Math. Ann. (2007) 337:557–590
DOI 10.1007/s00208-006-0047-1

Mathematische Annalen

Gradient estimates for the Perona–Malik equation

Marina Ghisi · Massimo Gobbino

Received: 12 May 2005 / Revised: 31 May 2006 /
Published online: 11 October 2006

Abstract We consider the Cauchy problem for the Perona–Malik equation

$$u_t = \operatorname{div}\left(\frac{\nabla u}{1 + |\nabla u|^2}\right)$$

in a bounded open set $\Omega \subseteq \mathbb{R}^n$, with Neumann boundary conditions.

If $n = 1$, we prove some a priori estimates on u and u_x. Then we consider the semi-discrete scheme obtained by replacing the space derivatives by finite differences. Extending the previous estimates to the discrete setting we prove a compactness result for this scheme and we characterize the possible limits in some cases. Finally, for $n > 1$ we give examples to show that the corresponding estimates on ∇u are in general false.

Mathematics Subject Classification (2000) 35B45 · 35B50 · 35K55

1 Introduction

In this paper we consider the initial boundary value problem

$$u_t = (\varphi'(u_x))_x = \varphi''(u_x)u_{xx} \quad \text{in } (-1, 1) \times [0, T), \tag{1.1}$$

M. Ghisi (✉)
Dipartimento di Matematica “Leonida Tonelli”, Università degli Studi di Pisa,
Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
e-mail: ghisi@dm.unipi.it

M. Gobbino
Dipartimento di Matematica Applicata “Ulisse Dini”, Università degli Studi di Pisa,
Via Diotisalvi 2, 56126 Pisa, Italy
e-mail: m.gobbino@dma.unipi.it

$$u_x(-1,t) = u_x(1,t) = 0 \quad \forall t \in [0,T), \tag{1.2}$$
$$u(x,0) = u_0(x) \quad \forall x \in (-1,1), \tag{1.3}$$

which is the formal gradient flow of the integral functional

$$PM_\varphi(u) = \int_{-1}^{1} \varphi(u_x)\, \mathrm{d}x. \tag{1.4}$$

Since we are interested in smooth solutions, we require that (1.1) and (1.2) are satisfied also for $t = 0$. In particular, Eq. 1.2 for $t = 0$ gives a compatibility condition on u_0.

We assume that $\varphi \in C^2(\mathbb{R})$ and for simplicity $\varphi'(0) = 0$, but we do *not* assume that φ is convex: therefore Eq. 1.1 is a forward-parabolic PDE where $\varphi''(u_x) > 0$, and a backward-parabolic PDE where $\varphi''(u_x) < 0$. The interesting case is of course when the initial condition u_0 is such that $\varphi''(u_{0x})$ changes its sign in $[-1,1]$: in this case we say that u_0 is *transcritical*.

We also consider the n-dimensional generalization of (1.4), i.e. the functional

$$PM_\varphi(u) = \int_\Omega \varphi(|\nabla u|)\, \mathrm{d}x, \tag{1.5}$$

whose gradient flow is the initial boundary value problem

$$u_t = \operatorname{div}\left(\varphi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) \quad \text{in } \Omega \times [0,T), \tag{1.6}$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega \times [0,T), \tag{1.7}$$
$$u(x,0) = u_0(x) \quad \forall x \in \Omega, \tag{1.8}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set with piecewise C^1 boundary, and n is the exterior normal to $\partial\Omega$. Note that (1.6) is well defined since $\varphi'(0) = 0$.

The typical example is the so called Perona–Malik equation, corresponding to $\varphi(\sigma) = 2^{-1}\log(1+\sigma^2)$, introduced in [15] in the context of image processing (see also [11]), hence in the case where $n = 2$, Ω is a rectangle, and u_0 represents the gray level of an image. A different choice is $\varphi(\sigma) = (\sigma^2 - 1)^2$, considered for example in [1] in the context of nonlinear elasticity and phase transitions.

From the analytical point of view, the forward–backward character of these PDEs induces a general skepticism (see [2]), partially supported by some negative results [9,10,12]. In particular, it is known that problem (1.1), (1.2), and (1.3) has no global solution ($T = +\infty$) if u_0 is transcritical (see [9,12]). Existence of local solutions, even for special classes of transcritical initial data, is still an open problem.

On the other hand, numerical experiments exhibit much better stability properties than expected (see [3,4,6,7,13]).

The existence of good reasons for being pessimistic and good reasons for being optimistic is usually referred as "the Perona–Malik paradox".

In this paper we partially support both positions. Concerning the one dimensional problem (1.1), (1.2), and (1.3), in Theorem 2.2 we prove some a priori estimates on u and u_x. In particular, we prove that the L^1 norm of u_x in the space variable (i.e. the total variation of u) is a non-increasing function of time, while the L^∞ norm of u_x in the space variable is a non-decreasing function of time if u_0 is transcritical.

Then we consider the semi-discrete scheme for (1.1), (1.2), and (1.3), obtained by replacing space derivatives with finite differences. In Theorem 2.5 we show that some of the estimates proved in the continuous case, and in particular the estimate of the total variation, can be extended to the discrete setting. Such estimates are enough to prove (Theorem 2.7) that the solutions of the discrete problems converge to a limit (up to subsequences), as the discretization step goes to 0. This suggests to consider these limits as solutions of (1.1), (1.2), and (1.3) in some weak sense, as we do in Definition 2.8. Then we show that the properties of these limits are consistent with what observed in numerical experiments (Theorem 2.9). We also prove that they are classical solutions of Eq. 1.1 in the forward region of the initial condition (Theorem 2.10). We do not think that the same is true in the backward region, due to the "fibrillation" phenomenon conjectured in [5] as a consequence of the "staircasing" effect observed in numerical experiments. This remains a challenging problem.

Finally, we consider the n-dimensional problem (1.6), (1.7), and (1.8). The estimates on u can be generalized almost word by word (Theorem 2.14), but for gradients the situation is different. In Theorem 2.15 we prove that the total variation of the solution is a non-increasing function of time in the case of radial solutions, but then we show with Theorem 2.17 that this cannot be true in general, also for the Perona–Malik equation in a rectangle. A consequence is that the argument used in dimension one to prove the compactness for the semi-discrete scheme cannot be passed to dimension $n > 1$. We also show with Theorem 2.18 that it is no more true that the L^∞ norm of the gradient is non-decreasing if u_0 is transcritical. This prevents us from extending in dimension $n > 1$ the proofs of non-existence of global solutions for transcritical data given in [9] or [12].

This paper is organized as follows: in Sect. 2 we state our results the counter-examples, in Sect. 3 we give proofs, in Sect. 4 we present the counter-examples.

2 Statements

Before we state our estimates, we collect some assumptions on φ, which will be used in several statements.

(φ0) φ is an even non-negative function of class C^2 such that $\varphi'(0) = 0$.

(φ1) $\sigma \cdot \varphi'(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$.

(φ2) φ is convex in a neighborhood of 0, i.e. there exists $\sigma_0 > 0$ such that $\varphi''(\sigma) \geq 0$ if $|\sigma| \leq \sigma_0$.

(φ3) φ is convex-concave, i.e. there exists $\sigma_0 > 0$ such that $\varphi''(\sigma) \geq 0$ if $|\sigma| \leq \sigma_0$, and $\varphi''(\sigma) \leq 0$ if $|\sigma| \geq \sigma_0$.

We remark that all these assumptions are satisfied in the Perona–Malik case.

Unless otherwise stated, we always consider C^2 solutions (this means that u_t and the second order derivatives in the space variables are assumed to be continuous functions), even if most statements do not involve second derivatives, and so we expect them to be true also for C^1 solutions.

2.1 A priori estimates in dimension one

Let us consider problem (1.1), (1.2), and (1.3). The following equalities are our main tools to derive estimates on u and u_x.

Proposition 2.1 *Let us assume that φ satisfies (φ0), let $u : [-1,1] \times [0,T) \to \mathbb{R}$ be a C^2 solution of* (1.1), (1.2), (1.3), *and let $\psi \in C^2(\mathbb{R})$. Then*

(1) *for every $t \in [0,T)$ we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} \psi(u(x,t))\,\mathrm{d}x = -\int_{-1}^{1} \psi''(u(x,t)) \cdot u_x(x,t) \cdot \varphi'(u_x(x,t))\,\mathrm{d}x; \quad (2.1)$$

(2) *if $\psi'(0) \cdot \varphi''(0) = 0$, then for every $t \in [0,T)$ we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} \psi(u_x(x,t))\,\mathrm{d}x = -\int_{-1}^{1} \psi''(u_x(x,t)) \cdot \varphi''(u_x(x,t)) \cdot [u_{xx}(x,t)]^2\,\mathrm{d}x. \quad (2.2)$$

Using Proposition 2.1 with suitable choices of ψ, we obtain the following estimates.

Theorem 2.2 *Let us assume that φ satisfies* (φ0), *and let $u : [-1,1] \times [0,T) \to \mathbb{R}$ be a C^2 solution of* (1.1), (1.2), *and* (1.3).

Then we have the following estimates on u and u_x.

(1) Classical gradient flow estimates. *For every $0 \leq t_1 \leq t_2 < T$ we have that*

$$\begin{aligned} PM_{\varphi}(u(x,t_1)) - PM_{\varphi}(u(x,t_2)) &= \int_{t_1}^{t_2}\int_{-1}^{1} \left[\varphi''(u_x)u_{xx}\right]^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_{-1}^{1} [u_t]^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \quad (2.3)$$

and in particular the function $t \to PM_\varphi(u(x,t))$ *is non-increasing. Moreover*

$$\|u(x,t_1) - u(x,t_2)\|_{L^2((-1,1))} \le \left\{PM_\varphi(u_0)\right\}^{1/2} \cdot |t_1 - t_2|^{1/2}. \qquad (2.4)$$

(2) L^p estimates on u. *If* φ *satisfies also* (φ1) *then for every* $p \in [1,\infty]$ *and every* $t \in [0,T)$ *we have that*

$$\|u(x,t)\|_{L^p((-1,1))} \le \|u_0(x)\|_{L^p((-1,1))}. \qquad (2.5)$$

(3) Maximum principle for u. *If* φ *satisfies also* (φ1) *then for every* $(x,t) \in [-1,1] \times [0,T)$ *we have that*

$$\min\{u_0(x) : x \in [-1,1]\} \le u(x,t) \le \max\{u_0(x) : x \in [-1,1]\}. \qquad (2.6)$$

(4) Total variation estimate on u. *If* φ *satisfies also* (φ2) *then for every* $t \in [0,T)$ *we have that*

$$\|u_x(x,t)\|_{L^1((-1,1))} \le \|u_{0x}(x)\|_{L^1((-1,1))}. \qquad (2.7)$$

From now on, let us set

$$m(t) := \min\{u_x(x,t) :\ x \in [-1,1]\}, \quad M(t) := \max\{u_x(x,t) :\ x \in [-1,1]\}.$$

(5) Barriers for u_x. *Let* $\sigma_1 < \sigma_2$ *be such that* $\varphi''(\sigma) \ge 0$ *in the interval* $[\sigma_1,\sigma_2]$. *Then we have the following implications*

$$M(0) \le \sigma_2 \implies M(t) \le \sigma_2 \quad \forall t \in [0,T); \qquad (2.8)$$
$$m(0) \ge \sigma_1 \implies m(t) \ge \sigma_1 \quad \forall t \in [0,T). \qquad (2.9)$$

Similarly, if $\varphi''(\sigma) \le 0$ *in the interval* $[\sigma_1,\sigma_2]$, *then we have the following implications*

$$M(0) \ge \sigma_2 \implies M(t) \ge \sigma_2 \quad \forall t \in [0,T); \qquad (2.10)$$
$$m(0) \le \sigma_1 \implies m(t) \le \sigma_1 \quad \forall t \in [0,T). \qquad (2.11)$$

(6) Subcritical maximum principle for u_x. *If* $\varphi''(M(0)) > 0$, *then* $M(t)$ *is a non-increasing function, and* $\varphi''(M(t)) \ge 0$ *for every* $t \in [0,T)$.
(7) Supercritical (reverse) maximum principle for u_x. *If* $\varphi''(M(0)) < 0$, *then* $M(t)$ *is a non-decreasing function, and* $\varphi''(M(t)) \le 0$ *for every* $t \in [0,T)$.
(8) Critical maximum principle for u_x. *If* $\varphi''(M(0)) = 0$, *then there are three cases.*
   (8.1) *If there exists* $\delta > 0$ *such that* $\varphi''(\sigma) \ge 0$ *for every* $\sigma \in [M(0) - \delta, M(0)]$, *then* $M(t)$ *is a non-increasing function, and* $\varphi''(M(t)) \ge 0$ *for every* $t \in [0,T)$.

