *Estimates on $|Au|$.* We have three types of estimates for $|Au(t)|$.
  - Let us assume that A is coercive. Using the coerciveness and (3.8) we have that

$$\nu\left|A^{1/2}u(t)\right|^2 \leqslant \left|Au(t)\right|^2 \leqslant \frac{|Au_0|^2}{|A^{1/2}u_0|^2}\left|A^{1/2}u(t)\right|^2, \tag{3.13}$$

    which allows to obtain upper and lower bounds for $|Au(t)|^2$ from the corresponding bounds for $|A^{1/2}u(t)|^2$. If the bounds on $|A^{1/2}u(t)|^2$ are optimal, then also the bounds on $|Au(t)|^2$ are optimal.
  - If A is noncoercive the estimate from above on $|Au|$ coming from (3.7) and (3.8) is not optimal. Better estimates are indeed provided by (3.11) in the nondegenerate case, and by (3.12) in the general case.

    In the noncoercive case we do not have estimates for $|Au(t)|$ from below.
- *Estimates on $|u'|$.* Due to (1.4) they can be easily derived from the estimates on $|A^{1/2}u(t)|$ and $|Au(t)|$.

Corollary 3.4. *Let A be a nonnegative operator, let $u_0 \in D(A)$ with $A^{1/2}u_0 \neq 0$. Let us assume that Eq.* (1.4) *is nondegenerate* ($\mu_1 > 0$).

- *If A is coercive there exist positive constants α_1, α_2, c_1, c_2 such that*

$$c_1 e^{-\alpha_1 t} \leqslant \left|A^{1/2}u(t)\right|^2 + \left|Au(t)\right|^2 + \left|u'(t)\right|^2 \leqslant c_2 e^{-\alpha_2 t} \quad \forall t \geqslant 0. \tag{3.14}$$

- *If A is noncoercive there exist positive constants α_1, c_1, c_2, c_3 such that*

$$c_1 e^{-\alpha_1 t} \leqslant \left|A^{1/2}u(t)\right|^2 \leqslant \frac{c_2}{1+t} \quad \forall t \geqslant 0, \tag{3.15}$$

$$\left|u'(t)\right|^2 + \left|Au(t)\right|^2 \leqslant \frac{c_3}{(1+t)^2} \quad \forall t \geqslant 0. \tag{3.16}$$

Corollary 3.5. *Let A be a nonnegative operator, let $m(\sigma) = \sigma^\gamma$ with $\gamma \geqslant 1$, and let $u_0 \in D(A)$ with $A^{1/2}u_0 \neq 0$.*

- *If A is coercive there exist positive constants $c_1, \dots, c_4$ such that*

$$\frac{c_1}{(1+t)^{1/\gamma}} \leqslant \left|A^{1/2}u(t)\right|^2 + \left|Au(t)\right|^2 \leqslant \frac{c_2}{(1+t)^{1/\gamma}} \quad \forall t \geqslant 0, \tag{3.17}$$

$$\frac{c_3}{(1+t)^{2+1/\gamma}} \leqslant \left|u'(t)\right|^2 \leqslant \frac{c_4}{(1+t)^{2+1/\gamma}} \quad \forall t \geqslant 0. \tag{3.18}$$

- *If A is noncoercive there exist positive constants $c_1, \dots, c_4$ such that*

$$\frac{c_1}{(1+t)^{1/\gamma}} \leqslant \left|A^{1/2}u(t)\right|^2 \leqslant \frac{c_2}{(1+t)^{1/(\gamma+1)}} \quad \forall t \geqslant 0, \tag{3.19}$$

$$\left|Au(t)\right|^2 \leqslant \frac{c_3}{(1+t)^{1/\gamma}} \quad \forall t \geqslant 0, \tag{3.20}$$

$$\left|u'(t)\right|^2 \leqslant \frac{c_4}{(1+t)^{1+(\gamma^2+1)/(\gamma^2+\gamma)}} \quad \forall t \geqslant 0. \tag{3.21}$$

3.3. Decay estimates for the hyperbolic equation

The following result is the hyperbolic counterpart of Theorem 3.2.

Theorem 3.6. *Let A be a nonnegative operator, and let $m \in C^1([0,+\infty),[0,+\infty))$. Let us assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the nondegeneracy condition* (2.1).

Then there exist $\varepsilon_\star > 0$, and positive constants $k_1, \dots, k_{10}$ such that for every $\varepsilon \in (0, \varepsilon_\star)$ we have the following estimates.

(1) *If $\psi \in C^1([\sigma_1, \sigma_2])$ satisfies* (3.4) *and* (3.5)*, then*

$$t \cdot \psi\left(\varepsilon \frac{|u_\varepsilon'(t)|^2}{c_\varepsilon(t)} + \left|A^{1/2}u_\varepsilon(t)\right|^2\right) \leqslant \frac{|u_0|^2}{2} + k_1\varepsilon \quad \forall t \geqslant 0. \tag{3.22}$$

(2) *Let us assume that u_0 satisfies* (2.2)*. Then*

$$\left|Au_\varepsilon(t)\right|^2 \leqslant \left[\frac{|Au_0|^2}{|A^{1/2}u_0|^2} + \varepsilon k_2\right] \cdot \left|A^{1/2}u_\varepsilon(t)\right|^2 \quad \forall t \geqslant 0, \tag{3.23}$$

$$\frac{|u_\varepsilon'(t)|^2}{c_\varepsilon^2(t)} \leqslant k_3\left|A^{1/2}u_\varepsilon(t)\right|^2 \quad \forall t \geqslant 0. \tag{3.24}$$

(3) *Let us assume that* u_0 *satisfies* (2.2)*, and let* $y : (-t_0, +\infty) \to (\sigma_1, \sigma_2)$ *be the solution of the Cauchy problem* (3.6)*. Then*

$$\left|A^{1/2}u(t)\right|^2 \geqslant y\left(\left(\frac{|Au_0|^2}{|A^{1/2}u_0|^2} + k_4\varepsilon\right)t + k_5\varepsilon\right) \quad \forall t \geqslant 0. \tag{3.25}$$

If moreover A *is coercive with constant* $\nu > 0$ *then*

$$\left|A^{1/2}u(t)\right|^2 \leqslant y\big((\nu - k_6\varepsilon)t - k_5\varepsilon\big) \quad \forall t \geqslant 0. \tag{3.26}$$

(4) *If* $\mu_1 > 0$ *then*

$$t^2 \cdot \left(\varepsilon\frac{|A^{1/2}u_\varepsilon'(t)|^2}{c_\varepsilon(t)} + \left|Au_\varepsilon(t)\right|^2\right) \leqslant \frac{|u_0|^2}{2\mu_1^2} + k_7\varepsilon \quad \forall t \geqslant 0, \tag{3.27}$$

$$t^2 \cdot \left|u_\varepsilon'(t)\right|^2 \leqslant \mu_2^2\frac{|u_0|^2}{2\mu_1^2} + k_8\varepsilon \quad \forall t \geqslant 0. \tag{3.28}$$

(5) *Let us assume that* u_0 *satisfies* (2.2)*, and let* $\phi_\varepsilon : [0, +\infty) \to [0, +\infty)$ *be defined by*

$$\phi_\varepsilon(t) := \int_0^t \frac{m(|A^{1/2}u_\varepsilon(s)|^2)}{|A^{1/2}u_\varepsilon(s)|^2}\, ds \quad \forall t \geqslant 0.$$

Then

$$\phi_\varepsilon(t) \cdot \left(\varepsilon\frac{|A^{1/2}u_\varepsilon'(t)|^2}{c_\varepsilon(t)} + \left|Au_\varepsilon(t)\right|^2\right) \leqslant \frac{1}{2} + k_9\sqrt{\varepsilon} \quad \forall t \geqslant 0, \tag{3.29}$$

$$\phi_\varepsilon(t) \cdot \frac{|u_\varepsilon'(t)|^2}{c_\varepsilon^2(t)} \leqslant k_{10} \quad \forall t \geqslant 0. \tag{3.30}$$

We remark that setting formally $\varepsilon = 0$ in (3.22), (3.23), (3.25), (3.26), (3.27), (3.29) we obtain the corresponding estimates of Theorem 3.2. Also the comments contained in Remark 3.3 can be easily transposed to the hyperbolic setting.

Corollary 3.7. *Let* A *be a nonnegative operator, and let* $(u_0, u_1) \in D(A) \times D(A^{1/2})$*. Let us assume that Eq.* (1.1) *is nondegenerate* ($\mu_1 > 0$)*.*

- *If* A *is coercive and* $A^{1/2}u_0 \neq 0$*, then for every small enough* ε *the solution* u_ε *of* (1.1)*,* (1.2) *satisfies all the estimates quoted in the appropriate section of Table* 1.

- *If A is noncoercive, then for every small enough ε the solution u_ε of* (1.1), (1.2) *satisfies all the estimates quoted in the appropriate section of Table* 2 (*the estimate from below for $|A^{1/2}u_\varepsilon|^2$ requires that $A^{1/2}u_0 \neq 0$*).

In both cases $\varepsilon|A^{1/2}u'_\varepsilon(t)|^2$ decays as $|Au_\varepsilon(t)|^2$.

Corollary 3.8. *Let A be a nonnegative operator, let $m(\sigma) = \sigma^\gamma$, and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $A^{1/2}u_0 \neq 0$.*

Then for every small enough ε the solution u_ε of (1.1), (1.2) *satisfies all the estimates quoted in the appropriate section of Table* 1 (*if A is coercive*) *or Table* 2 (*if A is noncoercive*).

As for $\varepsilon|A^{1/2}u'_\varepsilon(t)|^2$ we have that $\varepsilon|A^{1/2}u'_\varepsilon(t)|^2 \leqslant c(1+t)^{-1-1/\gamma}$ in the coercive case, and $\varepsilon|A^{1/2}u'_\varepsilon(t)|^2 \leqslant c(1+t)^{-1-1/(\gamma^2+\gamma)}$ in the noncoercive case.

