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Local C^1 solutions to some non-linear PDE system

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SUMMARY

We prove that the quasilinear initial value problem

$$\begin{cases} U_t = \sum_{i=1}^m A_i(t,x,U)U_{x_i} + f(t,x,U) \\ U(0,x) = U_0(x) \in C^{1,b}(\mathbb{R}^m), \quad x \in \mathbb{R}^m \end{cases}$$

has a unique, local in time, C^1 solution, if the matrices A_i are diagonalizable and commute with each other.

KEY WORDS: hyperbolic systems; non-linear PDE systems; local C^1 solutions

1. INTRODUCTION

Let us denote by $C^{1,b}(A)$ the space of $u \in C^1(A)$ such that u, $\nabla u \in L^\infty(A)$. In all the cases we omit the target space, which can easily deduced from the context. Let us consider for $U = (u^j)_{j=1}^n$ the Cauchy problem

$$\begin{cases} U_t = \sum_{i=1}^m A_i(t,x,U)U_{x_i} + f(t,x,U) \\ U(0,x) = U_0(x) \in C^{1,b}(\mathbb{R}^m), \quad x \in \mathbb{R}^m \end{cases} \tag{1}$$

where A_i, $f \in C^{1,b}([0,T_0] \times \mathbb{R}^m \times B_{R,n})$ $\forall R > 0$ and $B_{R,n} := \{\alpha \in R^n : |\alpha| \leqslant R\}$. It was proved that (1) has a unique C^1 solution if $m = 1$ and $x \in [a,b]$ (see References [1,2]). In the general case, without supplementary assumptions on the matrices A_i we cannot expect the existence

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of C^1 solutions, even in the strictly hyperbolic symmetrizable case, unless we take the initial data in suitable Sobolev spaces.

The aim of this paper is to prove that (1) has a local C^1 solution in the case $m>1$ under suitable assumption on the matrices A_i. In detail we shall prove the following theorem.

Theorem 1.1
Let us assume that the matrices A_i are diagonalizable (i.e. they have a C^1 diagonalizer with C^1 inverse) and that they commute with each other. Moreover, let us suppose that $A_i,\ f \in C^{1,b}([0,T_0]\times\mathbb{R}^m\times B_{R,n})$, $\forall R>0$, $U_0\in C^{1,b}(\mathbb{R}^m)$.

Then (1) admits a unique local solution $U\in C^{1,b}([0,T]\times\mathbb{R}^m)$.

We plane to treat in a first time the classical linear case when A_i and f do not depend on U, and then we prove that the map $V\to U$, where U satisfies

$$U_t=\sum_{i=1}^{m}A_i(t,x,V(t,x))U_{x_i}+f(t,x,V(t,x)),\qquad U(0,x)=U_0(x),\qquad x\in\mathbb{R}^m$$

admits at least a fixed point in a suitable function's space, and that this fixed point is unique.

2. THE LINEAR CASE

Let us consider the problem

$$\begin{cases} U_t=\sum_{i=1}^{m}B_i(t,x)U_{x_i}+G(t,x)\\ U(0,x)=U_0(x)\in C^{1,b}(\mathbb{R}^m)\end{cases}\tag{2}$$

where $B_i,\ G\in C^{1,b}([0,T_0]\times\mathbb{R}^m)$. In addition, let us assume that the matrices B_i are diagonalizable and commute each other. Hence, B_i admit a joint diagonalizer (see Reference [3]) that we indicate by $K(t,x)\in C^{1,b}([0,T_0]\times\mathbb{R}^m)$. Moreover, let us set $D_i:=KB_iK^{-1}$, and denote the element of place jj of D_i by d^i_{jj}.

If we carry out the change of variables $V:=KU$ and set

$$L(t,x):=\left(K_t-\sum_{i=1}^{m}D_iK_{x_i}\right)K^{-1},\qquad g:=KG$$

we can rewrite (2) as

$$\begin{cases} V_t=\sum_{i=1}^{m}D_iV_{x_i}+LV+g\\ V(0,x)=K(0,x)U_0(x)=:V_0(x)\end{cases}\tag{3}$$

Under the considered hypotheses it is true.