(8.2) *If the assumption of* (8.1) *is not satisfied, and there exists* $\delta > 0$ *such that the set* $\{\sigma \in [M(0), M(0) + \delta] : \varphi''(\sigma) \geq 0\}$ *has empty interior, then* $M(t)$ *is a non-decreasing function, and* $\varphi''(M(t)) \leq 0$ *for every* $t \in [0, T)$.

(8.3) *In any other case* $M(t) = M(0)$ *for every* $t \in [0, T)$.

Remark 2.3 A few comments and consequences of the conclusions of Theorem 2.2.

(a) Estimates (1) are the standard decay of the energy and 1/2-Hölder continuity in time of the gradient flow of a non-negative functional.

(b) Estimates such as (2) have been proved independently by many authors, at least for the Perona–Malik equation (see [6,12]).

(c) Estimate (3) is a refinement of the case $p = \infty$ in estimate (2).

(d) Estimate (4) proves that (φ2) is enough to give an a priori bound on the total variation of u, also in the transcritical case. Here $p = 1$ is crucial: indeed it is not difficult to see that there are no L^p estimates ($p > 1$) on u_x in the transcritical case.

(e) From estimate (4) one can easily deduce that the total variation of $u(x, t)$ in the space variable in a non-increasing function of time. Analogous statements can be obtained from estimates (2) and (3).

(f) Conclusions (6), (7), and (8) are stated for the maximum $M(t)$. Symmetric statements are true for the minimum $m(t)$.

(g) Let us assume that φ satisfies (φ0) and (φ3), as in the Perona–Malik case. Then (6), (7), and (8) can be read as follows: if $0 \leq M(0) \leq \sigma_0$, then $M(t)$ is non-increasing; if $M(0) > \sigma_0$, then $M(t)$ is non-decreasing. A symmetric statement holds true for the minimum. In particular, if u_0 is transcritical, then the solution remains transcritical for all times: this is one of the key tools to prove non-existence of global classical solutions for transcritical initial data (see [9,12]).

(h) Let us assume that φ satisfies (φ0) and (φ2), as in the Perona–Malik case. If $u_0(x)$ is non-increasing (resp. non-decreasing) in x, then the same is true for $u(x, t)$ for any fixed $t \in [0, T)$. So the evolution preserves monotonicity.

(i) Let us assume that φ satisfies (φ0), and that there exists $\sigma_3 \in \mathbb{R}$ such that $\varphi''(\sigma) \geq 0$ if $|\sigma| \geq \sigma_3$, as in the case where $\varphi(\sigma) = (\sigma^2 - 1)^2$. Then for every $(x, t) \in [-1, 1] \times [0, T)$ we have that

$$\min\{m(0), -\sigma_3\} \leq u_x(x, t) \leq \max\{\sigma_3, M(0)\}.$$

We sketch a proof of the last three conclusions in Remark 3.1.

2.2 The semi-discrete scheme in dimension one

A natural approach to (1.1), (1.2), and (1.3) is to approximate it by discretizing in the space variable. To this end, given an integer $n > 0$, we divide $[-1, 1]$ in $2n$ intervals of length $h = 1/n$, and we consider the space PC_n of all functions which are constant in each subinterval. Given $f : [-1, 1] \to \mathbb{R}$, we approximate the derivative of f with the function

$$D^{1/n} f(x) = \begin{cases} \dfrac{f(x+h) - f(x)}{h} & \text{if } x \in [-1, 1-h], \\ 0 & \text{if } x \in (1-h, 1]. \end{cases}$$

Then we approximate the functional (1.4) with the functional $PM_{\varphi,n} : PC_n \to \mathbb{R}$ defined by

$$PM_{\varphi,n}(u) = \int_{-1}^{1-h} \varphi\left(D^{1/n} u(x)\right) \mathrm{d}x \quad \forall u \in PC_n.$$

Finally, we approximate (1.1), (1.2), and (1.3) with the gradient flow of $PM_{\varphi,n}$ in the space PC_n, which turns out to be a well posed ODE.

Proposition 2.4 *For every $n \in \mathbb{N}$ we have that*

(1) *PC_n, endowed with the L^2 norm, is an Euclidean space of dimension* $2n$;
(2) *the functional $PM_{\varphi,n} : PC_n \to \mathbb{R}$ is differentiable*;
(3) *if φ'' is bounded, then $\nabla PM_{\varphi,n} : PC_n \to PC_n$ is a Lipschitz continuous function*;
(4) *if φ'' is bounded, then for every $u_{0n} \in PC_n$ the Cauchy problem*

$$u_n'(t) = -\nabla PM_{\varphi,n}(u_n(t)) \tag{2.12}$$
$$u_n(0) = u_{0n} \tag{2.13}$$

has a unique global solution $u_n \in C^1([0, +\infty); PC_n)$.

Some estimates can be extended from the continuous to the discrete setting. In the following statement, with a little abuse of notation, we consider u_n both as a function $u_n : [0, +\infty) \to PC_n$, and as a function $u_n : [-1, 1] \times [0, +\infty) \to \mathbb{R}$.

Theorem 2.5 *Let us assume that φ satisfies* ($\varphi 0$), *and let u_n be the solution of the Cauchy problem* (2.12), (2.13).

(1) Classical gradient flow estimates. *For every $0 \leq t_1 \leq t_2 < T$ we have that*

$$PM_{\varphi,n}(u_n(t_1)) - PM_{\varphi,n}(u_n(t_2)) = \int_{t_1}^{t_2} \left\| u_n'(t) \right\|^2_{L^2((-1,1))} \mathrm{d}t, \tag{2.14}$$

and in particular the function $t \to PM_{\varphi,n}(u_n(t))$ is non-increasing. Moreover

$$\|u_n(x, t_1) - u_n(x, t_2)\|_{L^2((-1,1))} \leq \left\{PM_{\varphi,n}(u_{0n})\right\}^{1/2} \cdot |t_1 - t_2|^{1/2}. \tag{2.15}$$

(2) L^p estimates on u_n. *If φ satisfies also* (φ1)*, then for every $p \in [1,\infty]$, and every $t \geq 0$, we have that*

$$\|u_n(x,t)\|_{L^p((-1,1))} \leq \|u_{0n}(x)\|_{L^p((-1,1))}. \tag{2.16}$$

(3) Maximum principle for u_n. *If φ satisfies also* (φ1)*, then for every $(x,t) \in [-1,1] \times [0,+\infty)$ we have that*

$$\min\{u_{0n}(x) : x \in [-1,1]\} \leq u_n(x,t) \leq \max\{u_{0n}(x) : x \in [-1,1]\}. \tag{2.17}$$

(4) Total variation estimate for u_n. *If φ satisfies also* (φ1)*, then for every $t \geq 0$ we have that*

$$\|D^{1/n}u_n(x,t)\|_{L^1((-1,1))} \leq \|D^{1/n}u_{0n}(x)\|_{L^1((-1,1))}. \tag{2.18}$$

(5) Monotonicity of sub(and super)critical regions. *If σ_0 is a global maximum point for φ' (this is true for example under assumption* (φ3)*) and we set*

$$I_n^-(t) := \left\{x \in [-1,1] : \; |D^{1/n}u_n(x,t)| \leq \sigma_0\right\},$$

then $I_n^-(t_1) \subseteq I_n^-(t_2)$ whenever $0 \leq t_1 \leq t_2$. The same is true if the inequality in the definition of $I_n^-(t)$ is strict.

Remark 2.6 The last three statements in Theorem 2.2, and in particular the supercritical reverse maximum principle for u_x, do not have a straight forward discrete counterpart. Indeed, should it be true, the same argument of the continuous case (see [9,12]) would give the non existence of global solutions for the Cauchy problem (2.12) and (2.13), which is of course an absurd.

The estimates of Theorem 2.5 are what is needed to prove a compactness result for the semi-discrete scheme (see also [4,6,8] where similar strategies are applied).

Theorem 2.7 *Let us assume that φ satisfies* (φ0), (φ1)*, and that φ'' is bounded. Let $\{u_{0n}\}$ be a family of functions such that $u_{0n} \in PC_n$ for every $n \in \mathbb{N}$, and*

$$\sup_{n\in\mathbb{N}} \left\{PM_{\varphi,n}(u_{0n}) + \|u_{0n}\|_{L^\infty((-1,1))} + \|D^{1/n}u_{0n}\|_{L^1((-1,1))}\right\} < +\infty. \tag{2.19}$$

Then the sequence $\{u_n\}$ of the corresponding solutions of (2.12) *and* (2.13) *is relatively compact in $C^0([0,+\infty);L^2((-1,1)))$.*

This compactness result motivates the following weak notion of gradient flow for the functional (1.4).

Definition 2.8 *Let us assume that φ satisfies ($\varphi 0$), ($\varphi 1$), and that φ' and φ'' are bounded. Let $u_0 \in BV((-1,1))$ with total variation $TV(u_0)$. We say that a function $u \in C^0([0,+\infty); L^2((-1,1)))$ belongs to $GF_\varphi(u_0)$ if there exists a sequence $\{n_k\}$ of positive integers and a sequence $\{u_{0k}\}$ such that $u_{0k} \in PC_{n_k}$ for every $k \in \mathbb{N}$,*

$$\|u_{0k} - u_0\|_2 + |\|u_{0k}\|_\infty - \|u_0\|_\infty| + \left| \|D^{1/n_k} u_{0k}\|_1 - TV(u_0) \right| \to 0, \tag{2.20}$$

and finally $u_k \to u$ in $C^0([0,+\infty); L^2((-1,1)))$, where u_k is the solution of (2.12) *with $n = n_k$ and initial condition u_{0k}.*

Now we show that the elements of $GF_\varphi(u_0)$ have most of the properties observed in numerical simulations.

Theorem 2.9 *Let φ be as in Definition* 2.8, *and let $u_0 \in BV((-1,1))$.*
Then $GF_\varphi(u_0) \neq \emptyset$, and every $u \in GF_\varphi(u_0)$ has the following properties.

(1) Initial condition. $u(0) = u_0$.
(2) Time regularity. $u \in H^1_{\text{loc}}((0,+\infty); L^2((-1,1)))$, $u' \in L^2((0,+\infty); L^2((-1,1)))$, *and in particular* $u \in C^{1/2}([0,+\infty); L^2((-1,1)))$.
(3) Space regularity. *For every $t \geq 0$ we have that $u(t) \in BV((-1,1))$ and*

$$\|u(t)\|_{L^\infty((-1,1))} \leq \|u_0\|_{L^\infty((-1,1))}, \tag{2.21}$$
$$TV(u(t)) \leq TV(u_0). \tag{2.22}$$

(4) Weak equation solved. *There exists $g \in L^\infty((-1,1) \times (0,+\infty))$ such that*
   - *for almost every $t \geq 0$ we have that the function $x \to g(x,t)$ belongs to $H^1((-1,1))$ and satisfies the boundary conditions $g(-1,t) = g(1,t) = 0$;*
   - *u is a solution of $u_t = g_x$.*

(5) Weak convergences. *If n_k and u_k are as in Definition* 2.8 *then*

$$u_k' \rightharpoonup u' \qquad \text{weakly in } L^2((0,+\infty), L^2((-1,1))); \tag{2.23}$$
$$\varphi'\left(D^{1/n_k} u_k\right) \rightharpoonup g \qquad \text{weakly * in } L^\infty((-1,1) \times (0,+\infty)); \tag{2.24}$$
$$D^{1/n_k} u_k(t) \rightharpoonup Du(t) \qquad \text{as Radon measures for every } t \geq 0, \tag{2.25}$$

where $Du(t)$ denotes the distributional derivative of $u(t)$ in the space variable.

(6) Stability. *If $\{u_{0h}\} \subseteq BV((-1,1))$ is a sequence such that*

$$\|u_{0h} - u_0\|_2 + |\|u_{0h}\|_\infty - \|u_0\|_\infty| + |TV(u_{0h}) - TV(u_0)| \to 0, \tag{2.26}$$

and $u_h \in GF_\varphi(u_{0h})$ for every $h \in \mathbb{N}$, then the sequence $\{u_h\}$ is relatively compact in $C^0([0,+\infty); L^2((-1,1)))$ and all its limit points belong to $GF_\varphi(u_0)$.