Corollary 3.9. *Let A be a nonnegative operator, let $m : [0, +\infty) \to [0, +\infty)$ be a function of class C^1 such that $m(\sigma) = 0$ if and only if $\sigma = 0$, and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $A^{1/2}u_0 \neq 0$.*

Then there exists a function $\varphi : [0, +\infty) \to (0, +\infty)$ such that

- *$\varphi(t) \to 0$ as $t \to +\infty$;*
- *for every small enough ε the solution u_ε of* (1.1), (1.2) *satisfies*

$$\left|A^{1/2}u_\varepsilon(t)\right|^2 + \left|Au_\varepsilon(t)\right|^2 + \left|u'_\varepsilon(t)\right|^2 + \varepsilon\left|A^{1/2}u'_\varepsilon(t)\right|^2 \leqslant \varphi(t) \quad \forall t \geqslant 0.$$

3.4. Energy estimates

Our proofs rely on suitable energy estimates. In the parabolic case they follow from the monotonicity of the classical quantities

$$E_k(t) := \left|A^{k/2}u(t)\right|^2, \qquad P(t) := \frac{|Au(t)|^2}{|A^{1/2}u(t)|^2}.$$

There are several ways to adapt these energies to the hyperbolic setting. We consider three extensions of $E_k(t)$ (we actually need only the cases $k = 0$ and $k = 1$)

$$D_{\varepsilon,k} := \frac{|A^{k/2}u_\varepsilon|^2}{2} + \varepsilon\langle A^{k/2}u_\varepsilon, A^{k/2}u'_\varepsilon\rangle, \tag{3.31}$$

$$E_{\varepsilon,k} := \varepsilon\frac{|A^{k/2}u'_\varepsilon|^2}{c_\varepsilon} + \left|A^{(k+1)/2}u_\varepsilon\right|^2, \tag{3.32}$$

$$G_\varepsilon := \frac{|u'_\varepsilon|^2}{c_\varepsilon^2}, \tag{3.33}$$

and the following three extensions of $P(t)$

$$P_\varepsilon := \frac{\varepsilon}{c_\varepsilon}\frac{|A^{1/2}u_\varepsilon|^2|A^{1/2}u'_\varepsilon|^2 - \langle Au_\varepsilon, u'_\varepsilon\rangle^2}{|A^{1/2}u_\varepsilon|^4} + \frac{|Au_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2}, \tag{3.34}$$

$$Q_\varepsilon := \frac{|u_\varepsilon'|^2}{c_\varepsilon^2|A^{1/2}u_\varepsilon|^2}, \tag{3.35}$$

$$R_\varepsilon := \varepsilon\frac{|A^{1/2}u_\varepsilon'|^2}{c_\varepsilon|A^{1/2}u_\varepsilon|^2} + \frac{|Au_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2}. \tag{3.36}$$

We point out that the first summand in the definition of P_ε is nonnegative by Cauchy–Schwarz inequality. As far as we know, $D_{\varepsilon,k}$ and $E_{\varepsilon,k}$ have been largely used in the literature, G_ε appeared in [6], P_ε and Q_ε where introduced in [5], and R_ε seems to be new. Most of the first order energies used in literature in the particular cases $m(\sigma)=\sigma$ or $m(\sigma)=\sigma^\gamma$ are special instances of those defined above.

The following result contains the estimates we need on these energies. We state them in the setting of linear equations.

Proposition 3.10. *Let A be a nonnegative operator, let $(u_0,u_1)\in D(A)\times D(A^{1/2})$, and let $\varepsilon_0>0$. For every $\varepsilon\in(0,\varepsilon_0)$ let $c_\varepsilon:[0,+\infty)\to[0,+\infty)$. Let us assume that* (3.2) *and* (3.3) *are satisfied for every $\varepsilon\in(0,\varepsilon_0)$ for suitable nonnegative constants μ_1, μ_2, δ_1. Let u_ε be the solution of the linear problem*

$$\varepsilon u_\varepsilon''(t)+u_\varepsilon'(t)+c_\varepsilon(t)Au_\varepsilon(t)=0 \quad \forall t\geqslant 0, \tag{3.37}$$

with initial data (1.2)*. Then we have the following estimates.*

(1) *Let us define $D_{\varepsilon,k}$, $E_{\varepsilon,k}$ (for $k\in\{0,1\}$), and G_ε according to* (3.31), (3.32), (3.33)*. Then there exists $\varepsilon_1\in(0,\varepsilon_0]$ such that for every $\varepsilon\in(0,\varepsilon_1)$ we have that*

$$\frac{|A^{k/2}u_\varepsilon(t)|^2}{4}+\int_0^t c_\varepsilon(s)\left|A^{(k+1)/2}u_\varepsilon(s)\right|^2 ds \leqslant D_{\varepsilon,k}(0)+2\varepsilon\mu_2E_{\varepsilon,k}(0) \quad \forall t\geqslant 0, \tag{3.38}$$

$$E_{\varepsilon,k}(t)+\int_0^t \frac{|A^{k/2}u_\varepsilon'(s)|^2}{c_\varepsilon(s)}\,ds \leqslant E_{\varepsilon,k}(0) \quad \forall t\geqslant 0, \tag{3.39}$$

$$G_\varepsilon(t)\leqslant \max\left\{G_\varepsilon(0),4E_{\varepsilon,1}(0)\right\} \quad \forall t\geqslant 0. \tag{3.40}$$

(2) *Let us assume in addition that $A^{1/2}u_0\neq 0$. Then there exist $\varepsilon_2\in(0,\varepsilon_1]$ and $\delta_2>0$ such that for every $\varepsilon\in(0,\varepsilon_2)$ we have that*

$$A^{1/2}u_\varepsilon(t)\neq 0 \quad \forall t\geqslant 0, \tag{3.41}$$

$$\frac{|\langle Au_\varepsilon(t),u_\varepsilon'(t)\rangle|}{|A^{1/2}u_\varepsilon(t)|^2}\leqslant \delta_2 \quad \forall t\geqslant 0. \tag{3.42}$$

In particular the functions $P_\varepsilon(t)$, $Q_\varepsilon(t)$, $R_\varepsilon(t)$ introduced in (3.34), (3.35), (3.36) *are well defined. Moreover for every $\varepsilon\in(0,\varepsilon_2)$ they satisfy the following estimates*

$$P_\varepsilon(t) \leqslant P_\varepsilon(0) \quad \forall t \geqslant 0, \tag{3.43}$$

$$Q_\varepsilon(t) \leqslant \max\bigl\{Q_\varepsilon(0), 4P_\varepsilon(0)\bigr\} \quad \forall t \geqslant 0, \tag{3.44}$$

$$R_\varepsilon(t) + \int_0^t \frac{|A^{1/2}u'_\varepsilon(s)|^2}{c_\varepsilon(s)|A^{1/2}u_\varepsilon(s)|^2}\,ds \leqslant R_\varepsilon(0) + 2\delta_2 P_\varepsilon(0)t \quad \forall t \geqslant 0. \tag{3.45}$$

4. Proofs

4.1. ODE lemmata

The following comparison result has already been used in [6].

Lemma 4.1. *Let $T > 0$, and let $f \in C^0([0,T]) \cap C^1([0,T))$. Let us assume that $f(t) \geqslant 0$ in $[0,T)$, and that there exist two constants $c_1 > 0$, $c_2 \geqslant 0$ such that*

$$f'(t) \leqslant -\sqrt{f(t)}\bigl(c_1\sqrt{f(t)} - c_2\bigr) \quad \forall t \in [0,T).$$

Then we have that $f(t) \leqslant \max\{f(0), (c_2/c_1)^2\}$ for every $t \in [0,T]$.

Lemma 4.2. *Let $t_0 > 0$, let $y : (-t_0, +\infty) \to (\sigma_1, \sigma_2)$ be the solution of Cauchy problem* (3.6)*, and let $w : [0,+\infty) \to \mathbb{R}$ be a function of class C^1 with $w(0) = \sigma_0$.*

Let $f \in C^0([0,+\infty))$, and let us assume that there exist constants $c_1 \geqslant 0$ and $c_2 \in [0,t_0)$ such that

$$\left|\int_0^t f(s)\,ds\right| \leqslant c_1 t + c_2 \quad \forall t \geqslant 0. \tag{4.1}$$

Then for every $\alpha \geqslant c_1$ we have the following implications.

(1) *If w satisfies the differential inequality*

$$w'(t) \leqslant 2w(t)m\bigl(w(t)\bigr)\bigl\{-\alpha + f(t)\bigr\} \quad \forall t \geqslant 0, \tag{4.2}$$

then we have the following estimate

$$w(t) \leqslant y\bigl((\alpha - c_1)t - c_2\bigr) \quad \forall t \geqslant 0. \tag{4.3}$$

(2) *If w satisfies the differential inequality*

$$w'(t) \geqslant 2w(t)m\bigl(w(t)\bigr)\bigl\{-\alpha + f(t)\bigr\} \quad \forall t \geqslant 0, \tag{4.4}$$

then we have the following estimate

$$w(t) \geqslant y\bigl((\alpha + c_1)t + c_2\bigr) \quad \forall t \geqslant 0. \tag{4.5}$$

Proof. For every $t \geqslant 0$ let us set

$$F(t) := \int_0^t f(s)\,ds, \qquad z(t) := y\bigl(\alpha t - F(t)\bigr).$$

We point out that $z(t)$ is well defined because our assumptions on c_1 and c_2 imply that

$$\alpha t - F(t) \geqslant (\alpha - c_1)t - c_2 \geqslant -c_2 > -t_0 \quad \forall t \geqslant 0.$$

Moreover $z(t)$ is a solution of the differential equation $z' = 2zm(z)\{-\alpha + f(t)\}$, while assumption (4.2) is equivalent to say that $w(t)$ is a subsolution of the same equation. Since $w(0) = z(0)$, the standard comparison principle implies that

$$w(t) \leqslant z(t) = y\bigl(\alpha t - F(t)\bigr) \leqslant y(\alpha t - c_1 t - c_2),$$

where in the last inequality we exploited assumption (4.1) and the fact that $y(t)$ is a decreasing function. This proves that (4.2) implies (4.3).