Theorem 2.1
Problem (3) admits a global solution $V\in C^{1,b}([0,T_0]\times\mathbb{R}^m)$.

In a first time let us consider for $j=1,\dots,n$ the problems:

$$\begin{cases} \phi^j_t(t,s,x)=-(d^1_{jj},\dots,d^m_{jj})(t,\phi^j(t,s,x))\\ \phi^j(s,s,x)=x\end{cases}\tag{4}$$

It is well known that such problems admit a unique global C^1 solution (see Reference [4]). In the sequel we denote $e_j = (0, 0, \ldots, 1, \ldots, 0)$, and by $\phi^{j,i}$ the ith component of ϕ^j; moreover for a function $h(t,x)$ we use the notation

$$h^{\phi}(\tau,t,x) = h(\tau, \phi(\tau,t,x))$$

Therefore, for a function $V = (v^j)_j \in C^{1,b}([0,T[\times \mathbb{R}^m)$ we have the following lemma.

Lemma 2.2
V is a solution of (3) if verifies for $j = 1, \ldots, n$:

$$v^j(t,x) := v_0^j(\phi^j(0,t,x)) + \int_0^t e_j[(LV+g)(\tau, \phi^j(\tau,t,x))]\,d\tau \tag{5}$$

Proof (*Sketch*)
Since $e_j L^{\phi^j}(\tau,t,x) = e_j((\partial/\partial\tau)K^{\phi^j}(\tau,t,x))(K^{-1})^{\phi^j}(\tau,t,x)$ using integration by part, we have

$$\begin{aligned}
v_t^j(t,x) = \lim_{h\to 0}\frac{1}{h}(v^j(t+h,x) - v^j(t,x)) &= \sum_{i=1}^m v_{0,x_i}^j(\phi^j(0,t,x))\phi_s^{j,i}(0,t,x)\\
&+ e_j \int_0^t \sum_{i=1}^m \left[g_{x_i}^{\phi^j} + \frac{\partial}{\partial\tau}K^{\phi^j}(K^{-1}V)_{x_i}^{\phi^j}\right](\tau,t,x)\,\phi_s^{j,i}(\tau,t,x)\,d\tau\\
&+ e_j \sum_{i=1}^m [\phi_s^{j,i}(t,t,x)(K_{x_i}K^{-1}V)(t,x) - \phi_s^{j,i}(0,t,x)(K_{x_i}K^{-1}V)(0,\phi^j(0,t,x))]\\
&+ e_j L(t,x)V(t,x) + e_j g(t,x)\\
&- e_j \int_0^t \sum_{i=1}^m K_{x_i}(\tau,\phi^j(\tau,t,x))\phi_s^{j,i}(\tau,t,x)\frac{\partial}{\partial\tau}(K^{-1}V)^{\phi^j}(\tau,t,x)\,d\tau
\end{aligned}$$

and also

$$\begin{aligned}
v_{x_h}^j(t,x) = &\sum_{i=1}^m v_{0,x_i}^j(\phi^j(0,t,x))\phi_{x_h}^{j,i}(0,t,x)\\
&+ e_j \int_0^t \sum_{i=1}^m \left[g_{x_i}^{\phi^j} + \frac{\partial}{\partial\tau}K^{\phi^j}(K^{-1}V)_{x_i}^{\phi^j}\right](\tau,t,x)\phi_{x_h}^{j,i}(\tau,t,x)\,d\tau\\
&+ e_j \sum_{i=1}^m [\phi_{x_h}^{j,i}(t,t,x)(K_{x_i}K^{-1}V)(t,x) - \phi_{x_h}^{j,i}(0,t,x)(K_{x_i}K^{-1}V)(0,\phi^j(0,t,x))]\\
&- e_j \left(\int_0^t \sum_{i=1}^m K_{x_i}(\tau,\phi^j(\tau,t,x))\phi_{x_h}^{j,i}(\tau,t,x)\frac{\partial}{\partial\tau}(K^{-1}V)^{\phi^j}(\tau,t,x)\,d\tau\right)
\end{aligned} \tag{6}$$