At this point a natural question is whether the function $g(x,t)$ found in statement (4) above is equal to $\varphi'(Du)$, up to a natural extension of φ' to Radon measures.

The staircasing effect observed in numerical experiments suggests that the answer is no. In the convex–concave case we can however give a positive answer in the subcritical region of the initial condition (note that we make no assumption on u_{0k} outside Ω).

Theorem 2.10 *Let us assume that φ satisfies the assumptions of Definition* 2.8, *that σ_0 is a global maximum point for φ', and that $\varphi'(\sigma)$ is strictly increasing in* $[-\sigma_0,\sigma_0]$. *Let $u_0 \in BV((-1,1))$, let $u \in GF_{\varphi}(u_0)$, and let n_k, u_{0k}, and u_k be as in Definition* 2.8.

Let us assume that there exists an open set $\Omega \subseteq (-1,1)$ such that $|D^{1/n_k}u_{0k}(x)| \leq \sigma_0$ for every $k \in \mathbb{N}$ and every $x \in \Omega$.

Then u has the following properties in Ω.

(1) More space regularity. *For every $t \geq 0$ the function $x \to u(x,t)$ belongs to $C^1(\Omega)$ and $|u_x(x,t)| \leq \sigma_0$ in Ω.*
(2) Strong convergences. *For every $t \geq 0$ we have that*

$$D^{1/n_k}u_k(x,t) \to u_x(x,t) \quad \text{uniformly on compact subsets of } \Omega, \tag{2.27}$$

$$\varphi'\left(D^{1/n_k}u_k(x,t)\right) \to \varphi'(u_x(x,t)) \text{ uniformly on compact subsets of } \Omega. \tag{2.28}$$

(3) Classical solution. *u is a classical solution of (1.1) in the open set $\Omega \times (0,+\infty)$.*

Corollary 2.11 *Let φ be as in Theorem* 2.10, *and let u_0 be a Lipschitz continuous function with Lipschitz constant less or equal than σ_0.*

Then the unique classical solution of (1.1), (1.2), *and* (1.3) *is the unique element of $GF_{\varphi}(u_0)$ which can be obtained by a subcritical approximating sequence,* i.e. *by choosing n_k and u_{0k} in such a way that $|D^{1/n_k}u_{0k}(x)| \leq \sigma_0$ for every $k \in \mathbb{N}$ and every $x \in (-1,1)$.*

Remark 2.12 The result of Corollary 2.11 can be extended to piecewise subcritical data, i.e. functions $u_0 \in BV((-1,1))$ with a finite number m of jumps located at points $-1 < x_1 < \cdots < x_m < 1$, and which are Lipschitz continuous in each connected component of $(-1,1) \setminus \{x_1,\ldots,x_m\}$ with Lipschitz constant less or equal than σ_0. In this case indeed we can take n_k and u_{0k} in such a way that $|D^{1/n_k}u_{0k}| > \sigma_0$ only in the m intervals containing a jump point of u_0. This allows to apply Theorem 2.9 in every open set of the form

$$\Omega_\varepsilon := (-1,1) \setminus \bigcup_{i=1}^{m}[x_i-\varepsilon, x_i+\varepsilon].$$

Letting $\varepsilon \to 0^+$ we get that u satisfies (1.1) for every $t > 0$ and every $x \in (-1, 1) \setminus \{x_1, \ldots, x_m\}$.

The behavior of u at jump points depends on the behavior of $\varphi'(\sigma)$ at infinity. We don't work out the details in this paper, referring the interested reader to Sect. 5 of [8] where the analogous problem is studied for the Mumford–Shah functional. What happens in the Perona–Malik case is that u satisfies the homogeneous Neumann boundary condition on both sides of discontinuity points. If during the evolution the jump height at x_i vanishes in a finite time T_i, then for generic u_0 there is no jump at x_i for every $t \geq T_i$, and this uniquely characterizes u. Nevertheless for special choices of u_0 there is a continuum of possible limits, also if u_{0k} has been chosen according to the limitations described at the beginning of this remark.

There is one more case in which we can easily characterize $GF_\varphi(u_0)$.

Theorem 2.13 *Let us assume that φ satisfies the assumptions of Definition* 2.8, *and that*

$$\lim_{\sigma \to +\infty} \frac{\varphi(\sigma)}{\sigma} = 0. \tag{2.29}$$

Let $u_0 \in BV((-1, 1))$ be a function whose distributional derivative has the absolutely continuous part equal to zero.

Then the unique element of $GF_\varphi(u_0)$ is the stationary solution $u(t) \equiv u_0$.

This result, even if in accordance with numerical experiments, has an unpleasant effect when combined with the stability property stated in (6) of Theorem 2.9. Indeed, since piecewise constant initial data are dense in $BV((-1, 1))$ in the sense of (2.26), it is simple to conclude that the stationary solution $u(t) \equiv u_0$ belongs to $GF_\varphi(u_0)$ for every $u_0 \in BV((-1, 1))$!

Of course one can eliminate unwanted stationary solutions and even get uniqueness by restricting the choice of the approximating sequence u_{0k}, as we did in Theorem 2.10, Corollary 2.11, and Remark 2.12. However, the same argument shows that *any* notion of solution for which Theorem 2.13 and the stability property hold true has this intrinsic non-uniqueness. In conclusion, if one doesn't want too much stationary solutions, either one looses the good stationary solutions of Theorem 2.13, or one looses the stability property!

2.3 A priori estimates in higher dimension

Now we consider the extent to which the estimates obtained in the one dimensional case can be extended to higher dimension. Unfortunately, negative answers are more than positive ones.

The positive answers concern the classical gradient flow estimates (of course), and the estimates on u.

Theorem 2.14 *Let us assume that φ satisfies* (φ0), *and let $u : \overline{\Omega} \times [0, T) \to \mathbb{R}$ be a C^2 solution of* (1.6), (1.7), *and* (1.8).

(1) Classical gradient flow estimates. *For every* $0 \leq t_1 \leq t_2 < T$ *we have that*

$$PM_\varphi(u(x,t_1)) - PM_\varphi(u(x,t_2)) = \int_{t_1}^{t_2} \int_\Omega [u_t(x,t)]^2 \, \mathrm{d}x \, \mathrm{d}t,$$

and in particular the function $t \to PM_\varphi(u(x,t))$ *is non-increasing. Moreover*

$$\|u(x,t_1) - u(x,t_2)\|_{L^2(\Omega)} \leq \left\{PM_\varphi(u_0)\right\}^{1/2} \cdot |t_1 - t_2|^{1/2}.$$

(2) L^p estimates on u. *If* φ *satisfies also* ($\varphi 1$) *then for every* $p \in [1,\infty]$ *and every* $t \in [0,T)$ *we have that*

$$\|u(x,t)\|_{L^p(\Omega)} \leq \|u_0(x)\|_{L^p(\Omega)}.$$

(3) Maximum principle for u. *If* φ *satisfies also* ($\varphi 1$) *then for every* $(x,t) \in \overline{\Omega} \times [0,T)$ *we have that*

$$\min\left\{u_0(x) : x \in \overline{\Omega}\right\} \leq u(x,t) \leq \max\left\{u_0(x) : x \in \overline{\Omega}\right\}.$$

When passing to gradients, the positive answer is that the estimate of the total variation of u is still true for *radial* solutions.

Theorem 2.15 *Let* Ω *be an open disc in* $\mathbb{R}^n$. *Let us assume that* φ *satisfies* ($\varphi 0$), ($\varphi 1$), ($\varphi 2$), *and let* $u : \overline{\Omega} \times [0,T) \to \mathbb{R}$ *be a radial* C^2 *solution of* (1.6), (1.7), *and* (1.8).

Then for every $t \in [0,T)$ *we have that*

$$\|\nabla u(x,t)\|_{L^1(\Omega)} \leq \|\nabla u_0(x)\|_{L^1(\Omega)}.$$

If u is not radial, things are more complex. A first reason is that the geometry of Ω plays a role. Consider indeed the simplest example, i.e. a solution u of the heat equation in a bounded regular open set $\Omega \subseteq \mathbb{R}^2$, with Neumann boundary conditions. By the classical gradient flow estimates, the L^2 norm of ∇u is a non-increasing function of time. The L^p norm ($p \neq 2$) of ∇u is known to be non-increasing provided that Ω is *convex*: the main tool is the so called Bernstein method, described at the end of Chapter IV of [14].

The convexity of Ω is in any case essential, as shown by the following result.

Theorem 2.16 *There exist a bounded* (*non-convex*) *open set* $\Omega \subseteq \mathbb{R}^2$, *and a function* $u_0 \in C^\infty(\Omega)$, *such that, if* u *is the solution of the heat equation in* Ω, *with Neumann boundary conditions on* $\partial\Omega$, *and initial condition* u_0, *and* $F(t) := \|\nabla u(x,t)\|_{L^1(\Omega)}$, *then* $F'(0) > 0$.

For the Perona–Malik equation, the answer is negative also if Ω is a rectangle.

Theorem 2.17 *There exist a rectangular open set $\Omega = (0, A) \times (0, B)$, and $u_0 \in C^\infty(\Omega)$, such that, if $u \in C^1([0,T]; H^1(\Omega)) \cap C^0([0,T]; H^3(\Omega))$ is a solution of the Perona–Malik equation in Ω, with Neumann boundary conditions on $\partial\Omega$, and initial condition u_0, and $F(t) := \|\nabla u(x,t)\|_{L^1(\Omega)}$, then $F'(0) > 0$.*

Finally, we draw our attention on the supercritical reverse maximum principle for the gradient. Once again the answer is negative, also in the radial case.

Theorem 2.18 *Let Ω be the unit disc with center in the origin of $\mathbb{R}^2$. There exists a radial function $u_0 \in C^\infty(\Omega)$ with*

$$\max\left\{|\nabla u_0(x)| : x \in \overline{\Omega}\right\} > 1, \tag{2.30}$$

such that, if u is a radial C^2 solution of the Perona–Malik equation in Ω, with Neumann boundary conditions on $\partial\Omega$, and initial condition u_0, then, for some $\delta > 0$,

$$\max\left\{|\nabla u(x,t)| : x \in \overline{\Omega}\right\} < \max\left\{|\nabla u_0(x)| : x \in \overline{\Omega}\right\} \quad \forall t \in (0, \delta). \tag{2.31}$$

We remind that the corresponding estimate in dimension one was a key tool in the proof of non-existence of global C^1 solutions with transcritical initial data.

It is of course interesting to consider semi-discrete schemes for (1.6), (1.7), and (1.8). In this case it should be easy to prove a discrete version of Theorem 2.14, but Theorem 2.17 shows that the total variation of the solution cannot be estimated by the total variation of the initial condition. For this reason we are skeptical about a simple extension of Theorem 2.7 and its consequences to higher dimension.

3 Proofs

Proof of Proposition 2.1 Computing the time derivative, and integrating by parts, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} \psi(u)\,\mathrm{d}x &= \int_{-1}^{1} \psi'(u)u_t\,\mathrm{d}x\\ &= \int_{-1}^{1} \psi'(u)\left(\varphi'(u_x)\right)_x\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= -\int_{-1}^{1} \big(\psi'(u)\big)_x \, \varphi'(u_x)\,\mathrm{d}x + \big[\psi'(u)\varphi'(u_x)\big]_{x=-1}^{x=1} \\
&= -\int_{-1}^{1} \psi''(u)\, u_x\, \varphi'(u_x)\,\mathrm{d}x,
\end{aligned}
$$

where the boundary terms are zero due to the boundary condition (1.2) and our assumption that $\varphi'(0) = 0$. This establishes equality (2.1).

Similarly

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} \psi(u_x)\,\mathrm{d}x &= \int_{-1}^{1} \psi'(u_x)u_{tx}\,\mathrm{d}x \\
&= \int_{-1}^{1} \psi'(u_x)\,\big(\varphi'(u_x)\big)_{xx}\,\mathrm{d}x \\
&= -\int_{-1}^{1} \big(\psi'(u_x)\big)_x\,(\varphi'(u_x))_x\,\mathrm{d}x + \big[\psi'(u_x)\varphi''(u_x)u_{xx}\big]_{x=-1}^{x=1} \\
&= -\int_{-1}^{1} \psi''(u_x)\varphi''(u_x)[u_{xx}]^2\,\mathrm{d}x,
\end{aligned}
$$

where the boundary terms are zero due to the boundary condition (1.2) and our assumption that $\psi'(0)\cdot\varphi''(0) = 0$. This establishes equality (2.2).