Under assumption (4.4) $w(t)$ is a supersolution of the same equation, hence

$$w(t) \geqslant z(t) = y\bigl(\alpha t - F(t)\bigr) \geqslant y(\alpha t + c_1 t + c_2),$$

which implies (4.5). □

4.2. Proof of Theorem 3.2 and corollaries

Statement (1). Since $\psi' \geqslant 0$ we have that

$$\frac{d}{dt}\bigl[t\psi\bigl(\bigl|A^{1/2}u\bigr|^2\bigr)\bigr] = \psi\bigl(\bigl|A^{1/2}u\bigr|^2\bigr) - 2t\psi'\bigl(\bigl|A^{1/2}u\bigr|^2\bigr)c(t)|Au|^2$$
$$\leqslant \psi\bigl(\bigl|A^{1/2}u\bigr|^2\bigr) \leqslant m\bigl(\bigl|A^{1/2}u\bigr|^2\bigr)\bigl|A^{1/2}u\bigr|^2 = -\frac{d}{dt}\left[\frac{1}{2}|u|^2\right].$$

Integrating in $[0, t]$ we obtain (3.7).

Statement (2). By Cauchy–Schwarz inequality we have that

$$|Au|^4 = \bigl(\bigl\langle A^{3/2}u, A^{1/2}u\bigr\rangle\bigr)^2 \leqslant \bigl|A^{3/2}u\bigr|^2\bigl|A^{1/2}u\bigr|^2, \tag{4.6}$$

hence

$$\frac{d}{dt}\left[\frac{|Au|^2}{|A^{1/2}u|^2}\right] = -2\frac{c(t)}{|A^{1/2}u|^4}\bigl(\bigl|A^{3/2}u\bigr|^2\bigl|A^{1/2}u\bigr|^2 - |Au|^4\bigr) \leqslant 0,$$

and therefore

$$\frac{|Au(t)|^2}{|A^{1/2}u(t)|^2} \leqslant \frac{|Au_0|^2}{|A^{1/2}u_0|^2} \quad \forall t \geqslant 0,$$

which is equivalent to (3.8).

Statement (3). Let us consider the function $w(t) := |A^{1/2}u(t)|^2$. Computing the time derivative and using (3.8) we have that

$$w' = -2m(w)\frac{|Au|^2}{|A^{1/2}u|^2}\left|A^{1/2}u\right|^2 \geqslant -2w \cdot m(w)\frac{|Au_0|^2}{|A^{1/2}u_0|^2}.$$

Applying the second statement of Lemma 4.2 with $\alpha = |Au_0|^2|A^{1/2}u_0|^{-2}$ and $f = 0$ we obtain (3.9).

If the operator is coercive with constant ν, then

$$w' = -2m(w)\frac{|Au|^2}{|A^{1/2}u|^2}\left|A^{1/2}u\right|^2 \leqslant -2\nu w \cdot m(w).$$

Therefore (3.10) follows from statement (1) of Lemma 4.2 (with $\alpha = \nu$ and $f = 0$).

Statement (4). We have that

$$\frac{d}{dt}\left[t\left|A^{1/2}u\right|^2\right] + 2tc(t)|Au|^2 = \left|A^{1/2}u\right|^2 \leqslant \frac{1}{\mu_1}c(t)\left|A^{1/2}u\right|^2 = -\frac{1}{2\mu_1}\frac{d}{dt}|u|^2,$$

hence

$$t\left|A^{1/2}u(t)\right|^2 + 2\int_0^t s \cdot c(s)\left|Au(s)\right|^2 ds \leqslant \frac{1}{2\mu_1}|u_0|^2 \quad \forall t \geqslant 0,$$

and therefore

$$2\int_0^t s\left|Au(s)\right|^2 ds \leqslant \frac{2}{\mu_1}\int_0^t s \cdot c(s)\left|Au(s)\right|^2 ds \leqslant \frac{1}{2\mu_1^2}|u_0|^2 \quad \forall t \geqslant 0. \tag{4.7}$$

Since

$$\frac{d}{dt}\left[t^2|Au|^2\right] = 2t|Au|^2 - 2t^2c(t)\left|A^{3/2}u\right|^2 \leqslant 2t|Au|^2,$$

integrating in $[0, t]$ and using (4.7) we obtain that

$$t^2\left|Au(t)\right|^2 \leqslant 2\int_0^t s\left|Au(s)\right|^2 ds \leqslant \frac{1}{2\mu_1^2}|u_0|^2 \quad \forall t \geqslant 0,$$

which is (3.11).

Statement (5). By (4.6) we have that

$$\frac{d}{dt}\left[\frac{1}{|Au|^2}\right]=\frac{2c(t)|A^{3/2}u|^2}{|Au|^4}=2\frac{c(t)}{|A^{1/2}u|^2}\frac{|A^{3/2}u|^2|A^{1/2}u|^2}{|Au|^4}\geqslant 2\frac{c(t)}{|A^{1/2}u|^2}=2\phi'(t).$$

Integrating in $[0,t]$ we obtain that

$$2\phi(t)\leqslant\frac{1}{|Au(t)|^2}-\frac{1}{|Au_0|^2}\leqslant\frac{1}{|Au(t)|^2}\quad\forall t\geqslant 0,$$

which is equivalent to (3.12).

Proof of Corollary 3.4. By (1.4) we easily obtain that

$$\mu_1\big|Au(t)\big|\leqslant\big|u'(t)\big|\leqslant\mu_2\big|Au(t)\big|. \tag{4.8}$$

In the nondegenerate case we have that $\mu_1>0$ and therefore any lower or upper bound on $|Au|$ yields a similar lower or upper bound on $|u'|$.

Let us assume now that A is coercive, hence (3.13) is satisfied. By (4.8) and (3.13) we have that any upper or lower bound for $|A^{1/2}u(t)|^2$ yields the same upper or lower bound for $|Au(t)|^2$ and $|u'(t)|^2$. In order to estimate $|A^{1/2}u(t)|^2$ we apply (3.9) and (3.10). In the nondegenerate case the solution $y(t)$ of (3.6) satisfies $\sigma_0e^{-\mu_2t}\leqslant y(t)\leqslant\sigma_0e^{-\mu_1t}$, which proves (3.14).

Let us assume now that A is noncoercive. The exponential lower bound on $|A^{1/2}u|^2$ follows from (3.9) as in the coercive case. In order to obtain an upper bound for $|A^{1/2}u|^2$ we have to use (3.7). Since in this case we can take $\psi(\sigma)=\mu_1\sigma$ we obtain that $|A^{1/2}u(t)|^2\leqslant ct^{-1}$. Since of course $|A^{1/2}u(t)|^2\leqslant|A^{1/2}u_0|^2$, up to changing the constant we have (3.15).

As for $|Au|$ (hence also for $|u'|$), (3.11) gives $|Au(t)|^2\leqslant ct^{-2}$. Since of course we have also that $|Au(t)|\leqslant|Au_0|$, up to changing the constant we obtain (3.16). □

Proof of Corollary 3.5. Let us assume that A is coercive. Due to (3.13) estimates on $|Au|^2$ follow from estimates on $|A^{1/2}u|^2$. In order to obtain such estimates it is enough to apply (3.9) and (3.10). In the case $m(\sigma)=\sigma^\gamma$ the solution of the Cauchy problem (3.6) is $y(t)=\sigma_0(1+2\gamma\sigma_0^\gamma t)^{-1/\gamma}$, from which we obtain (3.17). Since $|u'(t)|^2=|A^{1/2}u(t)|^{4\gamma}|Au(t)|^2$, (3.18) follows form (3.17).

Let us assume now that A is noncoercive. The lower bound on $|A^{1/2}u|^2$ can be proved as in the coercive case. In order to obtain an upper bound for $|A^{1/2}u|^2$ we have to use (3.7). Since in this case we can take $\psi(\sigma)=\sigma^{\gamma+1}$ we obtain that $|A^{1/2}u(t)|^2\leqslant ct^{-1/(\gamma+1)}$. Since of course $|A^{1/2}u(t)|^2\leqslant|A^{1/2}u_0|^2$, up to changing the constant we have (3.19).

In order to estimate $|Au|^2$, let us examine (3.12). Up to constants we have that

$$\phi'(t)=m\big(\big|A^{1/2}u(t)\big|^2\big)\big|A^{1/2}u(t)\big|^{-2}=\big|A^{1/2}u(t)\big|^{2(\gamma-1)}\geqslant c(1+t)^{-1+1/\gamma},$$

hence $\phi(t)\geqslant c_1(1+t)^{1/\gamma}-c_2$. Since of course $|Au(t)|^2\leqslant|Au_0|^2$, (3.12) implies (3.20).