To conclude the proof it is enough to remark that $\forall i,j$ one has

$$\phi_s^{j,i}(\tau,t,x) - \sum_{h=1}^m d_{jj}^h(t,x)\phi_{x_h}^{j,i}(\tau,t,x) = 0 \quad \forall \tau,t,x$$

□

To show that (3) admits a unique solution local in t, it is possible for example to prove with standard techniques that the map $F: C^{1,b}_{V_0,T} \to C^{1,b}_{V_0,T}$ definite by

$$(F(V))^j(t,x) := v_0^j(\phi^j(0,t,x)) + \int_0^t e_j[(LV+g)(\tau,\phi^j(\tau,t,x))]\,d\tau$$

is a contraction for some T, where $C^{1,b}_{V_0,T} := \{V \in C^{1,b}([0,T],\mathbb{R}^n) : V(0,x) = V_0(x)\}$. With classical techniques one can then prove that this solution is global in time.

3. THE NON-LINEAR PROBLEM

For sake of simplicity we treat the case in which $A_i(t,x,U) = A_i(U)$, $f(t,x,U) = f(U)$. We use the notations introduced in Section 2, only we recall that $K(\alpha)$ denote the joint diagonalizer of $A_i(\alpha)$ and $D_i(\alpha)$ are the respective diagonal matrices.

3.1. Preliminaries

3.1.1. Definitions—Notations. For a matrix A we denote by A^{T} the transpose matrix, moreover $|\cdot|$ denotes the usual norm for vector and matrices, $\|\cdot\|$ is the usual C^1 norm and $\|\cdot\|_\infty$ is the L^∞ norm. Let us set $|z(t,x)|_1 := |z(t,x)| + |\nabla z(t,x)|$. We said that $f \in C^0(A)$ has *modulo* of uniform continuity $\geqslant \delta(\varepsilon)$ on A if: for all $\varepsilon > 0$, $x, y \in A$ and $|x-y| \leqslant \delta(\varepsilon)$ imply $|f(x) - f(y)| \leqslant \varepsilon$.
Definitions of σ_0, C, T. Let us consider $\sigma_0 \geqslant 1$ such that

$$\max_{|\alpha| \leqslant 2\|U_0\|} \max_{i \leqslant m}((|K|_1 + |K^{-1}|_1 + |A_i|_1 + |D_i|_1 + |f|_1)(\alpha)) \leqslant \sigma_0 \tag{7}$$

moreover, let us put

$$C := 32(m+1)^3\sigma_0^3 n^2(1 + \|U_0\|^2), \quad T = \min\left\{T_0, \frac{\|U_0\|}{C+\sigma_0}, \frac{1}{(1+m\sigma_0)^5 C^2(C+1)}\right\}$$

Definitions of N_R, L_R, $\delta_{R,0}$, ε_0. Let us set

$$N_R := \{(x,t) \in [0,T] \times \mathbb{R}^m : m\sigma_0 t \leqslant R - |x|\}$$
$$L_R := \{(\tau,t,x) : (x,t) \in N_R,\ 0 \leqslant \tau \leqslant t\}$$

Let us consider $W \in C^1([0,T] \times \mathbb{R}^m)$ with $\|\nabla W\|_\infty \leqslant C$. Let us set $A_{i,w} := A_i(W)$, $f_w := f(W)$, $K_w := K(W)$, $D_{i,w} := D_i(W)$ and let ϕ_w^j be the corresponding solutions of (4). Let us fix $\delta_{R,0}(\varepsilon) \leqslant$ of the modulus of uniform continuity relative to ε on L_R of the following functions ($i \leqslant m, j \leqslant n$):

$$K_w(t,x),\quad K_w^{-1}(t,x),\quad K_w^{\phi_w^j}(\tau,t,x),\quad (\nabla_\alpha K)_w^{\phi_w^j}(\tau,t,x),\quad f_w(t,x),\quad A_{i,w}(t,x)$$

$$(K_{U_0}U_0)_{x_i}(\phi_w^j(0,t,x)),\quad U_0(\phi_w^j(\tau,t,x)),\quad D_{i,w}^{\phi_w^j}(\tau,t,x),\quad 6\sigma_0^2 n^2 m^2(1+C)Ct$$

$$(\nabla_\alpha D_i)_w^{\phi_w^j}(\tau,t,x),\quad f_w^{\phi_w^j}(\tau,t,x),\quad \nabla_\alpha(f_w)(\tau,\phi_w^j(\tau,t,x)),\quad \nabla U_0(\phi_w^j(\tau,t,x))$$