To be precise, the argument used in the proof of (2.2) requires that u is of class C^3, because of the term $(\varphi'(u_x))_{xx}$ which is involved. If u is only a C^2 solution, then a standard approximation procedure is necessary. To begin with, one takes a family $\{u_\varepsilon\}$ of C^3 approximations of u. It turns out that u_ε solves an approximate equation such as

$$
u_{\varepsilon t} = \varphi''(u_{\varepsilon x})u_{\varepsilon xx} + \rho_\varepsilon,
$$

where $\rho_\varepsilon(x,t)$ tends to zero uniformly on compact sets. Arguing as before, one proves that u_ε satisfies an equality similar to (2.2), with some extra terms depending on ρ_ε, which disappear as $\varepsilon \to 0^+$. We spare the reader from the details. □

Proof of Theorem 2.2

Gradient flow estimates Applying (2.2) with $\psi = \varphi$, we immediately get (2.3). Since φ is non-negative, using (2.3) and Hölder's inequality we obtain that

$$
\begin{aligned}
\|u(x,t_1)-u(x,t_2)\|_{L^2((-1,1))} &\leq \int\limits_{t_1}^{t_2} \|u_t(x,\tau)\|_{L^2((-1,1))}\,\mathrm{d}\tau \\
&\leq |t_1-t_2|^{1/2}\left\{\int\limits_{t_1}^{t_2} \|u_t(x,\tau)\|^2_{L^2((-1,1))}\,\mathrm{d}\tau\right\}^{1/2} \\
&= |t_1-t_2|^{1/2}\left\{PM_\varphi(u(x,t_1))-PM_\varphi(u(x,t_2))\right\}^{1/2} \\
&\leq |t_1-t_2|^{1/2}\left\{PM_\varphi(u_0)\right\}^{1/2},
\end{aligned}
$$

which proves (2.4).

L^p* estimates on *u If $p \in [2,\infty)$ the function $\psi(\sigma) = |\sigma|^p$ is C^2 and convex. By (φ1) this implies that $\psi''(\sigma)\cdot\sigma\cdot\varphi'(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$. From (2.1) we deduce that

$$
t \to \int\limits_{-1}^{1} |u(x,t)|^p\,\mathrm{d}x
$$

is a non-increasing function of time, which proves (2.5).

If $p \in [1,2)$, then the function $\psi(\sigma) = |\sigma|^p$ is convex but not of class C^2. So we approximate it with $\psi_\varepsilon(\sigma) = (\sigma^2+\varepsilon^2)^{p/2}$: applying (2.1) to ψ_ε, and letting $\varepsilon \to 0^+$, we prove (2.5) also in this case.

Finally, the case $p=\infty$ can be proved by letting $p \to +\infty$, or simply deduced from the maximum principle below.

Maximum principle for u Let $K := \max\{u_0(x) :\ x \in [-1,1]\}$, and let

$$
\psi(\sigma) := \begin{cases} 0 & \text{if } \sigma \leq K, \\ (\sigma-K)^4 & \text{if } \sigma \geq K. \end{cases}
$$

It is easy to see that ψ is a convex function of class C^2. Arguing as above, we have that

$$
F(t) := \int\limits_{-1}^{1} \psi(u(x,t))\,\mathrm{d}x
$$

is a non-negative and non-increasing function. Since $F(0) = 0$, then necessarily $F(t) = 0$ for every $t \in [0,T)$, which proves the inequality for the maximum in (2.6).

The proof for the minimum is completely analogous.

Total variation estimate for u Let σ_0 be as in (φ2). It is not difficult to find a family $\{\psi_\varepsilon(\sigma)\}_{\varepsilon>0}$ of functions such that

(4a) for every $\varepsilon > 0$, ψ_ε is a convex function of class C^2 such that $\psi_\varepsilon'(0) = 0$, and $\psi_\varepsilon''(\sigma) = 0$ if $|\sigma| \geq \sigma_0$;
(4b) $\psi_\varepsilon(\sigma) \to |\sigma|$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0^+$.

By (4a) and (φ2) we have that $\psi_\varepsilon''(\sigma)\varphi''(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$, hence by (2.2)

$$\int_{-1}^{1} \psi_\varepsilon(u_x(x,t))\,\mathrm{d}x \leq \int_{-1}^{1} \psi_\varepsilon(u_{0x}(x))\,\mathrm{d}x, \tag{3.1}$$

so that the conclusion follows from (4b) by letting $\varepsilon \to 0^+$.

Pointwise barriers for the derivative Let us prove (2.8). First of all, we remark that $M(0) \geq 0$ (by the Neumann boundary condition), hence $\sigma_2 \geq 0$. Now we distinguish two cases.

Case $\sigma_2 > 0$. Consider the function $\psi(\sigma) = \max\{\sigma - \sigma_2, 0\}$, and a family $\{\psi_\varepsilon(\sigma)\}_{\varepsilon>0}$ such that

(5a) for every $\varepsilon > 0$, ψ_ε is a convex function of class C^2 with $\psi_\varepsilon''(\sigma) = 0$ if $\sigma \notin [\sigma_1, \sigma_2]$;
(5b) $\psi_\varepsilon \to \psi$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0^+$;
(5c) $\psi_\varepsilon'(0) = 0$ for every $\varepsilon > 0$ (here we need that $\sigma_2 > 0$).

Once again $\psi_\varepsilon''(\sigma)\varphi''(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$, hence by (2.2) we get (3.1) also for this family $\{\psi_\varepsilon\}$. Passing to the limit as $\varepsilon \to 0^+$, we obtain that

$$0 \leq \int_{-1}^{1} \psi(u_x(x,t))\,\mathrm{d}x \leq \int_{-1}^{1} \psi(u_{0x}(x))\,\mathrm{d}x = 0,$$

where the last equality follows from the assumption that $M(0) \leq \sigma_2$. This proves that $u_x(x,t) \leq \sigma_2$ for every $(x,t) \in [-1,1] \times [0,T)$.

Case $\sigma_2 = 0$. Since $\varphi''(\sigma_2) \geq 0$, it may happen that $\varphi''(0) = 0$ or $\varphi''(0) > 0$.

- Assume that $\varphi''(0) = 0$. We can find a family $\{\psi_\varepsilon(\sigma)\}_{\varepsilon>0}$ satisfying (5a) and (5b), but not (5c). In any case, $\varphi''(0) = 0$ is enough to apply (2.2), and so we can conclude exactly as in the case $\sigma_2 > 0$.
- If $\varphi''(0) > 0$, then $\varphi''(\sigma) > 0$ in the interval $[\sigma_1, \eta]$ for every $\eta > 0$ small enough. Arguing as in the case $\sigma_2 > 0$, we obtain that $M(t) \leq \eta$ for every $t \in [0,T)$, and then we conclude by letting $\eta \to 0^+$.

This completes the proof of (2.8).

The proof of (2.9) is completely analogous.

Now let us prove (2.10). If $\sigma_2 \leq 0$, the conclusion is trivial by the Neumann boundary condition. If $\sigma_2 > 0$, let $\varepsilon > 0$ be such that $\sigma_2 - \varepsilon \geq \max\{0, \sigma_1\}$. Let $\psi_\varepsilon(\sigma)$ be a convex function of class C^2 such that $\psi_\varepsilon(\sigma) = 0$ for every $\sigma \leq \sigma_2 - \varepsilon$, $\psi_\varepsilon(\sigma) > 0$ for every $\sigma > \sigma_2 - \varepsilon$, and $\psi_\varepsilon''(\sigma) = 0$ for every $\sigma \geq \sigma_2$.

In this case $\psi_\varepsilon'(0) = 0$ and $\psi_\varepsilon''(\sigma)\varphi''(\sigma) \le 0$ for every $\sigma \in \mathbb{R}$, hence by (2.2)

$$\int_{-1}^{1} \psi_\varepsilon(u_x(x,t))\,\mathrm{d}x \ge \int_{-1}^{1} \psi_\varepsilon(u_{0x}(x))\,\mathrm{d}x > 0,$$

where the last equality follows from our assumption that $M(0) \ge \sigma_2$. This proves that $M(t) \ge \sigma_2 - \varepsilon$ for every $t \in [0, T)$. We conclude by letting $\varepsilon \to 0^+$.

The proof of (2.11) is completely analogous.

Maximum principles for the derivative Let us consider the two sets

$$B^- := \{\sigma \le M(0) : \ \varphi''(\sigma) < 0\},$$

$$B^+ := \{\sigma \ge M(0) : \ \exists\delta > 0 \text{ such that } \varphi''(\xi) \ge 0\ \forall\xi \in [\sigma - \delta, \sigma]\},$$

and let us set

$$\xi_1 := \sup B^- \in [-\infty, M(0)], \qquad \xi_2 := \inf B^+ \in [M(0), +\infty],$$

where $\sup B^- = -\infty$ if $B^- = \emptyset$ (resp. $\inf B^+ = +\infty$ if $B^+ = \emptyset$).

From (2.8) we have that any element of B^+ is an upper barrier for $M(t)$, while from (2.10) it is easy to deduce that any element of B^- is a lower barrier for $M(t)$. It follows that

$$\xi_1 \le M(t) \le \xi_2 \quad \forall t \in [0, T). \tag{3.2}$$

Moreover, either $\xi_1 = M(0)$ or $\xi_2 = M(0)$. Assume indeed that $\xi_1 < M(0)$; then necessarily $\varphi''(\sigma) \ge 0$ in $[\xi_1, M(0)]$, and therefore $\xi_2 = M(0)$. We can therefore distinguish three cases.

- If $\xi_1 < M(0)$ (this happens under the assumptions of statements (6) and (8.1)), then $\xi_2 = M(0)$. From (3.2) we deduce that $M(t)$ belongs the interval $[\xi_1, M(0)]$, and we already know that $\varphi''(\sigma) \ge 0$ in this interval. If $t_1 \in [0, T)$, and $M(t_1) > \xi_1$, then we can use $u(x, t_1)$ as a new initial condition: applying (2.8) with $[\sigma_1, \sigma_2] = [\xi_1, M(t_1)]$, we deduce that $M(t) \le M(t_1)$ for every $t \in [t_1, T)$. By the continuity of $M(t)$, this is enough to conclude that $M(t)$ is a non-increasing function.
- If $\xi_2 > M(0)$ (this happens under the assumptions of statements (7) and (8.2)), then $\xi_1 = M(0)$. From (3.2) we deduce that $M(t)$ belongs the interval $[M(0), \xi_2]$, and it is easy to see that $\varphi''(\sigma) \le 0$ in this interval. If $t_1 \in [0, T)$, and $M(t_1) > M(0)$, then we can use $u(x, t_1)$ as a new initial condition: applying (2.10) with $[\sigma_1, \sigma_2] = [M(0), M(t_1)]$, we deduce that $M(t) \ge M(t_1)$ for every $t \in [t_1, T)$, and this is enough to conclude that $M(t)$ is a non-decreasing function.

- In the remaining case we have that $\xi_1 = \xi_2 = M(0)$, and therefore $M(t)$ is constant by (3.2). □

Remark 3.1 Let us prove statement (g) of Remark 2.3. To this end, it is enough to apply statement (6) of Theorem 2.2 if $0 \leq M(0) < \sigma_0$, statement (7) if $M(0) > \sigma_0$, and statement (8.1) if $M(0) = \sigma_0$.

Let us prove now statement (h). Let us assume, without loss of generality, that u_0 is a non-increasing function of x, hence $M(0) = 0$ (remember the Neumann boundary condition). By the subcritical maximum principle for u_x, it follows that $u_x(x,t) \leq 0$ for every $(x,t) \in [-1,1] \times [0,T)$, which proves that $u(x,t)$ is a non-increasing function of x.

In order to prove statement (i), we just observe that in this case all real numbers $\sigma > \max\{\sigma_3, M(0)\}$ are upper barriers for $M(t)$, and analogously for the minimum.