As in the coercive case, (3.21) follows from (3.19) and (3.20). □

4.3. Proof of Propositions 3.1 and 3.10

Derivatives of energies. Let us consider the energies defined in (3.31) through (3.36). With simple computations (well, not so simple in the case of P_ε) we obtain that

$$D'_{\varepsilon,k} = -c_\varepsilon \left|A^{(k+1)/2}u_\varepsilon\right|^2 + \varepsilon\left|A^{k/2}u'_\varepsilon\right|^2, \tag{4.9}$$

$$E'_{\varepsilon,k} = -\left(2+\varepsilon\frac{c'_\varepsilon}{c_\varepsilon}\right)\frac{|A^{k/2}u'_\varepsilon|^2}{c_\varepsilon}, \tag{4.10}$$

$$G'_\varepsilon = -\frac{2}{\varepsilon}\left(1+\varepsilon\frac{c'_\varepsilon}{c_\varepsilon}\right)G_\varepsilon - \frac{2}{\varepsilon}\frac{\langle u'_\varepsilon, Au_\varepsilon\rangle}{c_\varepsilon}, \tag{4.11}$$

$$P'_\varepsilon = -\left(2+\varepsilon\frac{c'_\varepsilon}{c_\varepsilon}+4\varepsilon\frac{\langle u'_\varepsilon, Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2}\right)\frac{|A^{1/2}u_\varepsilon|^2|A^{1/2}u'_\varepsilon|^2-\langle Au_\varepsilon, u'_\varepsilon\rangle^2}{c_\varepsilon|A^{1/2}u_\varepsilon|^4}, \tag{4.12}$$

$$Q'_\varepsilon = -\frac{1}{\varepsilon}\left(2+2\varepsilon\frac{c'_\varepsilon}{c_\varepsilon}+2\varepsilon\frac{\langle u'_\varepsilon, Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2}\right)Q_\varepsilon - \frac{2}{\varepsilon}\frac{\langle u'_\varepsilon, Au_\varepsilon\rangle}{c_\varepsilon|A^{1/2}u_\varepsilon|^2}, \tag{4.13}$$

$$R'_\varepsilon = -\left(2+\varepsilon\frac{c'_\varepsilon}{c_\varepsilon}+2\varepsilon\frac{\langle u'_\varepsilon, Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2}\right)\frac{|A^{1/2}u'_\varepsilon|^2}{c_\varepsilon|A^{1/2}u_\varepsilon|^2} - 2\frac{\langle u'_\varepsilon, Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2}\frac{|Au_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2}. \tag{4.14}$$

Proof of Proposition 3.1. We know from Theorem 2.2 that a global solution exists for every $\varepsilon \in (0, \varepsilon_0)$. Now let us choose

$$\delta_1 > 2L \cdot \left(E_{\varepsilon_0,1}(0)\cdot \max\left\{G_{\varepsilon_0}(0), 4E_{\varepsilon_0,1}(0)\right\}\right)^{1/2},$$

and let us choose ε_1 in such a way that the following three conditions are satisfied

$$0<\varepsilon_1 \leqslant \varepsilon_0, \qquad 2\varepsilon_1\delta_1 \leqslant 1, \qquad E_{\varepsilon_1,0}(0) \leqslant \sigma_2.$$

From now on let $\varepsilon < \varepsilon_1$. When $c_\varepsilon(t) \neq 0$ we have that

$$\begin{aligned}\left|\frac{c'_\varepsilon(t)}{c_\varepsilon(t)}\right| &= \left|m'\left(\left|A^{1/2}u_\varepsilon(t)\right|^2\right)\frac{2\langle Au_\varepsilon(t), u'_\varepsilon(t)\rangle}{c_\varepsilon(t)}\right| \\ &\leqslant 2\left|m'\left(\left|A^{1/2}u_\varepsilon(t)\right|^2\right)\right|\cdot\left|Au_\varepsilon(t)\right|\cdot\frac{|u'_\varepsilon(t)|}{c_\varepsilon(t)} \\ &\leqslant 2\left|m'\left(\left|A^{1/2}u_\varepsilon(t)\right|^2\right)\right|\cdot\left\{E_{\varepsilon,1}(t)\cdot G_\varepsilon(t)\right\}^{1/2}.\end{aligned} \tag{4.15}$$

By definition of δ_1 this expression is less than δ_1 for $t=0$. We can therefore define

$$t_\varepsilon := \sup\left\{\tau>0\colon\ c_\varepsilon(t)>0 \text{ and } \left|\frac{c'_\varepsilon(t)}{c_\varepsilon(t)}\right| \leqslant \delta_1\ \forall t\in[0,\tau]\right\}.$$

We claim that $t_\varepsilon = +\infty$. Let us assume by contradiction that $t_\varepsilon \in \mathbb{R}$. This means that either $c(t_\varepsilon)=0$ or $|c'_\varepsilon(t_\varepsilon)/c_\varepsilon(t_\varepsilon)| = \delta_1$. From $|c'_\varepsilon(t)/c_\varepsilon(t)| \leqslant \delta_1$ we easily deduce that $c_\varepsilon(t) \geqslant c_\varepsilon(0)e^{-\delta_1 t} > 0$, which rules out the first possibility. In order to rule out the second one we estimate the three factors in (4.15).

- Since $\delta_1\varepsilon \leqslant 1/2 \leqslant 1$, from (4.10) we have that

$$E'_{\varepsilon,k}(t) \leqslant -\frac{|A^{k/2}u'_\varepsilon(t)|^2}{c_\varepsilon(t)} \tag{4.16}$$

in $[0, t_\varepsilon]$, and in particular $E_{\varepsilon,k}(t) \leqslant E_{\varepsilon,k}(0)$ for every $t \in [0, t_\varepsilon]$.
- Using once more that $\delta_1\varepsilon \leqslant 1/2$, from (4.11) we have that

$$G'_\varepsilon \leqslant -\frac{1}{\varepsilon}G_\varepsilon + \frac{2}{\varepsilon}|Au_\varepsilon|\frac{|u'_\varepsilon|}{c_\varepsilon} \leqslant -\sqrt{G_\varepsilon(t)}\left\{\frac{1}{\varepsilon}\sqrt{G_\varepsilon(t)} - \frac{2}{\varepsilon}\sqrt{E_{\varepsilon,1}(0)}\right\}, \tag{4.17}$$

and therefore from Lemma 4.1 we deduce that $G_\varepsilon(t) \leqslant \max\{G_\varepsilon(0), 4E_{\varepsilon,1}(0)\}$ for every $t \in [0, t_\varepsilon]$.
- We prove that (3.1) holds true for every $t \in [0, t_\varepsilon]$, hence in particular $\sigma_1 \leqslant |A^{1/2}u_\varepsilon(t_\varepsilon)|^2 \leqslant \sigma_2$, and therefore $|m'(|A^{1/2}u_\varepsilon(t_\varepsilon)|^2)| \leqslant L$.
Indeed the inequality on the left is trivial if $\sigma_1 = 0$ and follows from the fact that $c_\varepsilon(t) > 0$ in $[0, t_\varepsilon]$ if $\sigma_1 > 0$. The inequality on the right follows from the estimate $|A^{1/2}u_\varepsilon(t)|^2 \leqslant E_{\varepsilon,0}(t) \leqslant E_{\varepsilon_1,0}(0)$ and our assumption that $E_{\varepsilon_1,0}(0) \leqslant \sigma_2$.

We have therefore that

$$\left|\frac{c'_\varepsilon(t_\varepsilon)}{c_\varepsilon(t_\varepsilon)}\right| \leqslant 2L \cdot \left(E_{\varepsilon_0,1}(0) \cdot \max\left\{G_{\varepsilon_0}(0), 4E_{\varepsilon_0,1}(0)\right\}\right)^{1/2} < \delta_1,$$

which rules out the second possibility and shows that $t_\varepsilon = +\infty$.

Now we know that (3.3) holds true for every $t \geqslant 0$, and therefore all the estimates stated in this proof hold true for every $t \geqslant 0$. This proves (3.1). Finally, (3.2) is a simple consequence of (3.1). □

Proof of statement (1) of Proposition 3.10. As soon as $\varepsilon\delta_1 \leqslant 1/2$ we have that (4.16) holds true for every $t \geqslant 0$. Integrating in $[0, t]$ we obtain (3.39). Also (4.17) holds true for every $t \geqslant 0$, so that (3.40) follows from Lemma 4.1.

Finally, integrating (4.9) in $[0, t]$ we obtain that

$$\int_0^t c_\varepsilon(s)\left|A^{(k+1)/2}u_\varepsilon(s)\right| ds = D_{\varepsilon,k}(0) - D_{\varepsilon,k}(t) + \varepsilon\int_0^t \left|A^{k/2}u'_\varepsilon(s)\right|^2 ds. \tag{4.18}$$

Now let us estimate the last two terms in the right-hand side. By (3.39) we have that

$$\varepsilon\int_0^t \left|A^{k/2}u'_\varepsilon(s)\right|^2 ds \leqslant \varepsilon\mu_2\int_0^t \frac{|A^{k/2}u'_\varepsilon(s)|^2}{c_\varepsilon(s)} ds \leqslant \varepsilon\mu_2 E_{\varepsilon,k}(0). \tag{4.19}$$

Moreover

$$-D_{\varepsilon,k}(t) = -\frac{|A^{k/2}u_\varepsilon(t)|^2}{2} - \varepsilon\left\langle A^{k/2}u_\varepsilon(t), A^{k/2}u'_\varepsilon(t)\right\rangle$$

$$\leqslant -\frac{|A^{k/2}u_\varepsilon(t)|^2}{2} + \frac{|A^{k/2}u_\varepsilon(t)|^2}{4} + \varepsilon^2 \frac{|A^{k/2}u'_\varepsilon(t)|^2}{c_\varepsilon(t)} c_\varepsilon(t)$$
$$\leqslant -\frac{|A^{k/2}u_\varepsilon(t)|^2}{4} + \varepsilon\mu_2 E_{\varepsilon,k}(0). \tag{4.20}$$

Replacing (4.19) and (4.20) in (4.18) we obtain (3.38). □

Proof of statement (2) of Proposition 3.10. Let ε_1 be given by statement (1). Let us choose

$$\delta_2 > \mu_2\big(P_{\varepsilon_1}(0) \cdot \max\big\{Q_{\varepsilon_1}(0), 4P_{\varepsilon_1}(0)\big\}\big)^{1/2},$$

and let us choose ε_2 in such a way that

$$0 < \varepsilon_2 \leqslant \varepsilon_1, \qquad 2 - 2\varepsilon_2\delta_1 - 4\varepsilon_2\delta_2 \geqslant 1.$$

For every $\varepsilon \in (0, \varepsilon_2)$ let us set for simplicity $d_\varepsilon(t) := |A^{1/2}u_\varepsilon(t)|^2$. When $d_\varepsilon(t) \neq 0$ we have that

$$\left|\frac{d'_\varepsilon(t)}{d_\varepsilon(t)}\right| = 2\left|\frac{\langle Au_\varepsilon(t), u'_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^2}\right|$$
$$\leqslant 2\frac{|Au_\varepsilon(t)|}{|A^{1/2}u_\varepsilon(t)|} \cdot \frac{|u'_\varepsilon(t)|}{c_\varepsilon(t)|A^{1/2}u_\varepsilon(t)|} \cdot c_\varepsilon(t)$$
$$\leqslant 2\big\{P_\varepsilon(t) \cdot Q_\varepsilon(t)\big\}^{1/2} \cdot \mu_2. \tag{4.21}$$

It is easy to see that for $t = 0$ this is less than $2\delta_2$. We can therefore define

$$t_\varepsilon := \sup\left\{\tau > 0\colon\ d_\varepsilon(t) > 0 \text{ and } \left|\frac{d'_\varepsilon(t)}{d_\varepsilon(t)}\right| \leqslant 2\delta_2\ \forall t \in [0, \tau]\right\}.$$

We claim that $t_\varepsilon = +\infty$. Let us assume by contradiction that $t_\varepsilon \in \mathbb{R}$. This means that either $d_\varepsilon(t_\varepsilon) = 0$ or $|d'_\varepsilon(t_\varepsilon)/d_\varepsilon(t_\varepsilon)| = 2\delta_2$.