Let us remark that $\delta_{R,0}(\varepsilon)$ do not depend on W, since the functions W are equilipschitz continuous (by $\|\nabla W\|_\infty \leqslant C$), and then the same hold true for ϕ_w^j.

Let us set $\delta_R(\varepsilon)=\delta_{R,0}(\varepsilon_0)$ where $\varepsilon_0=\varepsilon(16mC)^{-1}$.

The space X and the Map F. If $W\in C^{1,b}([0,T]\times\mathbb{R}^m)$, we say that $W\in C_{\delta_R}$ if $\forall R>0$, ∇W has modulus of uniform continuity $\geqslant \delta_R$ on N_R. We can now define

$$X:=\{W\in C^{1,b}([0,T]\times\mathbb{R}^m): W(0,x)=U_0(x),\ \ \|\nabla W\|_\infty\leqslant C,\ \ W\in C_{\delta_R}\}$$

Problem (2) with $B_i=A_{i,w}$, $G=f_w$ admits a unique global solution $U=K_w^{-1}V$, where V is the corresponding solution of (3). Let $F:W\to U$, we shall prove that $F:X\to X$ and that F is a continuous function. Since X is a convex compact set with the C^1 topology, then F has at least a fixed point. Next we prove that, since $A_i,\ f\in C^1(\mathbb{R}^n)$, such a solution is unique.

3.1.2. Preliminar estimates. Since $W(0,x)=U_0$ we have $\|W\|_\infty\leqslant 2\|U_0\|$, hence by (7)

$$\max_{(t,x)\in[0,T]\times\mathbb{R}^m}\ \max_{i\leqslant m}((|K|_1+|K^{-1}|_1+|A_i|_1+|D_i|_1+|f|_1)(W(t,x)))\leqslant\sigma_0$$

Let us set $H_{j,w}^{\mathrm{T}}:=-(d^i_{jj,w})_{i=1}^m$. Using

$$\phi^j_{w,x_i}(t,s,x)=e_i+\int_s^t \phi^j_{w,x_i}(\tau,s,x)[\nabla_x H_{j,w}]^{\mathrm{T}}(\tau,\phi^j_w(\tau,s,x))\,\mathrm{d}\tau$$

we get:

$$\|\phi^j_{w,x_i}\|_\infty\leqslant \mathrm{e}^{nm\sigma_0 CT}\leqslant 3 \tag{8}$$

and in the same way

$$\|\phi^j_{w,s}\|_\infty\leqslant m\sigma_0\mathrm{e}^{nm\sigma_0 CT}\leqslant 3m\sigma_0,\quad \|\phi^j_{w,t}\|_\infty\leqslant m\sigma_0 \tag{9}$$

Since $K_{w,x_i}=\sum_{j=1}^n K_{\alpha_j}W^j_{x_i}$ (the same is true for $K_{w,t}$), $K_w(0,x)=K_{U_0}(0,x)$, $g_{w,x_i}=K_{w,x_i}f_w+K_w f_{w,x_i}$, and $(\partial/\partial\tau)K_w^{\phi^j_w}=(K_{w,t}-\sum_{i=1}^m d^i_{jj,w}K_{w,x_i})^{\phi^j_w}$, we obtain

$$\|K_{w,x_i}\|_\infty, \|K_{w,t}\|_\infty\leqslant n\sigma_0 C,\quad \|K_{w,x_i}(0,x)\|_\infty\leqslant n\sigma_0\|U_0\| \tag{10}$$

$$\|g_{w,x_i}\|_\infty\leqslant 2n\sigma_0^2 C,\quad \left\|\frac{\partial}{\partial\tau}K_w^{\phi^j_w}\right\|_\infty\leqslant(1+m\sigma_0)n\sigma_0 C\leqslant 2mn\sigma_0^2 C \tag{11}$$