Proof of Proposition 2.4 Let $n > 0$ be a fixed positive integer, and let $h = 1/n$ be the discretization step. An element $u \in PC_n$ can be identified with the $2n$-tuple $(a_1, \ldots, a_{2n}) \in \mathbb{R}^{2n}$, where $u(x) = a_i$ in the interval $(-1 + (i-1)h, -1 + ih)$.

The Euclidean norm on $\mathbb{R}^{2n}$ corresponds to the L^2 norm on PC_n multiplied by $\sqrt{n}$. With this notation we have that

$$PM_{\varphi,n}(u) = h \sum_{i=1}^{2n-1} \varphi\left(\frac{a_{i+1} - a_i}{h}\right),$$

and the gradient flow of $PM_{\varphi,n}$ with respect to the L^2 norm of PC_n reduces to the following system of $2n$ ODEs (the dot denotes time derivatives)

$$\dot{a}_i(t) = \frac{1}{h}\left\{\varphi'\left(\frac{a_{i+1}(t) - a_i(t)}{h}\right) - \varphi'\left(\frac{a_i(t) - a_{i-1}(t)}{h}\right)\right\} \quad i = 1, \ldots, 2n, \tag{3.3}$$

where by definition $a_0(t) = a_1(t)$ and $a_{2n+1}(t) = a_{2n}(t)$ for every $t \geq 0$.

If φ'' is bounded, then the right hand side of (3.3) is Lipschitz continuous, hence this system has a unique global solution for every initial condition by the Cauchy–Lipschitz–Picard Theorem for ODEs. □

Proof of Theorem 2.5

Gradient flow estimates Classical gradient flow techniques.

***L^p* estimates and maximum principle** We follow the same strategy used in the continuous setting. To this end, we need a discrete version of (2.1). Given $\psi \in C^1(\mathbb{R})$, using (3.3) and simple algebraic manipulations on the sums, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(h\sum_{i=1}^{2n}\psi(a_i)\right)=h\sum_{i=1}^{2n}\psi'(a_i)\dot{a}_i=-\sum_{i=1}^{2n}\left\{\psi'(a_{i+1})-\psi'(a_i)\right\}\varphi'\left(\frac{a_{i+1}-a_i}{h}\right). \tag{3.4}$$

In particular, if ψ is convex and φ satisfies ($\varphi 1$), then each summand in the right hand side of (3.4) is the product of two factors with the same sign. It follows that

$$t\to h\sum_{i=1}^{2n}\psi(a_i(t))$$

is a non-increasing function. Now estimates (2.16) and (2.17) follow with the same choices of the convex function ψ used in the continuous setting.

Total variation estimate We need a discrete version of (2.2). Given $\psi\in C^1(\mathbb{R})$ with $\psi'(0)=0$, using (3.3) and simple algebraic manipulations on the sums, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(h\sum_{i=1}^{2n}\psi\left(\frac{a_{i+1}-a_i}{h}\right)\right)&=h\sum_{i=1}^{2n}\psi'\left(\frac{a_{i+1}-a_i}{h}\right)\frac{\dot{a}_{i+1}-\dot{a}_i}{h}\\&=-\frac{1}{h}\sum_{i=1}^{2n}\left\{\psi'\left(\frac{a_{i+1}-a_i}{h}\right)-\psi'\left(\frac{a_i-a_{i-1}}{h}\right)\right\}\\&\quad\cdot\left\{\varphi'\left(\frac{a_{i+1}-a_i}{h}\right)-\varphi'\left(\frac{a_i-a_{i-1}}{h}\right)\right\}.\end{aligned}$$

Now let us apply this equality to the family $\psi_\varepsilon(\sigma)=\sqrt{\sigma^2+\varepsilon}$. Let $F_\varepsilon(t)$ be the corresponding sum in the right hand side. Integrating in $[0,t]$ we have that

$$h\sum_{i=1}^{2n}\psi_\varepsilon\left(\frac{a_{i+1}(t)-a_i(t)}{h}\right)=h\sum_{i=1}^{2n}\psi_\varepsilon\left(\frac{a_{i+1}(0)-a_i(0)}{h}\right)-\frac{1}{h}\int_0^t F_\varepsilon(\tau)\,\mathrm{d}\tau. \tag{3.5}$$

Unfortunately, we cannot prove that $F_\varepsilon(t)$ is non-negative (and examples may be given where this is false). However, we prove that the limit of $F_\varepsilon(t)$ as $\varepsilon\to 0^+$ is non-negative. To begin with, we observe that

$$\lim_{\varepsilon\to 0^+}\psi'_\varepsilon(\sigma)=\begin{cases}-1 & \text{if } \sigma<0,\\ 0 & \text{if } \sigma=0,\\ 1 & \text{if } \sigma>0.\end{cases}$$

Now let us examine each summand in the sum defining $F_\varepsilon(t)$. The limit of the factor involving ψ'_ε can only be $-2,-1,0,1,2$, depending on the signs of $a_{i+1}-a_i$ and a_i-a_{i-1}. Let us assume that this limit is 2: this means that $a_{i+1}-a_i>0$

and $a_i - a_{i-1} < 0$. In this case by (φ1) the term involving φ' is positive, and therefore the limit of the summand is positive. Examining in the same way the other cases, we prove that the limit of each summand is always non-negative.

Now we pass to the limit in (3.5): this can be done by Lebesgue's theorem since $|\psi'_\varepsilon(\sigma)| \leq 1$ for every $\varepsilon > 0$ and every $\sigma \in \mathbb{R}$, and the terms involving φ' are equibounded by continuity. We obtain that

$$\sum_{i=1}^{2n} |a_{i+1}(t) - a_i(t)| \leq \sum_{i=1}^{2n} |a_{i+1}(0) - a_i(0)|,$$

which is exactly (2.18).

Monotonicity of sub(and super)critical regions The main tool is the following comparison result for ODEs (see [8, Lemma 4.10]).

Lemma 3.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function, and let C, V be real numbers such that $f(C) = -f(-C) = V$. Let $y \in C^1([0, +\infty); \mathbb{R})$ be a function such that $|y'(t) + f(y(t))| \leq V$ for every $t \geq 0$, and $|y(T)| \leq C$ (resp. $|y(T)| < C$) for some $T \geq 0$.*

Then $|y(t)| \leq C$ (resp. $|y(t)| < C$) for every $t \geq T$. □

Let us set $d_i(t) := (a_{i+1}(t) - a_i(t))/h$ for $i = 1, \ldots, 2n-1$, and $d_0(t) = d_{2n}(t) = 0$ for every $t \geq 0$. Thesis is equivalent to show that for every index i we have that if $|d_i(T)| \leq \sigma_0$ (resp. $|d_i(T)| < \sigma_0$) for some $T \geq 0$, then $|d_i(t)| \leq \sigma_0$ (resp. $|d_i(t)| < \sigma_0$) for every $t \geq T$. From (3.3) it is easy to deduce that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} d_i(t) + \frac{2}{h^2} \varphi'(d_i(t)) \right| = \frac{1}{h^2} \left| \varphi'(d_{i+1}(t)) - \varphi'(d_{i-1}(t)) \right| \leq \frac{2}{h^2} \varphi'(\sigma_0),$$

and therefore we can conclude applying Lemma 3.2 with $y(t) = d_i(t)$, $f(\sigma) = 2\varphi'(\sigma)/h^2$, $C = \sigma_0$, $V = 2\varphi'(\sigma_0)/h^2$. □

Proof of Theorem 2.7 By the standard Ascoli Theorem, it is enough to prove that

(A1) for every $t \geq 0$, the sequence $\{u_n(t)\}$ is relatively compact in $L^2((-1, 1))$;
(A2) there exists a constant $C \in \mathbb{R}$ such that

$$\|u_n(t_1) - u_n(t_2)\|_{L^2((-1,1))} \leq C|t_1 - t_2|^{1/2}$$

for every $n \in \mathbb{N}$, and every $t_2 \geq t_1 \geq 0$.

Let M be the supremum in (2.19). Then by (2.15) we have that (A2) is satisfied with $C = M^{1/2}$. Now let us fix $t \geq 0$. By (2.16) with $p = \infty$ and (2.18) we obtain that

$$\|u_n(x,t)\|_{L^\infty((-1,1))} + \|D^{1/n}u_n(x,t)\|_{L^1((-1,1))} \leq M,$$

and so we have a bound on the L^∞ norm and on the total variation of $u_n(x,t)$ as a function of x. By a well known result, this implies that the family $\{u_n(t)\}$ is relatively compact in $L^p((-1,1))$ for every $p \in [1,\infty)$ and therefore also assumption (A1) is satisfied. □

Proof of Theorem 2.9 Let n_k, u_{0k}, and u_k be as in Definition 2.8. Since we assumed that φ' is bounded, the boundedness of $\|D^{1/n_k}u_{0k}\|_{L^1((-1,1))}$ implies the boundedness of $PM_{\varphi,n_k}(u_{0k})$. By (2.20) we have therefore that

$$M := \sup_{k\in\mathbb{N}} \left\{PM_{\varphi,n_k}(u_{0k}) + \|u_{0k}\|_{L^\infty((-1,1))} + \|D^{1/n_k}u_{0k}\|_{L^1((-1,1))}\right\} < +\infty. \tag{3.6}$$

So we can apply Theorem 2.7 and deduce that $GF_\varphi(u_0)$ is nonempty.

Initial condition, time and space regularity The initial condition is trivial. From (2.14) we have that

$$\int_0^{+\infty} \|u_k'(x,t)\|^2_{L^2((-1,1))}\,\mathrm{d}t \leq M. \tag{3.7}$$

This proves that u_k', up to subsequences, has a weak limit as $k \to +\infty$. It is completely standard to see that this limit is indeed u' and does not depend on the subsequence. This proves both the conclusion of statement (2) and (2.23).

Passing to the limit in (2.16) with $p = \infty$ and using (2.20) we obtain (2.21).

Passing to the limit in (2.18) and using (2.20) we obtain both (2.22) and (2.25).

Weak equation solved From the boundedness of φ' we get a uniform bound on the L^∞ norm of $\varphi'\left(D^{1/n_k}u_k(x,t)\right)$. It follows that, up to extracting a further subsequence (not relabeled), $\varphi'\left(D^{1/n_k}u_k(x,t)\right)$ converges to a function $g(x,t)$ in the weak * topology of $L^\infty((-1,1)\times(0,+\infty))$.

Now we multiply both sides of (2.12) by a test function $\phi \in C^\infty((-1,1))$ and we integrate in $(-1,1)\times(0,T)$. With some simple algebra we find that

$$\begin{aligned}\int_0^T \mathrm{d}t \int_{-1}^1 u_k' \cdot \phi\,\mathrm{d}x &= -\int_0^T \mathrm{d}t \int_{-1}^1 \nabla PM_{\varphi,n_k}(u_k)\cdot\phi\,\mathrm{d}x\\ &= -\int_0^T \mathrm{d}t \int_{-1}^1 \varphi'\left(D^{1/n_k}u_k\right)\cdot D^{1/n_k}\phi\,\mathrm{d}x.\end{aligned}$$

Now we pass to the limit as $k \to +\infty$ using that $u'_k \rightharpoonup u'$, $\varphi'\left(D^{1/n_k}u_k\right) \rightharpoonup g(x,t)$, and $D^{1/n_k}\phi \to \phi_x$ strongly. We obtain that

$$\int_0^T \mathrm{d}t \int_{-1}^1 u' \cdot \phi \,\mathrm{d}x = -\int_0^T \mathrm{d}t \int_{-1}^1 g \cdot \phi_x \,\mathrm{d}x.$$

Since T is arbitrary, we get that for almost every $t \geq 0$ the integrals in the space variable coincide. This is equivalent to say that for almost every $t \geq 0$ the function $x \to g(x,t)$ belongs to $H^1((-1,1))$, $g(-1,t) = g(1,t) = 0$, and $u_t = g_x$. Since this uniquely characterizes g in terms of u, we can conclude that the whole sequence $\varphi'(D^{1/n_k}u_k)$ weakly * converges to g, without extracting further subsequences. Therefore all the conclusions of statement (4) are proved.

Stability Thanks to the preceding estimates, assumption (2.26) implies uniform bounds on the L^∞ norm, the total variation, and the Hölder constant of u_h. Therefore the compactness of $\{u_h\}$ follows from Ascoli's Theorem.