From $|d'_\varepsilon(t)/d_\varepsilon(t)| \leqslant 2\delta_2$ we easily deduce that $d_\varepsilon(t) \geqslant d_\varepsilon(0)e^{-2\delta_2 t} > 0$, which rules out the first possibility.

In order to rule out the second one we estimate the two factors in (4.21).

- Let us consider (4.12). Due to our assumption on ε_2 the term in the parentheses is nonnegative. The remaining fraction is nonnegative by the Cauchy–Schwarz inequality. It follows that $P'_\varepsilon(t) \leqslant 0$ for every $t \in [0, t_\varepsilon)$, hence $P_\varepsilon(t) \leqslant P_\varepsilon(0)$ for every $t \in [0, t_\varepsilon]$.
- Using once more our assumptions on ε_2, from (4.13) we have that

$$Q'_\varepsilon \leqslant -\frac{1}{\varepsilon}Q_\varepsilon + \frac{2}{\varepsilon}\frac{|Au_\varepsilon|}{|A^{1/2}u_\varepsilon|} \cdot \frac{|u'_\varepsilon|}{c_\varepsilon|A^{1/2}u_\varepsilon|} \leqslant -\sqrt{Q_\varepsilon}\left\{\frac{1}{\varepsilon}\sqrt{Q_\varepsilon} - \frac{2}{\varepsilon}\sqrt{P_\varepsilon(0)}\right\}, \tag{4.22}$$

and therefore from Lemma 4.1 we deduce that $Q_\varepsilon(t) \leqslant \max\{Q_\varepsilon(0), 4P_\varepsilon(0)\}$ for every $t \in [0, t_\varepsilon]$.

Coming back to (4.21) we have that

$$\left|\frac{d_\varepsilon'(t_\varepsilon)}{d_\varepsilon(t_\varepsilon)}\right| \leqslant 2\mu_2\big(P_{\varepsilon_1}(0)\cdot\max\big\{Q_{\varepsilon_1}(0), 4P_{\varepsilon_1}(0)\big\}\big)^{1/2} < 2\delta_2.$$

This shows that $t_\varepsilon = +\infty$ and proves estimates (3.41) and (3.42). At this point we have that $P_\varepsilon'(t) \leqslant 0$ for every $t \geqslant 0$, which proves (3.43). Moreover also (4.22) holds true for every $t \geqslant 0$, and so (3.44) follows from Lemma 4.1.

Finally, from (4.14) and our choice of ε_2 we deduce that

$$R_\varepsilon'(t) \leqslant -\frac{|A^{1/2}u_\varepsilon'(t)|^2}{c_\varepsilon(t)|A^{1/2}u_\varepsilon(t)|^2} - \frac{d_\varepsilon'(t)}{d_\varepsilon(t)}\cdot\frac{|Au_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2} \leqslant -\frac{|A^{1/2}u_\varepsilon'(t)|^2}{c_\varepsilon(t)|A^{1/2}u_\varepsilon(t)|^2} + 2\delta_2 P_\varepsilon(0).$$

Integrating in $[0, t]$ we obtain (3.45). □

4.4. *Proof of Theorem 3.6*

To begin with, let ε_1 be as in Proposition 3.1. For every small enough $\varepsilon \in (0, \varepsilon_1)$ all the estimates of Proposition 3.10 hold true. Further smallness assumptions are needed in the proof of the five statements. In any case all such assumptions are satisfied for every ε smaller than a given $\varepsilon_\star > 0$.

Statement (1). Let us consider the function $\psi(E_{\varepsilon,0}(t))$ which is well defined for every $t \geqslant 0$ because of (3.1). Since $\psi_\varepsilon'(\sigma) \geqslant 0$ and $E_{\varepsilon,0}'(t) \leqslant 0$ we have that

$$\frac{d}{dt}\big[t\psi\big(E_{\varepsilon,0}(t)\big)\big] = \psi\big(E_{\varepsilon,0}(t)\big) + t\psi'\big(E_{\varepsilon,0}(t)\big)E_{\varepsilon,0}'(t) \leqslant \psi\big(E_{\varepsilon,0}(t)\big),$$

and therefore integrating in $[0, t]$ we obtain that

$$t\psi\big(E_{\varepsilon,0}(t)\big) \leqslant \int_0^t \psi\big(E_{\varepsilon,0}(s)\big)\,ds. \tag{4.23}$$

Let Λ be the Lipschitz constant of ψ in $[\sigma_1, \sigma_2]$. Then

$$\psi\big(E_{\varepsilon,0}(t)\big) \leqslant \psi\big(\big|A^{1/2}u_\varepsilon(t)\big|^2\big) + \Lambda\varepsilon\frac{|u_\varepsilon'(t)|^2}{c_\varepsilon(t)}.$$

Thus we need to estimate the integral of the summands in the right-hand side. By (3.4) and (3.38) with $k = 0$ we have that

$$\int_0^t \psi\big(\big|A^{1/2}u_\varepsilon\big|^2\big)\,ds \leqslant \int_0^{+\infty} m\big(\big|A^{1/2}u_\varepsilon\big|^2\big)\big|A^{1/2}u_\varepsilon\big|^2\,ds \leqslant D_{\varepsilon,0}(0) + 2\mu_2\varepsilon E_{\varepsilon_1,0}(0). \tag{4.24}$$

By (3.39) with $k = 0$ we have that the integral of the second summand is less or equal than $\varepsilon\Lambda E_{\varepsilon_1,0}(0)$. Replacing this estimate and (4.24) in (4.23), and using the definition of $D_{\varepsilon,0}(0)$, we obtain (3.22).

Statement (2). Since $A^{1/2}u_0 \neq 0$ we can apply inequalities (3.43) and (3.44). We obtain that

$$\frac{|Au_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2} \leqslant P_\varepsilon(t) \leqslant P_\varepsilon(0) =: \frac{|Au_0|^2}{|A^{1/2}u_0|^2} + k_2\varepsilon$$

and

$$\frac{|u_\varepsilon'(t)|^2}{c_\varepsilon^2(t)|A^{1/2}u_\varepsilon(t)|^2} = Q_\varepsilon(t) \leqslant \max\bigl\{Q_{\varepsilon_1}(0), 4P_{\varepsilon_1}(0)\bigr\} =: k_3.$$

This proves (3.23) and (3.24).

Statement (3). Let us set $w_\varepsilon(t) := |A^{1/2}u_\varepsilon(t)|^2$. Then

$$\begin{aligned} w_\varepsilon' &= 2\langle Au_\varepsilon, u_\varepsilon'\rangle = -2m\bigl(\bigl|A^{1/2}u_\varepsilon\bigr|^2\bigr)|Au_\varepsilon|^2 - 2\varepsilon\langle Au_\varepsilon, u_\varepsilon''\rangle \\ &= 2w_\varepsilon m(w_\varepsilon)\left\{-\frac{|Au_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2} - \varepsilon\frac{\langle Au_\varepsilon, u_\varepsilon''\rangle}{m(|A^{1/2}u_\varepsilon|^2)|A^{1/2}u_\varepsilon|^2}\right\}. \end{aligned} \tag{4.25}$$

Now we plan to use Lemma 4.2. To this end we set for simplicity

$$f_\varepsilon(t) := -\frac{\langle Au_\varepsilon(t), u_\varepsilon''(t)\rangle}{m(|A^{1/2}u_\varepsilon(t)|^2)|A^{1/2}u_\varepsilon(t)|^2}.$$

Combining (4.25) and (3.23) we have that w_ε satisfies the differential inequality

$$w_\varepsilon' \geqslant 2w_\varepsilon m(w_\varepsilon)\left\{-\frac{|Au_0|^2}{|A^{1/2}u_0|^2} - k_2\varepsilon + \varepsilon f_\varepsilon(t)\right\}.$$

If we assume the coerciveness of A, then $|Au_\varepsilon|^2|A^{1/2}u_\varepsilon|^{-2} \geqslant \nu$, hence w_ε satisfies the differential inequality

$$w_\varepsilon' \leqslant 2w_\varepsilon m(w_\varepsilon)\bigl\{-\nu + \varepsilon f_\varepsilon(t)\bigr\}.$$

If we prove that there exist constants M_1 and M_2, independent on ε and ν, such that

$$\left|\int_0^t \frac{\langle Au_\varepsilon(s), u_\varepsilon''(s)\rangle}{m(|A^{1/2}u_\varepsilon(s)|^2)|A^{1/2}u_\varepsilon(s)|^2}\,ds\right| \leqslant M_1 t + M_2, \tag{4.26}$$

then (3.25) and (3.26) follow from Lemma 4.2. Indeed in the first case we apply the lemma with $\alpha = |Au_0|^2|A^{1/2}u_0|^{-2} + k_2\varepsilon$, and $f = \varepsilon f_\varepsilon$, so that $c_1 = \varepsilon M_1$, $c_2 = \varepsilon M_2$ (the assumptions $c_2 < t_0$ and $\alpha \geqslant c_1$ are trivially satisfied provided that ε is small enough). In the second case we apply the lemma with $\alpha = \nu$ and once again $f = \varepsilon f_\varepsilon$ (the assumptions on α, c_1, c_2 are satisfied as before for every small enough ε).