Let us set

$$z_{1,h}=\left[\sum_{i=1}^m (K_w(0,\cdot)U_0)^j_{x_i}(\phi^j_w(0,t,x))\phi^{j,i}_{w,x_h}(0,t,x)\right]_j^{\mathrm{T}}$$

$$z_{2,h}=\left[\sum_{i=1}^m\int_0^t e_j\left[g^{\phi^j_w}_{w,x_i}+\frac{\partial}{\partial\tau}K_w^{\phi^j_w}U_{x_i}^{\phi^j_w}\right](\tau,t,x)\phi^{j,i}_{w,x_h}(\tau,t,x)\,\mathrm{d}\tau\right]_j^{\mathrm{T}}$$

$$z_{3,h}=\left[e_j\sum_{i=1}^m\sum_{\sigma=1}^n(K_{w,\alpha_\sigma}W^\sigma_{x_i})(0,\phi^j_w(0,t,x))\phi^{j,i}_{w,x_h}(0,t,x)U_0(\phi^j_w(0,t,x))\right]_j^{\mathrm{T}}$$

$$z_{4,h}=\left[e_j\int_0^t\sum_{i=1}^m\left(K^{\phi^j_w}_{w,x_i}\frac{\partial}{\partial\tau}U^{\phi^j_w}\right)(\tau,t,x)\phi^{j,i}_{w,x_h}(\tau,t,x)\right]_j^{\mathrm{T}}$$

Using (8)–(11), we obtain

$$\|z_{1,h}\|_\infty \leqslant 3mn[n\sigma_0\|U_0\|^2 + \sigma_0\|U_0\|] \tag{12}$$

$$\|z_{2,h}\|_\infty \leqslant 3nmT[2\sigma_0^2 nC + n\sigma_0 C(1+m\sigma_0)\|\nabla U\|_\infty] \tag{13}$$

$$\|z_{3,h}\|_\infty \leqslant 3n^2 m\sigma_0\|U_0\|^2 \tag{14}$$

$$\|z_{4,h}\|_\infty \leqslant 3n^2 mT\sigma_0 C(1+m\sigma_0)\|\nabla U\|_\infty \tag{15}$$

3.2. *Equiboundedness of* ∇U

Since $K_w^{-1}V = U$ we have $U^j_{x_i} = e_j[K^{-1}_{w,x_i}V + K_w^{-1}V_{x_i}]$ where, by (6):

$$V_{x_h}(t,x) = z_{1,h} + K_{w,x_h}U(t,x) + z_{2,h} - z_{3,h} - z_{4,h} \tag{16}$$

Since $K_w^{-1}K_{w,x_h} = -K^{-1}_{w,x_h}K_w$ we get

$$U_{x_h} = K_w^{-1}(z_{1,h} + z_{2,h} - z_{3,h} - z_{4,h}), \quad h = 1,\ldots,m$$

By (12)–(15), thanks to the conditions on T and C we have

$$\begin{aligned}\|\nabla_x U\|_\infty &\leqslant 3m^2 n\sigma_0^2[2n\|U_0\|^2 + \|U_0\|] \\ &\quad + 6m^2n^2\sigma_0^3 CT + 6m^2n^2\sigma_0^2 CT(1+m\sigma_0)\|\nabla U\|_\infty \\ &\leqslant \frac{3C}{8(m+1)\sigma_0} + \frac{C}{16(1+m\sigma_0)} + \frac{1}{8(1+m\sigma_0)}\|\nabla U\|_\infty\end{aligned}$$

therefore by (2), since $\sigma_0 \leqslant C/16$ we obtain

$$\|\nabla U\|_\infty \leqslant \sigma_0 + (1+m\sigma_0)\|\nabla_x U\|_\infty \leqslant \frac{C}{2} + \frac{\|\nabla U\|_\infty}{8}$$

Hence we get $\|\nabla U\|_\infty \leqslant C$.