So up to subsequences (not relabeled) we may assume that u_h converges to some u_∞ in $C^0([0,+\infty); L^2((-1,1)))$. By definition of $GF_\varphi(u_{0h})$, for every $h \in \mathbb{N}$ there exists $n_h \geq h$ and $v_{0h} \in PC_{n_h}$ with corresponding solution v_h of (2.12) such that

$$\|v_{0h} - u_{0h}\|_2 + \left|\|v_{0h}\|_\infty - \|u_{0h}\|_\infty\right| + \left|\|D^{1/n_h}v_{0h}\|_1 - TV(u_{0h})\right| \leq \frac{1}{h},$$

$$\|v_h(x,t) - u_h(x,t)\|_{L^2((-1,1))} \leq \frac{1}{h} \quad \forall t \in [0,h].$$

This is enough to conclude that v_{0h} approximates u_0 as required in Definition 2.8 and $v_h \to u_\infty$, which is equivalent to say that $u_\infty \in GF_\varphi(u_0)$. □

Proof of Theorem 2.10 The strategy of the proof is the following. First of all we choose an open set $\Omega_1 \subset\subset \Omega$. In Proposition 3.3 we prove a pointwise estimate on $\|u'_k(t)\|_{L^2(\Omega_1)}$ for every $t > 0$ which improves the integral estimate coming from (2.14). In Lemma 3.4 we show that the preceding estimate gives the uniform convergence of $\varphi'(D^{1/n_k}u_k)$ in the compact subsets of Ω_1, improving the weak * convergence of (2.24). Finally we exploit the invertibility of φ' in $[-\sigma_0, \sigma_0]$ to deduce that $D^{1/n_k}u_k$ uniformly converges to u_x in the compact subsets of Ω_1, thus showing that the function $g(x,t)$ in statement (4) of Theorem 2.9 coincides with $\varphi'(u_x(x,t))$ in $\Omega_1 \times (0,+\infty)$. Since Ω_1 is arbitrary, this is enough to prove all the conclusions.

Proposition 3.3 *For every open set $\Omega_1 \subset\subset \Omega$ there exists a constant $c = c(\Omega_1)$ and a positive integer $k_0 = k_0(\Omega_1)$ such that for every $k \geq k_0$ we have that*

$$\left\|u'_k(T)\right\|^2_{L^2(\Omega_1)} \leq PM_{\varphi,n_k}(u_{0k})\left(\frac{1}{T} + c\right) \quad \forall T > 0. \tag{3.8}$$

Proof Let us choose open sets Ω_2 and Ω_3 such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. Let k_0 be such that the neighborhood of Ω_1 with radius $1/n_{k_0}$ is contained in Ω_2 and the neighborhood of Ω_3 with radius $1/n_{k_0}$ is contained in Ω. Let us choose a function $r \in C^\infty((-1,1))$ such that $r(x) \equiv 1$ in Ω_2, $r(x) \equiv 0$ outside Ω_3, and $0 \leq r(x) \leq 1$ otherwise. Given $k \geq k_0$, once again we set $h = 1/n_k$, and we identify $u_k(t)$ with the $2n_k$-tuple $(a_1(t), \dots, a_{2n_k}(t))$. We also consider the $2n_k$-tuple of real numbers $(r_1, \dots, r_{2n_k})$ where $r_i = r(-1+ih)$. Finally we set $a_0(t) = a_1(t)$, $a_{2n_k+1}(t) = a_{2n_k}(t)$, $r_0 = r_{2n_k+1} = 0$. With these notations estimate (2.14) may be rewritten as

$$h \int_0^{+\infty} \sum_{i=1}^{2n_k} \dot{a}_i^2(t)\,\mathrm{d}t = \int_0^{+\infty} \|u_k'(t)\|^2_{L^2((-1,1))}\,\mathrm{d}t \leq PM_{\varphi,n_k}(u_{0k}). \tag{3.9}$$

Moreover by our choice of k_0 and r we have that

$$\|u_k'(t)\|^2_{L^2(\Omega_1)} \leq h \sum_{i=1}^{2n_k} r_i^2 \dot{a}_i^2(t), \tag{3.10}$$

$$\left| \frac{r_{i+1} - r_i}{h} \right| \leq \|r_x\|_{L^\infty((-1,1))} =: c_1. \tag{3.11}$$

Finally, by the monotonicity of subcritical regions, we have that $|D^{1/n_k} u_k| \leq \sigma_0$ in $\Omega \times (0, +\infty)$, hence for every $t \geq 0$ and every $i = 1, \dots, 2n_k$ we have the implication

$$r_{i+1} + r_i \neq 0 \quad \Longrightarrow \quad \varphi''\left(\frac{a_{i+1}(t) - a_i(t)}{h}\right) \geq 0. \tag{3.12}$$

Now we are ready to estimate the norm of $u_k'(t)$ in $L^2(\Omega_1)$. To this end, we compute the time derivative of the right hand side of (3.10) multiplied by t.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t \cdot h \sum_{i=1}^{2n_k} r_i^2 \dot{a}_i^2 \right) = h \sum_{i=1}^{2n_k} r_i^2 \dot{a}_i^2 + t \cdot 2h \sum_{i=1}^{2n_k} r_i^2 \dot{a}_i \ddot{a}_i. \tag{3.13}$$

Let us concentrate on the second sum in the right hand side, where we use (3.3) to compute $\ddot{a}_i$. Manipulating the sums and using simple algebraic equalities it can be rewritten as

$$\begin{aligned} \sum_{i=1}^{2n_k} r_i^2 \dot{a}_i \ddot{a}_i &= \frac{1}{h} \sum_{i=1}^{2n_k} r_i^2 \dot{a}_i \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \varphi'\left(\frac{a_{i+1} - a_i}{h}\right) - \frac{\mathrm{d}}{\mathrm{d}t} \varphi'\left(\frac{a_i - a_{i-1}}{h}\right) \right\} \\ &= -\frac{1}{h} \sum_{i=1}^{2n_k} \left(r_{i+1}^2 \dot{a}_{i+1} - r_i^2 \dot{a}_i \right) \frac{\mathrm{d}}{\mathrm{d}t} \varphi'\left(\frac{a_{i+1} - a_i}{h}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{h^2}\sum_{i=1}^{2n_k-1}\left(r_{i+1}^2\dot{a}_{i+1} - r_i^2\dot{a}_i\right)(\dot{a}_{i+1}-\dot{a}_i)\varphi''\left(\frac{a_{i+1}-a_i}{h}\right)\\
&= -\frac{1}{2h^2}\sum_{i=1}^{2n_k-1}\left(r_{i+1}^2 + r_i^2\right)(\dot{a}_{i+1}-\dot{a}_i)^2\varphi''\left(\frac{a_{i+1}-a_i}{h}\right)\\
&\quad -\frac{1}{2h^2}\sum_{i=1}^{2n_k-1}\left(r_{i+1}+r_i\right)\left(r_{i+1}-r_i\right)(\dot{a}_{i+1}+\dot{a}_i)(\dot{a}_{i+1}-\dot{a}_i)\varphi''\left(\frac{a_{i+1}-a_i}{h}\right)\\
&=: S_1 + S_2
\end{aligned}
$$

Let us consider now the simple algebraic inequality

$$
\begin{aligned}
|(a+b)(a-b)(c+d)(c-d)| &\leq \frac{1}{2}\left[(a+b)^2(c-d)^2 + (c+d)^2(a-b)^2\right]\\
&\leq (a^2+b^2)(c-d)^2 + (c^2+d^2)(a-b)^2.
\end{aligned}
$$

Recalling (3.12) we can estimate every nonzero term in S_2 applying this inequality with $a = r_{i+1}$, $b = r_i$, $c = \dot{a}_{i+1}$, $d = \dot{a}_i$. We obtain that

$$
\begin{aligned}
S_2 \leq &\frac{1}{2h^2}\sum_{i=1}^{2n_k-1}\left(r_{i+1}^2 + r_i^2\right)(\dot{a}_{i+1}-\dot{a}_i)^2\varphi''\left(\frac{a_{i+1}-a_i}{h}\right)\\
&+\frac{1}{2h^2}\sum_{i=1}^{2n_k-1}\left(r_{i+1}-r_i\right)^2(\dot{a}_{i+1}^2+\dot{a}_i^2)\varphi''\left(\frac{a_{i+1}-a_i}{h}\right)
\end{aligned}
$$

The first term is exactly $-S_1$, so that coming back to (3.13) we have proved that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(t\cdot h\sum_{i=1}^{2n_k} r_i^2\dot{a}_i^2\right) &\leq h\sum_{i=1}^{2n_k} r_i^2\dot{a}_i^2 + t\cdot h\sum_{i=1}^{2n_k-1}\varphi''\left(\frac{a_{i+1}-a_i}{h}\right)\\
&\quad\times\left(\frac{r_{i+1}-r_i}{h}\right)^2\left(\dot{a}_{i+1}^2+\dot{a}_i^2\right)\\
&\leq h\sum_{i=1}^{2n_k}\dot{a}_i^2 + c_2c_1^2 t\cdot h\sum_{i=1}^{2n_k-1}\left(\dot{a}_{i+1}^2+\dot{a}_i^2\right),
\end{aligned}
$$

where c_1 is given by (3.11) and c_2 denotes the supremum of φ'' in $[-\sigma_0, \sigma_0]$. Integrating in $[0, T]$ and using (3.9) we obtain that

$$
T\cdot h\sum_{i=1}^{2n_k} r_i^2\dot{a}_i^2(T) \leq PM_{\varphi,n_k}(u_{0k})(1+2Tc_2c_1^2).
$$

Combining with (3.10), estimate (3.8) is proved with $c(\Omega_1) := 2c_1^2c_2$. □

Lemma 3.4 *Let $n_k \to +\infty$ be a sequence of positive integers, let v_k be a sequence of functions such that $v_k \in PC_{n_k}$ for every $k \in \mathbb{N}$, and let $\Omega_1 \subseteq (-1,1)$ be an open set such that*

$$\sup_{k\in\mathbb{N}} \|\nabla PM_{n_k,\varphi}(v_k)\|_{L^2(\Omega_1)} < +\infty. \tag{3.14}$$

Then there exists a function $\gamma \in H^1(\Omega_1)$ such that, up to subsequences,

$$\nabla PM_{n_k,\varphi}(v_k) \rightharpoonup -\gamma_x \quad \textit{weakly in } L^2(\Omega_1), \tag{3.15}$$

$$\varphi'\left(D^{1/n_k} v_k\right) \to \gamma \quad \textit{uniformly on compact subsets of } \Omega_1. \tag{3.16}$$

Proof By (3.14) and the boundedness of φ' we have that there exist $\gamma \in L^2(\Omega_1)$, $\psi \in L^2(\Omega_1)$, and a subsequence (not relabeled) such that

$$\nabla PM_{n_k,\varphi}(v_k) \rightharpoonup \psi \quad \text{weakly in } L^2(\Omega_1) \tag{3.17}$$

$$\varphi'\left(D^{1/n_k} v_k\right) \rightharpoonup \gamma \quad \text{weakly in } L^2(\Omega_1). \tag{3.18}$$

Now let $\phi \in C_0^\infty(\Omega_1)$. It is easy to see that for k large one has that

$$\int_{\Omega_1} \nabla PM_{n_k,\varphi}(v_k) \cdot \phi \, \mathrm{d}x = \int_{\Omega_1} \varphi'\left(D^{1/n_k} v_k\right) \cdot D^{1/n_k}\phi \, \mathrm{d}x.$$

Now we pass to the limit using (3.17), (3.18) and the fact that $D^{1/n_k}\phi \to \phi_x$ strongly. We obtain that

$$\int_{\Omega_1} \psi \cdot \phi \, \mathrm{d}x = \int_{\Omega_1} \gamma \cdot \phi_x \, \mathrm{d}x,$$

which is equivalent to say that $\gamma \in H^1(\Omega_1)$ and $\psi = -\gamma_x$, which proves (3.15).