In order to prove (4.26) we consider the identity

$$\frac{\langle Au_\varepsilon, u_\varepsilon''\rangle}{c_\varepsilon|A^{1/2}u_\varepsilon|^2} = \left(\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle}{c_\varepsilon|A^{1/2}u_\varepsilon|^2}\right)' - \frac{|A^{1/2}u_\varepsilon'|^2}{c_\varepsilon|A^{1/2}u_\varepsilon|^2} + \frac{c_\varepsilon'}{c_\varepsilon}\cdot\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle}{c_\varepsilon|A^{1/2}u_\varepsilon|^2} + 2\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle^2}{c_\varepsilon|A^{1/2}u_\varepsilon|^4}.$$

Integrating in $[0,t]$ we obtain that

$$\left|\int_0^t \frac{\langle Au_\varepsilon, u_\varepsilon''\rangle}{c_\varepsilon |A^{1/2}u_\varepsilon|^2}\,ds\right| \leqslant \frac{|\langle Au_\varepsilon(t), u_\varepsilon'(t)\rangle|}{c_\varepsilon(t)|A^{1/2}u_\varepsilon(t)|^2} + \frac{|\langle Au_0, u_1\rangle|}{c_\varepsilon(0)|A^{1/2}u_0|^2} + \int_0^t \frac{|A^{1/2}u_\varepsilon'|^2}{c_\varepsilon |A^{1/2}u_\varepsilon|^2}\,ds$$
$$+ \int_0^t \frac{|c_\varepsilon'|}{c_\varepsilon}\cdot\frac{|\langle Au_\varepsilon, u_\varepsilon'\rangle|}{c_\varepsilon|A^{1/2}u_\varepsilon|^2}\,ds + 2\int_0^t \frac{\langle Au_\varepsilon, u_\varepsilon'\rangle^2}{c_\varepsilon|A^{1/2}u_\varepsilon|^4}\,ds$$
$$=: I_1 + I_2 + I_3 + I_4 + I_5.$$

Now let us estimate the five terms separately. By (3.43) and (3.44) there exists a constant δ_3 such that

$$\frac{|\langle Au_\varepsilon, u_\varepsilon'\rangle|}{c_\varepsilon|A^{1/2}u_\varepsilon|^2} \leqslant \frac{|Au_\varepsilon|}{|A^{1/2}u_\varepsilon|}\cdot\frac{|u_\varepsilon'|}{c_\varepsilon|A^{1/2}u_\varepsilon|} \leqslant \sqrt{P_\varepsilon(0)\cdot\max\bigl\{Q_\varepsilon(0), 4P_\varepsilon(0)\bigr\}} \leqslant \delta_3 \tag{4.27}$$

for every $t \geqslant 0$. From (4.27) we have that $I_1 \leqslant \delta_3$ and $I_2 \leqslant \delta_3$. An estimate of I_3 is provided by (3.45). As for I_4 we have that $|c_\varepsilon'/c_\varepsilon| \leqslant \delta_1$ is bounded by (3.3), while the rest of the integrand can be estimated as in the case of I_1: it follows that $I_4 \leqslant \delta_1\delta_3 t$. Finally, using again (4.27) we have that

$$\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle^2}{c_\varepsilon|A^{1/2}u_\varepsilon|^4} = c_\varepsilon\cdot\left(\frac{|\langle Au_\varepsilon, u_\varepsilon'\rangle|}{c_\varepsilon|A^{1/2}u_\varepsilon|^2}\right)^2 \leqslant \mu_2\delta_3^2,$$

so that $I_5 \leqslant \mu_2\delta_3^2 t$. This completes the proof of (4.26).

Statement (4). Step 1. There exists a constant γ_1 such that

$$tE_{\varepsilon,1}(t) + \int_0^t s\cdot\frac{|A^{1/2}u_\varepsilon'(s)|^2}{c_\varepsilon(s)}\,ds \leqslant \gamma_1 \quad \forall t \geqslant 0. \tag{4.28}$$

Indeed from (4.16) with $k=1$ we have that

$$\bigl[tE_{\varepsilon,1}(t)\bigr]' = E_{\varepsilon,1}(t) + tE_{\varepsilon,1}'(t) \leqslant E_{\varepsilon,1}(t) - t\cdot\frac{|A^{1/2}u_\varepsilon'(t)|^2}{c_\varepsilon(t)},$$

hence integrating in $[0,t]$ we obtain that

$$tE_{\varepsilon,1}(t) + \int_0^t s\cdot\frac{|A^{1/2}u_\varepsilon'(s)|^2}{c_\varepsilon(s)}\,ds \leqslant \int_0^t E_{\varepsilon,1}(s)\,ds.$$

It remains to estimate the last integral. By (3.39) and (3.38) with $k=1$ we have that

$$\int_0^t E_{\varepsilon,1}(s)\,ds \leqslant \varepsilon \int_0^t \frac{|A^{1/2}u'_\varepsilon(s)|^2}{c_\varepsilon(s)}\,ds + \frac{1}{\mu_1}\int_0^t c_\varepsilon(s)\big|Au_\varepsilon(s)\big|^2\,ds$$

$$\leqslant \varepsilon E_{\varepsilon,1}(0) + \frac{1}{\mu_1}\big(D_{\varepsilon,1}(0) + 2\varepsilon\mu_2 E_{\varepsilon,1}(0)\big),$$

and it is clear that the last expression is bounded independently on ε.

Step 2. We show that there exists a constant γ_2 such that

$$\int_0^t s\cdot\big|Au_\varepsilon(s)\big|^2\,ds \leqslant \frac{|u_0|^2}{4\mu_1^2} + \varepsilon\gamma_2 \quad \forall t \geqslant 0. \tag{4.29}$$

Indeed from (4.9) with $k = 1$ we have that

$$\big[tD_{\varepsilon,1}(t)\big]' = D_{\varepsilon,1}(t) + tD'_{\varepsilon,1}(t) = D_{\varepsilon,1}(t) - tc_\varepsilon(t)\big|Au_\varepsilon(t)\big|^2 + \varepsilon t\big|A^{1/2}u'_\varepsilon(t)\big|^2,$$

hence integrating in $[0,t]$ we obtain that

$$tD_{\varepsilon,1}(t) + \int_0^t s\cdot c_\varepsilon(s)\big|Au_\varepsilon(s)\big|^2\,ds = \frac{1}{2}\int_0^t \big|A^{1/2}u_\varepsilon(s)\big|^2\,ds + \varepsilon\int_0^t \big\langle Au_\varepsilon(s), u'_\varepsilon(s)\big\rangle\,ds$$

$$+ \varepsilon\int_0^t s\big|A^{1/2}u'_\varepsilon(s)\big|^2\,ds.$$

Now let us estimate the three integrals in the right-hand side. For the first one we use (3.38) with $k = 0$ and we obtain that

$$\frac{1}{2}\int_0^t \big|A^{1/2}u_\varepsilon(s)\big|^2\,ds \leqslant \frac{1}{2\mu_1}\int_0^t c_\varepsilon(s)\big|A^{1/2}u_\varepsilon(s)\big|^2\,ds \leqslant \frac{D_{\varepsilon,0}(0)}{2\mu_1} + \varepsilon\frac{\mu_2 E_{\varepsilon_1,0}(0)}{\mu_1}.$$

The second one is the integral of a derivative, hence

$$\varepsilon\int_0^t \big\langle Au_\varepsilon(s), u'_\varepsilon(s)\big\rangle\,ds = \frac{\varepsilon}{2}\big\{\big|A^{1/2}u_\varepsilon(t)\big|^2 - \big|A^{1/2}u_0\big|^2\big\} \leqslant \frac{\varepsilon}{2}\big|A^{1/2}u_\varepsilon(t)\big|^2 \leqslant \frac{\varepsilon}{2}E_{\varepsilon_1,0}(0).$$

For the third one we use (4.28) and we obtain that

$$\varepsilon\int_0^t s\big|A^{1/2}u'_\varepsilon(s)\big|^2\,ds \leqslant \varepsilon\mu_2\int_0^t s\cdot\frac{|A^{1/2}u'_\varepsilon(s)|^2}{c_\varepsilon(s)}\,ds \leqslant \varepsilon\mu_2\gamma_1.$$

Recalling the definition of $D_{\varepsilon,0}(0)$ we have thus proved that

$$\int_0^t s\cdot\left|Au_\varepsilon(s)\right|^2 ds \leqslant \frac{1}{\mu_1}\int_0^t s\cdot c_\varepsilon(s)\left|Au_\varepsilon(s)\right|^2 ds \leqslant \frac{1}{4\mu_1^2}|u_0|^2+\varepsilon\gamma_3-\frac{tD_{\varepsilon,1}(t)}{\mu_1} \tag{4.30}$$

for a suitable constant γ_3. Using once more (4.28) we have that

$$\begin{aligned}-tD_{\varepsilon,1}(t) &= -t\left(\frac{|A^{1/2}u_\varepsilon(t)|^2}{2}+\varepsilon\left\langle A^{1/2}u_\varepsilon(t), A^{1/2}u'_\varepsilon(t)\right\rangle\right) \leqslant t\cdot\frac{\varepsilon^2}{2}\left|A^{1/2}u'_\varepsilon(t)\right|^2 \\ &= \frac{\varepsilon}{2}c_\varepsilon(t)\cdot t\frac{\varepsilon|A^{1/2}u'_\varepsilon(t)|^2}{c_\varepsilon(t)} \leqslant \frac{\varepsilon}{2}\mu_2\cdot tE_{\varepsilon,1}(t) \leqslant \frac{\varepsilon}{2}\mu_2\gamma_1.\end{aligned}$$

Together with (4.30) this inequality proves (4.29).