3.3. *Equicontinuity of* ∇U

In this section we fix $\varepsilon > 0$, $R > 0$ and we assume that (t,x), $(s,y) \in N_R$ with $|(t,x)-(s,y)| \leqslant \delta_R(\varepsilon)$ $(= \delta_{R,0}(\varepsilon_0))$ and $t \leqslant s$.

Preliminar estimates. First let us remark that for $j = 1,\ldots,n$

$$(t,x) \in N_R \Longrightarrow (\tau, \phi^j_w(\tau,t,x)) \in N_R \quad \forall 0 \leqslant \tau \leqslant t \tag{17}$$

Indeed hold true $m\sigma_0\tau = m\sigma_0 t - m\sigma_0(t-\tau) \leqslant R - |x| - m\sigma_0(t-\tau)$ and

$$|\phi^j_w(\tau,t,x)| \leqslant |x| + \int_\tau^t |H_{j,w}(r,\phi^j_w(r,t,x))|\,\mathrm{d}r \leqslant |x| + m\sigma_0(t-\tau)$$

By (8)–(9) we have, $\forall \tau \geqslant 0$:

$$|(t,x)-(s,y)| \leqslant \delta \Longrightarrow |\phi^j_w(\tau,t,x) - \phi^j_w(\tau,s,y)| \leqslant \delta 6m\sigma_0 =: \delta M$$

Let us divide the segment with end points (t,x) and (s,y) in $[M]+1$ equal parts with extremes $(t_\theta,x_\theta)\in N_R$ (N_R is convex). Let $\tau\leqslant t$, then we get ($\tau\leqslant t_\theta\ \forall\theta$):

$$|\phi_w^j(\tau,t_\theta,x_\theta)-\phi_w^j(\tau,t_{\theta+1},x_{\theta+1})|\leqslant\delta$$

As $\nabla_x W$ are uniformly equicontinuous in N_R, if $\delta=\delta_R$ we deduce that

$$|\nabla_x W(\tau,\phi_w^j(\tau,t,x))-\nabla_x W(\tau,\phi_w^j(\tau,s,y))|\leqslant 2M\varepsilon,\quad j=1,\dots,n \tag{18}$$

Now let us estimate

$$\gamma(r):=\max_{h\leqslant m,j\leqslant n}|\phi_{w,x_h}^j(r,t,x)-\phi_{w,x_h}^j(r,s,y)|(:=\gamma_{j,h}(r))$$

Let $r\leqslant t\leqslant s$, using (18) we can estimate

$$\begin{aligned}\gamma_{j,h}(r) &\leqslant 3mn\sigma_0 C|t-s|+mn\sigma_0 C\int_r^t\gamma(\tau)\,\mathrm{d}\tau\\ &\quad+3\int_r^t\sum_{\sigma=1}^n|(H_{j,w})_{\alpha_\sigma}\nabla_x W^\sigma(\tau,\phi_w^j(\tau,t,x))-(H_{j,w})_{\alpha_\sigma}\nabla_x W^\sigma(\tau,\phi_w^j(\tau,s,y))|\,\mathrm{d}\tau\\ &\leqslant \varepsilon_0+nm\sigma_0 C\int_r^t\gamma(\tau)\,\mathrm{d}\tau+3nT\varepsilon_0 C\\ &\quad+3nT\sigma_0\sup_{\sigma\leqslant n,r\leqslant\tau\leqslant t}|\nabla_x W^\sigma(\tau,\phi_w^j(\tau,t,x))-\nabla_x W^\sigma(\tau,\phi_w^j(\tau,s,y))|\\ &\leqslant \varepsilon_0+nm\sigma_0 CT\sup_{0\leqslant r\leqslant t}\gamma(r)+3nT[\varepsilon_0 C+12\sigma_0^2 m\varepsilon]\end{aligned}$$

Hence we have:

$$\sup_{0\leqslant r\leqslant t}\gamma(r)\leqslant 4\varepsilon_0+72\sigma_0^2 mnT\varepsilon$$