Now let I be a compact interval contained in Ω_1, and let $x, y \in I$. Let M be the supremum in (3.14), let us set $\gamma_k := \varphi'(D^{1/n_k} v_k)$, $h = 1/n_k$, and let us identify v_k with a $2n_k$-tuple $a_1, \ldots, a_{2n_k}$ so that

$$\gamma_k(y) = \varphi'\left(\frac{a_{j+1} - a_j}{h}\right), \quad \gamma_k(x) = \varphi'\left(\frac{a_{i+1} - a_i}{h}\right),$$

for some integers i and j. Without loss of generality we can assume that $y \geq x$, so that $j \geq i$. If the neighborhood of I with radius h is contained in Ω_1, then by the Cauchy Schwarz inequality we have that

$$\begin{aligned}
|\gamma_k(y)-\gamma_k(x)| &= \left|\varphi'\left(\frac{a_{j+1}-a_j}{h}\right)-\varphi'\left(\frac{a_{i+1}-a_i}{h}\right)\right| \\
&\leq \sum_{m=i+1}^{j}\left|\varphi'\left(\frac{a_{m+1}-a_m}{h}\right)-\varphi'\left(\frac{a_m-a_{m-1}}{h}\right)\right| \\
&\leq \sqrt{j-i}\cdot\left\{h^2\sum_{m=i+1}^{j}\frac{1}{h^2}\left|\varphi'\left(\frac{a_{m+1}-a_m}{h}\right)-\varphi'\left(\frac{a_m-a_{m-1}}{h}\right)\right|^2\right\}^{1/2} \\
&\leq \sqrt{h(j-i)}\cdot\left\|\nabla PM_{\varphi,n_k}(v_k)\right\|_{L^2(\Omega_1)} \\
&\leq M\left\{|y-x|+\frac{1}{n_k}\right\}^{1/2}.
\end{aligned}$$

At this point the uniform convergence in I (up to subsequences) of γ_k follows from a simple variant of the classical Ascoli Theorem (keeping into account the vanishing term $1/n_k$). □

We are now ready to conclude the proof of Theorem 2.10. Let $T > 0$. Form Proposition 3.3 we know that

$$\sup_{k\in\mathbb{N}}\left\|\nabla PM_{\varphi,n_k}(u_k(T))\right\|_{L^2(\Omega_1)} = \sup_{k\in\mathbb{N}}\left\|u_k'(T)\right\|_{L^2(\Omega_1)} < +\infty,$$

hence applying Lemma 3.4 with $v_k = u_k(T)$ we deduce that there exists $\gamma \in H^1(\Omega_1)$ such that, up to subsequences (not relabeled),

$$\varphi'\left(D^{1/n_k}u_k(T)\right) \to \gamma \quad \text{uniformly on compact subsets of } \Omega_1. \tag{3.19}$$

Since $|D^{1/n_k}u_k(T)| \leq \sigma_0$ (by the monotonicity of subcritical regions) and φ' is invertible in $[-\sigma_0,\sigma_0]$, we can apply $(\varphi')^{-1}$ to (3.19) obtaining that

$$D^{1/n_k}u_k(T) \to (\varphi')^{-1}(\gamma) \quad \text{uniformly on compact subsets of } \Omega_1.$$

Compared with (2.25) this proves that $(\varphi')^{-1}(\gamma) = Du(T)$. Therefore $u(T) \in C^1(\Omega_1)$ and $\gamma = \varphi'(u_x)$. Moreover this characterizes γ and therefore the whole sequence $\varphi'\left(D^{1/n_k}u_k\right)$ converges in Ω_1 to γ, without extracting further subsequences. Since Ω_1 is arbitrary, this is enough to conclude the proof. □

Proof of Theorem 2.13 If the absolutely continuous part of the Du_0 is zero and φ satisfies (2.29), then (2.20) implies that $PM_{\varphi,n_k}(u_{0k}) \to 0$. Now the conclusion follows by passing to the limit in (2.15). □

Proof of Theorem 2.14 The classical gradient flow estimates can be proved in the usual way.

The L^p estimates and the maximum principle for u can be proved as in dimension one, up to replacing (2.1) with the following equality: if $\psi \in C^2(\mathbb{R})$, and $u : \overline{\Omega} \times [0, T) \to \mathbb{R}$ is a solution of (1.6), (1.7), and (1.8), then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \psi(u(x,t))\, \mathrm{d}x = -\int_{\Omega} \psi''(u(x,t)) \cdot |\nabla u(x,t)| \cdot \varphi'(|\nabla u(x,t)|)\, \mathrm{d}x.$$

This equality can be proved integrating by parts as in the proof of (2.1). □

Proof of Theorem 2.15 Let $\Omega := \{x \in \mathbb{R}^n : |x| < R\}$. Writing u as a function of t and $r = |x|$, equations (1.6), (1.7) take the form

$$u_t = \varphi''(u_r)u_{rr} + (n-1)\frac{\varphi'(u_r)}{r} \quad \forall (r,t) \in (0,R) \times [0,T), \tag{3.20}$$

$$u_r(0,t) = u_r(R,t) = 0 \quad \forall t \in [0,T), \tag{3.21}$$

and moreover

$$\|\nabla u(x,t)\|_{L^1(\Omega)} = \omega_{n-1} \int_0^R r^{n-1} |u_r(r,t)|\, \mathrm{d}r,$$

where ω_{n-1} is the $(n-1)$-dimensional Hausdorff measure of the unit sphere in $\mathbb{R}^n$.

Now we need to extend (2.2) to radial solutions. Given $\psi \in C^2(\mathbb{R})$ with $\psi'(0) = 0$, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_0^R r^{n-1}\psi(u_r)\, \mathrm{d}r &= \int_0^R r^{n-1}\psi'(u_r)u_{rt}\, \mathrm{d}r \\
&= \int_0^R r^{n-1}\psi'(u_r) \left(\varphi''(u_r)u_{rr} + (n-1)\frac{\varphi'(u_r)}{r}\right)_r \mathrm{d}r \\
&= -\int_0^R \left(r^{n-1}\psi'(u_r)\right)_r \varphi''(u_r)u_{rr}\, \mathrm{d}r \\
&\quad +(n-1)\int_0^R r^{n-1}\psi'(u_r)\left(\frac{\varphi'(u_r)}{r}\right)_r \mathrm{d}r,
\end{aligned}$$

where we neglected the boundary terms in the integration by parts due to (3.21) and our assumption that $\psi'(0) = 0$. Computing the derivatives, two terms cancel and we finally obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^R r^{n-1}\psi(u_r)\,\mathrm{d}r = -\int_0^R r^{n-1}\psi''(u_r)\varphi''(u_r)u_{rr}^2\,\mathrm{d}r - (n-1)\int_0^R \psi'(u_r)\varphi'(u_r)r^{n-3}\,\mathrm{d}r \tag{3.22}$$

Now let us apply this identity to a family $\{\psi_\varepsilon\}_{\varepsilon>0}$ of functions satisfying (4a) and (4b) as in the proof of Theorem 2.2, and

(4c) $\psi_\varepsilon'(\sigma)\sigma \geq 0$ for every $\sigma \in \mathbb{R}$ and every $\varepsilon > 0$.

By (4a) and (φ2) we have that $\psi_\varepsilon''(\sigma)\varphi''(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$. By (4c) and (φ1) we have that $\psi_\varepsilon'(\sigma)\varphi'(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$. It follows that the right hand side of (3.22) is non-positive, and therefore

$$\int_0^R r^{n-1}\psi_\varepsilon(u_r(r,t))\,\mathrm{d}r \leq \int_0^R r^{n-1}\psi_\varepsilon(u_{0r}(r))\,\mathrm{d}r.$$

We finally conclude by letting $\varepsilon \to 0^+$. □

4 Counter-examples

In this section we show our counter-examples to gradient estimates in dimension 2 (or higher). In the first two examples, we consider the function

$$F(t) := \int_\Omega |\nabla u(x,t)|\,\mathrm{d}x, \tag{4.1}$$

where u is a solution of the heat equation, and of the Perona–Malik equation, respectively. What we need is to compute $F'(0)$, which is formally given by

$$F'(0) = \int_\Omega \frac{\nabla u(x,0)}{|\nabla u(x,0)|}\cdot \nabla u_t(x,0)\,\mathrm{d}x, \tag{4.2}$$

where the dot denotes the scalar product. We expect that this formula holds true for large classes of functions; just for the reader's (and our own) comfort, we rigorously justify it under a set of assumptions on $u(x,t)$ which are satisfied in our examples.

Lemma 4.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u : \Omega \times [0,T) \to \mathbb{R}$. Let us assume that $u \in C^1([0,T); H^1(\Omega))$ and that $\nabla u(x,0) \neq 0$ for every $x \in \Omega$.*

Then the function $F(t)$ defined in (4.1) *is* (*right*) *derivable at $t = 0$, and $F'(0)$ is given by* (4.2).

Proof Since $|\nabla u(x,0)| \neq 0$ in Ω, we have that

$$F'(0) = \lim_{t\to 0^+} \frac{1}{t} \int_\Omega \{|\nabla u(x,t)| - |\nabla u(x,0)|\}\, \mathrm{d}x$$

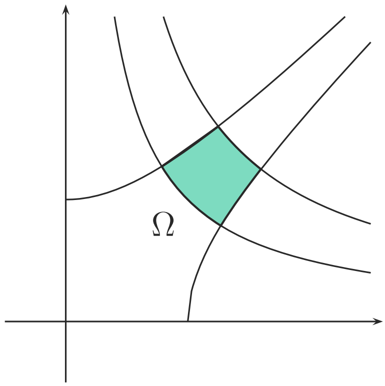
$$= \lim_{t\to 0^+} \frac{1}{t} \int_\Omega \frac{|\nabla u(x,t)|^2 - |\nabla u(x,0)|^2}{|\nabla u(x,t)| + |\nabla u(x,0)|}\, \mathrm{d}x$$

$$= \lim_{t\to 0^+} \int_\Omega \frac{\nabla u(x,t) + \nabla u(x,0)}{|\nabla u(x,t)| + |\nabla u(x,0)|} \cdot \frac{\nabla u(x,t) - \nabla u(x,0)}{t}\, \mathrm{d}x.$$

Since the first factor is bounded and $u \in C^0([0,T); H^1(\Omega))$, it follows that the first factor tends to $|\nabla u(x,0)|^{-1}\nabla u(x,0)$ in $L^2(\Omega;\mathbb{R}^n)$. Since $u \in C^1([0,T); H^1(\Omega))$, the second factor tends to $\nabla u_t(x,0)$ in $L^2(\Omega;\mathbb{R}^n)$. So we can pass to the limit in the integral, and this proves (4.2). □

We point out that in Lemma 4.1 we don't need that $\nabla u(x,0) \neq 0$ on $\partial\Omega$.

Example 1 (proof of Theorem 2.16) Let Ω be the shaded region in the following picture



analytically described by

$$\Omega := \{(x,y) \in \mathbb{R}^2 : \ x \geq 0,\ y \geq 0,\ 1 < xy < 2,\ -1 < x^2 - y^2 < 1\},$$

let $f \in C^\infty(\mathbb{R})$ be a non-decreasing function such that $f'(x) > 0$ if and only if $x \in (1,2)$, and let

$$u_0(x,y) := f(xy).$$

Let $u : \Omega \times [0, +\infty) \to \mathbb{R}$ be the solution of the heat equation with Neumann boundary conditions on $\partial\Omega$, and u_0 as initial condition, and let $F(t)$ be defined by (4.1). A direct computation shows that $\nabla u_0 \neq 0$ in Ω. Moreover, $u_0 \in H^3(\Omega)$, and its normal derivative is identically zero in $\partial\Omega$. By the standard regularity results for the heat equation, this implies that $u \in C^0([0, +\infty); H^3(\Omega)) \cap C^1([0, +\infty); H^1(\Omega))$. It follows that all the assumptions of Lemma 4.1 are satisfied, and $u_t(x, y, 0) = \Delta u_0(x, y)$ in Ω, and therefore by (4.2)

$$F'(0) = \int_\Omega \frac{\nabla u(x, y, 0)}{|\nabla u(x, y, 0)|} \cdot \nabla u_t(x, y, 0)\, \mathrm{d}x\, \mathrm{d}y = \int_\Omega \frac{\nabla u_0}{|\nabla u_0|} \cdot \nabla \Delta u_0\, \mathrm{d}x\, \mathrm{d}y.$$

With our u_0, this turns out to be

$$F'(0) = \int_\Omega \frac{4xy}{\sqrt{x^2 + y^2}} f''(xy)\, \mathrm{d}x\, \mathrm{d}y + \int_\Omega \left(x^2 + y^2\right)^{3/2} f'''(xy)\, \mathrm{d}x\, \mathrm{d}y.$$

Using the new variables $v = xy$, $w = x^2 - y^2$, these integrals become

$$F'(0) = \int_{-1}^{1} \mathrm{d}w \int_{1}^{2} \frac{2v}{(4v^2 + w^2)^{3/4}} f''(v)\, \mathrm{d}v + \frac{1}{2} \int_{-1}^{1} \mathrm{d}w \int_{1}^{2} \left(4v^2 + w^2\right)^{1/4} f'''(v)\, \mathrm{d}v.$$

Now, in the first integral we integrate by parts in v, and in the second integral we integrate by parts twice in v. We finally obtain that

$$F'(0) = \int_{-1}^{1} \mathrm{d}w \int_{1}^{2} \frac{2v^2 - w^2}{(4v^2 + w^2)^{7/4}} f'(v)\, \mathrm{d}v,$$

and this is positive because the integrand is positive in the given domain. □

Remark 4.2 A similar example can be given in an open set with boundary of class C^∞. To this end, it is enough to use the same initial condition u_0, but defined in an open set Ω' of class C^∞ which coincides with our Ω in the region $1 < xy < 2$ (note that Ω' is still a non-convex set). In order to use (4.2) in this case, one needs to extend Lemma 4.1, allowing ∇u_0 to be zero in $\Omega' \setminus \Omega$. We leave the details to the interested reader.