Step 3. We are now ready to prove (3.27). Since $E'_{\varepsilon,1}(t)\leqslant 0$ we have that

$$\left[t^2E_{\varepsilon,1}(t)\right]' = 2tE_{\varepsilon,1}(t)+t^2E'_{\varepsilon,1}(t) \leqslant 2tE_{\varepsilon,1}(t).$$

Integrating in $[0,t]$ and using (4.28) and (4.29) we obtain that

$$\begin{aligned}t^2E_{\varepsilon,1}(t) &\leqslant 2\int_0^t sE_{\varepsilon,1}(s)\,ds \\ &\leqslant 2\varepsilon\int_0^t s\cdot\frac{|A^{1/2}u'_\varepsilon(s)|^2}{c_\varepsilon(s)}\,ds+2\int_0^t s\left|Au_\varepsilon(s)\right|^2 ds \\ &\leqslant 2\varepsilon\gamma_1+\frac{|u_0|^2}{2\mu_1^2}+2\varepsilon\gamma_2 =: \frac{|u_0|^2}{2\mu_1^2}+k_7\varepsilon,\end{aligned}$$

which is (3.27).

Step 4. We prove (3.28). To this end, we consider the function $\mathcal{G}_\varepsilon(t) := t^2|u'_\varepsilon(t)|^2$. We have that

$$\begin{aligned}\mathcal{G}'_\varepsilon(t) &= -\frac{2}{\varepsilon}t^2\left|u'_\varepsilon(t)\right|^2-\frac{2}{\varepsilon}t^2c_\varepsilon(t)\left\langle u'_\varepsilon(t), Au_\varepsilon(t)\right\rangle+2t\left|u'_\varepsilon(t)\right|^2 \\ &\leqslant -\frac{2}{\varepsilon}t^2\left|u'_\varepsilon(t)\right|^2+\frac{2}{\varepsilon}t^2\mu_2\left|u'_\varepsilon(t)\right|\left|Au_\varepsilon(t)\right|+2t\left|u'_\varepsilon(t)\right|^2 \\ &= -\frac{2}{\varepsilon}t^2\left|u'_\varepsilon(t)\right|^2+\frac{2}{\varepsilon}t\left|u'_\varepsilon(t)\right|\left\{\mu_2t\left|Au_\varepsilon(t)\right|+\varepsilon\left|u'_\varepsilon(t)\right|\right\}.\end{aligned}$$

From (3.40) we have that $|u'_\varepsilon(t)| = c_\varepsilon(t)\sqrt{G_\varepsilon(t)} \leqslant \gamma_4$ for a suitable constant γ_4. From (3.27) we have that

$$\mu_2t\left|Au(t)\right|+\varepsilon\left|u'_\varepsilon(t)\right| \leqslant \mu_2\left(\frac{|u_0|^2}{2\mu_1^2}+k_7\varepsilon\right)^{1/2}+\varepsilon\gamma_4 =: \Gamma_\varepsilon.$$

Therefore we have that

$$\mathcal{G}'_\varepsilon(t) \leqslant -\frac{2}{\varepsilon}\sqrt{\mathcal{G}_\varepsilon(t)}\bigl\{\sqrt{\mathcal{G}_\varepsilon(t)} - \Gamma_\varepsilon\bigr\}.$$

Applying Lemma 4.1, and recalling that $\mathcal{G}_\varepsilon(0) = 0$, we finally have that

$$t^2\bigl|u_\varepsilon(t)\bigr|^2 = \mathcal{G}_\varepsilon(t) \leqslant \Gamma_\varepsilon^2 \leqslant \mu_2^2\frac{|u_0|^2}{2\mu_1^2} + k_8\varepsilon$$

for a suitable constant k_8. This proves (3.28).

Statement (5). Let $\psi_\varepsilon(t) := \phi_\varepsilon(t) + \sqrt{\varepsilon}$. Since $\psi_\varepsilon(t) \geqslant \phi_\varepsilon(t)$ it is enough to prove (3.29) and (3.30) with ϕ_ε replaced by ψ_ε.

Step 1. We prove that there exists a constant h_1 such that for every ε small enough and every $t \geqslant 0$ we have that

$$\psi_\varepsilon(t) \geqslant \frac{|A^{1/2}u_0|^2}{|A^{1/2}u_\varepsilon(t)|^2}\sqrt{\varepsilon}, \qquad \left|\frac{\psi'_\varepsilon(t)}{\psi_\varepsilon(t)}\right| \leqslant \frac{h_1}{\sqrt{\varepsilon}}. \tag{4.31}$$

Indeed let us assume that $2|A^{1/2}u_0|^2\delta_3\sqrt{\varepsilon} \leqslant 1$, where δ_3 is the constant which appears in (4.27). In order to prove the first inequality in (4.31) it is enough to check that it holds true for $t = 0$ and for $t \geqslant 0$ we have that

$$\sqrt{\varepsilon}\frac{d}{dt}\left(\frac{|A^{1/2}u_0|^2}{|A^{1/2}u_\varepsilon|^2}\right) = -2\bigl|A^{1/2}u_0\bigr|^2\sqrt{\varepsilon}\frac{\langle Au_\varepsilon, u'_\varepsilon\rangle}{c_\varepsilon|A^{1/2}u_\varepsilon|^2}\cdot\frac{c_\varepsilon}{|A^{1/2}u_\varepsilon|^2} \leqslant 2\bigl|A^{1/2}u_0\bigr|^2\sqrt{\varepsilon}\delta_3\psi'_\varepsilon \leqslant \psi'_\varepsilon.$$

This proves the first inequality, from which we have that

$$\left|\frac{\psi'_\varepsilon(t)}{\psi_\varepsilon(t)}\right| = \frac{c_\varepsilon}{|A^{1/2}u_\varepsilon|^2}\cdot\frac{1}{\psi_\varepsilon(t)} \leqslant \frac{c_\varepsilon}{|A^{1/2}u_0|^2\sqrt{\varepsilon}} \leqslant \frac{\mu_2}{|A^{1/2}u_0|^2\sqrt{\varepsilon}} =: \frac{h_1}{\sqrt{\varepsilon}}.$$

Step 2. We prove inequality (3.29) with ψ_ε instead of ϕ_ε.
To this end we compute

$$\begin{aligned}
\frac{d}{dt}\bigl(\psi_\varepsilon^2 E_{\varepsilon,1}\bigr) &= 2\psi_\varepsilon\psi'_\varepsilon E_{\varepsilon,1} - \psi_\varepsilon^2\frac{|A^{1/2}u'_\varepsilon|^2}{c_\varepsilon}\left[2 + \varepsilon\frac{c'_\varepsilon}{c_\varepsilon}\right] \\
&= 2\psi_\varepsilon\psi'_\varepsilon|Au_\varepsilon|^2 + 2\psi_\varepsilon\psi'_\varepsilon\varepsilon\frac{|A^{1/2}u'_\varepsilon|^2}{c_\varepsilon} - \psi_\varepsilon^2\frac{|A^{1/2}u'_\varepsilon|^2}{c_\varepsilon}\left[2 + \varepsilon\frac{c'_\varepsilon}{c_\varepsilon}\right] \\
&= 2\psi_\varepsilon\psi'_\varepsilon|Au_\varepsilon|^2 - \psi_\varepsilon^2\frac{|A^{1/2}u'_\varepsilon|^2}{c_\varepsilon}\left[2 + \varepsilon\frac{c'_\varepsilon}{c_\varepsilon} - 2\varepsilon\frac{\psi'_\varepsilon}{\psi_\varepsilon}\right].
\end{aligned}$$

By (3.3) and (4.31) the term in the brackets is greater than $2 - \sqrt{\varepsilon}h_2$ for some constant h_2, and therefore

$$\frac{d}{dt}\bigl(\psi_\varepsilon^2 E_{\varepsilon,1}\bigr) \leqslant 2\psi_\varepsilon\psi'_\varepsilon|Au_\varepsilon|^2 - (2 - \sqrt{\varepsilon}h_2)\psi_\varepsilon^2\frac{|A^{1/2}u'_\varepsilon|^2}{c_\varepsilon}.$$

In order to estimate the first term of the right-hand side we consider the identity

$$\begin{aligned}2\psi_\varepsilon\psi_\varepsilon'|Au_\varepsilon|^2 &= -2\varepsilon\frac{d}{dt}\left[\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle}{|A^{1/2}u_\varepsilon|^2}\psi_\varepsilon\right] + 2\varepsilon\frac{|A^{1/2}u_\varepsilon'|^2}{|A^{1/2}u_\varepsilon|^2}\psi_\varepsilon \\ &\quad + \frac{2\langle Au_\varepsilon, u_\varepsilon'\rangle}{|A^{1/2}u_\varepsilon|^2}\psi_\varepsilon\left[\varepsilon\frac{\psi_\varepsilon'}{\psi_\varepsilon} - 2\varepsilon\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle}{|A^{1/2}u_\varepsilon|^2} - 1\right] \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

By (4.31) we have that

$$I_2 = 2\varepsilon\frac{|A^{1/2}u_\varepsilon'|^2}{c_\varepsilon}\cdot\frac{c_\varepsilon}{|A^{1/2}u_\varepsilon|^2}\cdot\psi_\varepsilon = 2\varepsilon\frac{|A^{1/2}u_\varepsilon'|^2}{c_\varepsilon}\cdot\frac{\psi_\varepsilon'}{\psi_\varepsilon}\cdot\psi_\varepsilon^2 \leqslant 2\sqrt{\varepsilon}h_1\frac{|A^{1/2}u_\varepsilon'|^2}{c_\varepsilon}\cdot\psi_\varepsilon^2.$$