We remark that as in (18), if $\tau\leqslant t$

$$|K_{w,x_i}^{\phi_w^j}(\tau,t,x)-K_{w,x_i}^{\phi_w^j}(\tau,s,y)|\leqslant nC\varepsilon_0+12\sigma_0^2 nm\varepsilon$$

the same valuation is true for $K_{w,t}^{\phi_w^j}$, and by (4) and the definition of $\delta_{R,0}$ we get

$$\left|\frac{\partial}{\partial\tau}K_w^{\phi_w^j}(\tau,t,x)-\frac{\partial}{\partial\tau}K_w^{\phi_w^j}(\tau,s,y)\right|\leqslant nC\varepsilon_0(1+2\sigma_0 m)+12n\sigma_0^2 m\varepsilon(1+m\sigma_0)$$

Since $g_{w,x_i}=K_{w,x_i}f_w+K_w f_{w,x_i}$, using (18) we get for $\tau\leqslant t\leqslant s$

$$|g_{w,x_i}(\tau,\phi_w^j(\tau,t,x))-g_{w,x_i}(\tau,\phi_w^j(\tau,s,y))|\leqslant 4n\sigma_0 C\varepsilon_0+24nm\sigma_0^3\varepsilon$$

Estimates for quantities z_i. We are now able to estimate

$$\begin{aligned}|z_{1,h}(t,x)-z_{1,h}(s,y)| &\leqslant 3mn\varepsilon_0+mn[n\sigma_0\|U_0\|^2+\sigma_0\|U_0\|]\gamma(0)\\ &\leqslant mn\varepsilon_0[3+4\sigma_0[n\|U_0\|^2+\|U_0\|]]+72\varepsilon Tm^2n^2\sigma_0^3[n\|U_0\|^2+\|U_0\|]\\ &\leqslant\frac{C\varepsilon_0}{8\sigma_0^2}+3\varepsilon nTC\end{aligned}\tag{19}$$

Let us put

$$D_{R,\varepsilon}h := \sup\{|h(t,x) - h(s,y)| : (t,x),\ (s,y) \in N_R,\ |(t,x) - (s,y)| \leqslant \delta_R(\varepsilon)\}$$

Now we can estimate, as for (18)

$$\begin{aligned}
|z_{2,h}(t,x) - z_{2,h}(s,y)| &\leqslant 6mn^2|t-s|\sigma_0^2 C(1+mC) + 3Tnm\{4n\sigma_0 C\varepsilon_0 + 24mn\sigma_0^3\varepsilon\} \\
&\quad + 3TnmC\{nC\varepsilon_0(1+2\sigma_0 m) + 12n\sigma_0^2 m\varepsilon(1+m\sigma_0)\} \\
&\quad + 6Tn^2m^2C\sigma_0^2 \sup_{0\leqslant\tau\leqslant t\, j\leqslant n} |\nabla_x U(\tau,\phi_w^j(\tau,t,x)) - \nabla_x U(\tau,\phi_w^j(\tau,s,y))| \\
&\quad + 2Tn^2m\sigma_0^2 C(1+mC)\{4\varepsilon_0 + 72\sigma_0^2 mnT\varepsilon\} \\
&\leqslant \left(1+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)\varepsilon_0 + 2CT\varepsilon + 3\varepsilon TC^2 \\
&\quad + \varepsilon nT + 72Tn^2m^3\sigma_0^3 C D_{R,\varepsilon}\nabla_x U \\
&\leqslant 2\varepsilon_0 + 3CT(1+C)\varepsilon + \frac{D_{R,\varepsilon}\nabla_x U}{(1+m\sigma_0)^5} \\
&\leqslant 2\varepsilon_0 + \frac{\varepsilon}{32(1+m\sigma_0)^4} + \frac{D_{R,\varepsilon}\nabla_x U}{2(1+m\sigma_0)^4}
\end{aligned} \tag{20}$$

Besides, we get

$$\begin{aligned}
|z_{3,h}(t,x) - z_{3,h}(s,y)| &\leqslant 3mn^2\varepsilon_0(\|U_0\|^2 + 2\sigma_0\|U_0\|) + n^2m\|U_0\|^2\sigma_0[4\varepsilon_0 + 72nT\varepsilon\sigma_0^2 m] \\
&\leqslant 14mn^2\sigma_0(\|U_0\|^2+1)\varepsilon_0 + \frac{\varepsilon}{4(1+m\sigma_0)^5} \leqslant \frac{C\varepsilon_0}{8\sigma_0^2} + \frac{\varepsilon}{8(1+m\sigma_0)^4}
\end{aligned} \tag{21}$$