Example 2 (proof of Theorem 2.17) The construction of the counter-example is organized as follows: in the first paragraph we introduce a function $a \in C^\infty(\mathbb{R})$ and a constant $C > 0$; in the second paragraph we introduce a function $b \in C^\infty(\mathbb{R})$ depending on a parameter $\lambda > 0$; then in the third paragraph we consider the rectangular open set

$$\Omega := (0, \lambda + C + 2) \times (0, \lambda + 2), \tag{4.3}$$

and the initial conıtion $u_0 : \Omega \to \mathbb{R}$ defined by

$$u_0(x,y) := a(x - b(y)) + b(y), \tag{4.4}$$

and we show that, if λ is big enough, this provides a counter-example.

First ingredient We show that there exist a constant $C > 0$, and a function $a \in C^\infty(\mathbb{R})$ such that $a'(x) = 0$ if and only if $x \le 0$ or $x \ge C$, and

$$\int_0^C I(a'(x), a''(x))\,\mathrm{d}x > 0, \tag{4.5}$$

where

$$I(r,s) := \frac{s^2(2r^2 - 2r - 1)}{2(2r^2 - 2r + 1)^{3/2}(r^2 - r + 1)^2}.$$

Note that the integrand is negative near $x = 0$ and $x = C$.

In order to construct such a function, we first choose a function $f \in C^\infty(\mathbb{R})$ such that $f(x) > 2$ for every $x \in (0,1)$, and $f(x) = 2$ otherwise, and a function $g \in C^\infty(\mathbb{R})$ such that $g(x) = 0$ if and only if $x \le 0$, and $g(x) = 2$ for every $x \ge 1$. Given a positive integer n, we consider a function $a \in C^\infty(\mathbb{R})$ such that

$$a'(x) = \begin{cases} g(x) & \text{if } x \in [0,1], \\ f(x-i) & \text{if } x \in [i, i+1] \text{ for some } i = 1, 2, \ldots, n, \\ g(n+2-x) & \text{if } x \in [n+1, n+2], \\ 0 & \text{otherwise.} \end{cases}$$

With this choice we have that

$$\int_0^{n+2} I(a'(x), a''(x))\,\mathrm{d}x = 2\int_0^1 I(g(x), g'(x))\,\mathrm{d}x + n\int_0^1 I(f(x), f'(x))\,\mathrm{d}x.$$

The last integral is positive because $f(x) > 2$ in $(0,1)$. This means that if $C = n + 2$ is large enough, then (4.5) is satisfied.

Second ingredient Let $\lambda > 0$. It is not difficult to see that there exists a function $b \in C^\infty(\mathbb{R})$ such that $b(x) = 0$ for every $x \le 0$, $b(x) = \lambda + 2$ for every $x \ge \lambda + 2$, $b(x) = x$ for every $x \in [1, \lambda + 1]$, and $b'(x) > 0$ for every $x \in (0, \lambda + 2)$.

Conclusion Let Ω and u_0 be defined by (4.3) and (4.4), respectively. Let $u \in C^1([0,T]; H^1(\Omega)) \cap C^0([0,T]; H^3(\Omega))$ be a solution of the Perona–Malik equation in Ω, with Neumann boundary conditions on $\partial\Omega$, and initial datum u_0. Let $F(t)$ be defined as in (4.1).

It is easy to verify that $u_0 \in C^\infty(\Omega)$ and its normal derivative is identically zero in $\partial\Omega$, and moreover $\nabla u_0 \neq 0$ in Ω.

Applying Lemma 4.1, and integrating by parts, we therefore have that

$$F'(0) = \int_\Omega \frac{\nabla u_0}{|\nabla u_0|} \cdot \nabla u_t(x,y,0)\,\mathrm{d}x\,\mathrm{d}y = -\int_\Omega \operatorname{div}\left(\frac{\nabla u_0}{|\nabla u_0|}\right) u_t(x,y,0)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.6}$$

This integration by parts requires some justification. Indeed for the Perona–Malik equation we have that

$$u_t = \frac{u_{xx} + u_{yy} - u_x^2 u_{xx} - u_y^2 u_{yy} + u_y^2 u_{xx} + u_x^2 u_{yy} - 4u_x u_y u_{xy}}{(1 + u_x^2 + u_y^2)^2}, \tag{4.7}$$

and this is true also for $t = 0$ because of the regularity assumptions on u. Moreover

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{(u_x^2 + u_y^2)^{3/2}}. \tag{4.8}$$

With our choice of u_0, it is not difficult to see that (4.8) is unbounded near those points of $\partial\Omega$ where $|\nabla u_0| = 0$. However, in the same points we have that $u_t(x,y,0) = 0$, and the product turns out to be bounded, independently on λ (since a lot of terms are involved, this requires a lengthy, but elementary, calculation, which we leave to the interested reader).

In the same way, the vector function $v(x,y) := u_t(x,y,0)|\nabla u_0(x,y)|^{-1}\nabla u_0(x,y)$ can be continuously extended to $\overline{\Omega}$, and the scalar product between $v(x,y)$ and the exterior normal to $\partial\Omega$ turns out to be identically zero in $\partial\Omega$.

This justifies the integration by parts in (4.6).

Now we need to estimate the second integral in (4.6). Since u_{0x} and u_{0xx} are non-zero only when $0 \le x - b(y) \le C$, by (4.8) the integral reduces to

$$F'(0) = -\int_0^{\lambda+2} \mathrm{d}y \int_{b(y)}^{b(y)+C} \operatorname{div}\left(\frac{\nabla u_0(x,y)}{|\nabla u_0(x,y)|}\right) u_t(x,y,0)\,\mathrm{d}x.$$

Let us split the integration with respect to y in the three subintervals $[0,1]$, $[1, \lambda+1]$, and $[\lambda+1, \lambda+2]$.

Since the integrand is bounded, independently on λ, we have that the first and the third integral are constants, independent on λ, which we denote by c_1 and c_3, respectively. When $y \in [1, \lambda+1]$, we have that $b(y) \equiv y$, hence $u_0(x,y) = a(x-y) + y$. Computing the derivatives of u_0 in (4.8) and (4.7) in terms of the derivatives of a, and using the variable change $z = x - y$, we obtain that

$$-\int_1^{\lambda+1} dy \int_y^{y+C} \operatorname{div}\left(\frac{\nabla u_0}{|\nabla u_0|}\right) u_t(x,y,0)\, dx = \int_1^{\lambda+1} dy \int_0^C I(a'(z), a''(z))\, dz =: c_2\lambda.$$

In conclusion, we have that

$$F'(0) = c_1 + c_3 + c_2\lambda.$$

Since $c_2 > 0$ by (4.5), it follows that $F'(0) > 0$ when λ is large enough. □

Example 3 (proof of Theorem 2.18) Let $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and let $u_0(x,y) = f(r)$, where $r = (x^2+y^2)^{1/2}$, and $f \in C^\infty(\mathbb{R})$ is any function such that

- $f'(r) > 0$ for every $r \in (0,1)$, and $f'(r) = 0$ otherwise;
- $f'(r)$ has a unique maximum point r_0;
- $f'(r_0) > 1$, and $f''(r_0) = f'''(r_0) = 0$.

If u is a radial solution of the Perona–Malik equation in Ω, then u solves (3.20) and (3.21) with $\varphi(\sigma) = 2^{-1}\log(1+\sigma^2)$, and $u_r = |\nabla u|$.

The maximum of $|\nabla u_0|$ in $\overline{\Omega}$ is $f'(r_0) > 1$, hence (2.30) is satisfied.

If we prove that $u_{rt}(r_0, 0) < 0$, then (2.31) follows by a standard calculus argument. In order to compute this derivative, we differentiate (3.20) with respect to r in a neighborhood of $(r_0, 0)$: this can be done because (3.20) is a backward strictly parabolic equation in a cylinder of the form $(r_0 - \varepsilon, r_0 + \varepsilon) \times [0, \varepsilon)$, hence its solution is of class C^∞ in the cylinder, also for $t = 0$. So we obtain that

$$(u_r)_t = \varphi'''(u_r)u_{rr}^2 + \varphi''(u_r)u_{rrr} + \frac{\varphi''(u_r)}{r}u_{rr} - \frac{\varphi'(u_r)}{r^2}. \tag{4.9}$$

Now let us examine this expression for $t = 0$, and $r = r_0$. The first and the third summand are zero because $u_{rr}(r_0, 0) = f''(r_0) = 0$. The second summand is zero because $u_{rrr}(r_0,0) = f'''(r_0) = 0$. Therefore $u_{rt}(r_0, 0)$ is equal to the fourth summand, hence it is negative. □

Remark 4.3 Since no uniqueness result is known, it may happen that u_0 is radial, but the solution is not radial. Also in this case the computation of $\left(\sqrt{u_x^2 + u_y^2}\right)_t$ for $t = 0$ reduces to (4.9), and so the same argument works. We only need to assume that u is regular enough to compute this derivative.

References

1. Alberti, G., Müller, S.: A new approach to variational problems with multiple scales. Commun. Pure Appl. Math. **54**, 761–825 (2001)
2. Alvarez, L., Guichard, F., Lions, P.L., Morel, J.M.: Axioms and fundamental equations of image processing. Arch. Ration. Mech. Anal. **123**, 199–257 (1993)
3. Bellettini, G., Fusco, G., Guglielmi, N.: Numerical experiments and conjectures on the dynamics defined by some singularly perturbed non-convex functionals of the gradient (in preparation)

4. Bellettini, G., Novaga, M., Paolini, E.: The gradient flow of a non-convex functional in one dimension (preprint)
5. De Giorgi, E.: Su alcuni problemi instabili legati alla teoria della visione. In: Bruno, T., Buonocore, P., Carbone, L., Esposito, V. (eds). Atti del convegno in onore di Carlo Ciliberto (Napoli, 1995), La Città del Sole, Napoli, (1997), pp. 91–98
6. Esedoglu, S.: An analysis of the Perona–Malik scheme. Commun. Pure Appl. Math. **54**, 1442–1487 (2001)
7. Esedoglu, S.: Stability properties of Perona–Malik scheme (preprint)
8. Gobbino, M.: Gradient flow for the one-dimensional Mumford–Shah functional. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **27**, 145–193 (1998)
9. Gobbino, M.: Entire solutions of the one-dimensional Perona–Malik equation (preprint)
10. Höllig, K.: Existence of infinitely many solutions for a forward–backward heat equation. Trans. Am. Math. Soc. **278**, 299–316 (1983)
11. Kawohl, B.: From Mumford–Shah to Perona–Malik in image processing. Math. Methods Appl. Sci. **27**, 1803–1814 (2004)
12. Kawohl, B., Kutev, N.: Maximum and comparison principle for one-dimensional anisotropic diffusion. Math. Ann. **311**, 107–123 (1998)
13. Kichenassamy, S.: The Perona–Malik paradox. SIAM J. Appl. Math. **57**, 1328–1342 (1997)
14. Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and quasilinear equations of parabolic type, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, R.I. 1967
15. Perona, P., Malik, J.: Scale space and edge detection using anisotropic diffusion. IEEE Trans. Pattern Anal. Mach. Intell. **12**, 629–639 (1990)