In order to estimate I_3 we use that the absolute value of the term in the brackets is less than $1 + h_3\sqrt{\varepsilon}$ for a suitable constant h_3, and that for every $\delta_\varepsilon > 0$ we have that

$$\left|\frac{2\langle Au_\varepsilon, u_\varepsilon'\rangle}{|A^{1/2}u_\varepsilon|^2}\psi_\varepsilon\right| \leqslant 2\frac{|A^{1/2}u_\varepsilon'|}{\sqrt{c_\varepsilon}}\psi_\varepsilon\cdot\frac{\sqrt{c_\varepsilon}}{|A^{1/2}u_\varepsilon|} \leqslant \delta_\varepsilon\frac{|A^{1/2}u_\varepsilon'|^2}{c_\varepsilon}\psi_\varepsilon^2 + \frac{1}{\delta_\varepsilon}\psi_\varepsilon'.$$

It follows that

$$\begin{aligned}\frac{d}{dt}\left(\psi_\varepsilon^2 E_{\varepsilon,1}\right) &\leqslant -2\varepsilon\frac{d}{dt}\left[\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle}{|A^{1/2}u_\varepsilon|^2}\psi_\varepsilon\right] + \frac{1+\sqrt{\varepsilon}h_3}{\delta_\varepsilon}\psi_\varepsilon' \\ &\quad + \frac{|A^{1/2}u_\varepsilon'|^2}{c_\varepsilon}\psi_\varepsilon^2\left\{-2 + h_2\sqrt{\varepsilon} + 2h_1\sqrt{\varepsilon} + (1+\sqrt{\varepsilon}h_3)\delta_\varepsilon\right\}.\end{aligned}$$

Now we choose δ_ε in such a way that the last term is 0. It is not difficult to see that this implies that $\delta_\varepsilon \geqslant 2 - h_4\sqrt{\varepsilon}$, which is positive provided that ε is small enough. Therefore the previous inequality reduces to

$$\frac{d}{dt}\left(\psi_\varepsilon^2 E_{\varepsilon,1}\right) \leqslant -2\varepsilon\frac{d}{dt}\left[\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle}{|A^{1/2}u_\varepsilon|^2}\psi_\varepsilon\right] + \left(\frac{1}{2} + h_5\sqrt{\varepsilon}\right)\psi_\varepsilon'.$$

Integrating in $[0, t]$ we obtain that

$$\begin{aligned}\psi_\varepsilon^2(t)E_{\varepsilon,1}(t) &\leqslant \psi_\varepsilon^2(0)E_{\varepsilon,1}(0) - 2\varepsilon\frac{\langle Au_\varepsilon(t), u_\varepsilon'(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^2}\psi_\varepsilon(t) + 2\varepsilon\frac{\langle Au_0, u_1\rangle}{|A^{1/2}u_0|^2}\psi_\varepsilon(0) \\ &\quad + \left(\frac{1}{2} + h_5\sqrt{\varepsilon}\right)\left(\psi_\varepsilon(t) - \psi_\varepsilon(0)\right).\end{aligned}$$

Using the monotonicity of ψ_ε we have that $\psi_\varepsilon^2(0)E_{\varepsilon,1}(0) \leqslant \psi_\varepsilon(0)\psi_\varepsilon(t)E_{\varepsilon,1}(0) = \sqrt{\varepsilon}E_{\varepsilon,1}(0)\psi_\varepsilon(t)$. By (3.42) the second term is less than $2\delta_2\varepsilon\psi_\varepsilon(t)$. Using once more the monotonicity of ψ_ε also the third term is less than $2\delta_2\varepsilon\psi_\varepsilon(t)$. In conclusion there exists a constant k_9 such that

$$\psi_\varepsilon^2(t)E_{\varepsilon,1}(t) \leqslant \left(\frac{1}{2} + k_9\sqrt{\varepsilon}\right)\psi_\varepsilon(t). \tag{4.32}$$

Dividing by $\psi_\varepsilon(t)$ (which is positive) we obtain the required estimate.

Step 3. By (4.11) we have that

$$\begin{aligned}
\frac{d}{dt}[\psi_\varepsilon G_\varepsilon] &= \psi'_\varepsilon G_\varepsilon + \psi_\varepsilon G'_\varepsilon \\
&= \psi'_\varepsilon G_\varepsilon - \frac{2}{\varepsilon}\psi_\varepsilon\left(1+\varepsilon\frac{c'_\varepsilon}{c_\varepsilon}\right)G_\varepsilon - \frac{2}{\varepsilon}\psi_\varepsilon\frac{\langle u'_\varepsilon, Au_\varepsilon\rangle}{c_\varepsilon} \\
&\leqslant \psi'_\varepsilon G_\varepsilon - \frac{2}{\varepsilon}\psi_\varepsilon\left(1+\varepsilon\frac{c'_\varepsilon}{c_\varepsilon}\right)G_\varepsilon + \frac{2}{\varepsilon}\psi_\varepsilon\frac{|u'_\varepsilon||Au_\varepsilon|}{c_\varepsilon} \\
&= -\frac{2}{\varepsilon}\psi_\varepsilon G_\varepsilon\left(1+\varepsilon\frac{c'_\varepsilon}{c_\varepsilon} - \frac{\varepsilon}{2}\frac{\psi'_\varepsilon}{\psi_\varepsilon}\right) + \frac{2}{\varepsilon}\sqrt{\psi_\varepsilon G_\varepsilon}\sqrt{\psi_\varepsilon|Au_\varepsilon|^2}.
\end{aligned}$$

If ε is small enough the term in the parentheses is greater than $1/2$ and by (4.32) we have that $\psi_\varepsilon|Au_\varepsilon|^2 \leqslant \psi_\varepsilon E_{\varepsilon,1} \leqslant 1$. It follows that

$$\frac{d}{dt}[\psi_\varepsilon G_\varepsilon] \leqslant -\sqrt{\psi_\varepsilon G_\varepsilon}\left\{\frac{1}{\varepsilon}\sqrt{\psi_\varepsilon G_\varepsilon} - \frac{2}{\varepsilon}\right\}.$$

Applying Lemma 4.1 to the function $\psi_\varepsilon G_\varepsilon$ we obtain that

$$\psi_\varepsilon\frac{|u'_\varepsilon|^2}{c_\varepsilon^2} \leqslant \max\left\{\frac{\sqrt{\varepsilon}|u_1|^2}{\mu_0^2}, 4\right\},$$

which gives (3.30) with ψ_ε in place of ϕ_ε.

Proof of Corollary 3.7. Let us assume that A is coercive. The estimates on $|A^{1/2}u_\varepsilon|^2$ follow from (3.25) and (3.26) as in the parabolic case. Also the estimates on $|Au_\varepsilon|^2$ follow from (3.23) and the coercivity as in the parabolic case. The estimate on $|u'_\varepsilon|^2$ follows from (3.24). As for $\varepsilon|A^{1/2}u'_\varepsilon|^2$, we have to use (3.29). By the estimates on $|A^{1/2}u_\varepsilon|^2$ we have that ϕ'_ε (hence also ϕ_ε) grows exponentially, and therefore $\varepsilon|A^{1/2}u'_\varepsilon|^2$ decays exponentially.

Let us assume now that A is not coercive. If $A^{1/2}u_0 \neq 0$ the lower bound for $|A^{1/2}u_\varepsilon|^2$ follows from (3.25) as in the coercive case. The upper bound follows from (3.22) applied with $\psi(\sigma) = \mu_1\sigma$ as in the parabolic case. The estimates on $|Au_\varepsilon|^2$, $|u'_\varepsilon|^2$ and $\varepsilon|A^{1/2}u'_\varepsilon|^2$ follow from (3.27) and (3.28). □

Proof of Corollary 3.8. Let us assume that A is coercive. The estimates on $|A^{1/2}u_\varepsilon|^2$ follow from (3.25) and (3.26) as in the parabolic case. Also the estimates on $|Au_\varepsilon|^2$ follow from (3.23) and the coercivity as in the parabolic case. The estimate on $|u'_\varepsilon|^2$ follows from (3.24). As for $\varepsilon|A^{1/2}u'_\varepsilon|^2$, we have to use (3.29). Using the estimates from below for $|A^{1/2}u_\varepsilon|^2$ we have indeed that

$$\phi'_\varepsilon(t) = m\left(\left|A^{1/2}u_\varepsilon(t)\right|^2\right)\left|A^{1/2}u_\varepsilon(t)\right|^{-2} = \left|A^{1/2}u_\varepsilon(t)\right|^{2(\gamma-1)} \geqslant c(1+t)^{-1+1/\gamma},$$

hence $\phi_\varepsilon(t) \geqslant c_1(1+t)^{1/\gamma} - c_2$, from which the conclusion follows as in the parabolic case.

Let us assume now that A is not coercive. The lower bound for $|A^{1/2}u_\varepsilon|^2$ follows from (3.25) as in the coercive case. The upper bound follows from (3.22) applied with $\psi(\sigma) = \sigma^{\gamma+1}$ as in the parabolic case. For the remaining estimates we use (3.29) and (3.30) with the same estimate on $\phi_\varepsilon(t)$ found in the coercive case (as we have seen its proof requires only the lower bound for $|A^{1/2}u_\varepsilon|^2$, which is the same both in the coercive and in the noncoercive case). $\square$

Proof of Corollary 3.9. From (3.22), (3.1) and the monotonicity of ψ we have that

$$\psi\left(\left|A^{1/2}u_\varepsilon\right|^2\right) \leqslant \min\left\{\psi(\sigma_2), ct^{-1}\right\}.$$

Applying ψ^{-1} to both sides we obtain an ε-independent estimate on $|A^{1/2}u_\varepsilon|^2$ which tends to 0 as $t \to +\infty$. At this point (3.23) and (3.24) provide similar estimates for $|Au_\varepsilon|^2$ and $|u'_\varepsilon|^2$. As for $\varepsilon|A^{1/2}u'_\varepsilon|^2$, the fastest way to obtain a (nonoptimal!) estimate is to use (3.39) with $k = 1$ combined with the decay of $|A^{1/2}u_\varepsilon|^2$, hence of $c_\varepsilon(t)$. $\square$

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