Let us now calculate

$$\begin{aligned}
|z_{4,h}(t,x) - z_{4,h}(s,y)| &\leqslant 3mn^2|t-s|\sigma_0 C^2(1+m\sigma_0) \\
&\quad + Tn^2m\sigma_0 C^2(1+m\sigma_0)[4\varepsilon_0 + 72\sigma_0^2 mnT\varepsilon] \\
&\quad + 3TmnC(1+m\sigma_0)(nC\varepsilon_0 + 12n\sigma_0^2\varepsilon m) \\
&\quad + 3Tmn^2\sigma_0 C[12(1+m\sigma_0)m\sigma_0 D_{R,\varepsilon}\nabla U + mC\varepsilon_0] \\
&\leqslant 2\varepsilon_0 + nT\varepsilon + 36Tm^2n^2C(1+m\sigma_0)\sigma_0^2\varepsilon + \frac{D_{R,\varepsilon}\nabla U}{2(1+m\sigma_0)^4\sigma_0} \\
&\leqslant 2\varepsilon_0 + \frac{\varepsilon}{32(1+m\sigma_0)^4} + \frac{D_{R,\varepsilon}\nabla U}{2(1+m\sigma_0)^4\sigma_0}
\end{aligned} \tag{22}$$

Estimate of ∇U. By (19)–(22) we deduce, using (12)–(15):

$$|U_{x_h}(t,x) - U_{x_h}(s,y)| \leqslant \varepsilon_0 \sum_{j=1}^{4} \|z_{j,h}\| + \sigma_0 \sum_{j=1}^{4} |z_{j,h}(t,x) - z_{j,h}(s,y)|$$

$$\leqslant \varepsilon_0 \frac{C}{\sigma_0} + \frac{C\varepsilon_0}{8\sigma_0} + 3\varepsilon TnC\sigma_0 + 2\varepsilon_0\sigma_0 + \frac{\varepsilon}{16(1+m\sigma_0)^3}$$

$$+ \frac{C\varepsilon_0}{8\sigma_0} + \frac{\varepsilon}{8(1+m\sigma_0)^3} + 2\varepsilon_0\sigma_0 + \frac{\varepsilon}{32(1+m\sigma_0)^3} + \frac{D_{R,\varepsilon}\nabla U}{(1+m\sigma_0)^3}$$

$$\leqslant \varepsilon_0 \left[\frac{5C}{4\sigma_0} + 4\sigma_0\right] + \frac{D_{R,\varepsilon}\nabla U}{(1+m\sigma_0)^3} + \varepsilon \left[\frac{3}{32(1+m\sigma_0)^5} + \frac{3}{16(1+m\sigma_0)^3}\right]$$

Let us moreover note that, since U is a solution of (2), we obtain

$$|U_t(t,x) - U_t(s,y)| \leqslant (mC+1)\varepsilon_0 + \sigma_0 \sum_{i=1}^{m} |U_{x_i}(t,x) - U_{x_i}(s,y)|$$

It follows that

$$D_{R,\varepsilon}\nabla U \leqslant (mC+1)\varepsilon_0 + 4Cm\varepsilon_0 + \varepsilon \left[\frac{3}{16(1+m\sigma_0)^4} + \frac{3}{8(1+m\sigma_0)^2}\right]$$

$$+ \frac{2D_{R,\varepsilon}\nabla U}{(1+m\sigma_0)^2} \leqslant 6Cm\varepsilon_0 + \frac{\varepsilon}{8} + \frac{D_{R,\varepsilon}\nabla U}{2}$$

hence $D_{R,\varepsilon}\nabla U \leqslant 12mC\varepsilon_0 + (\varepsilon/4) \leqslant \varepsilon$.

3.4. Continuity of the map

Let us assume that $W_p \to W$ if $p \to +\infty$ in X with respect to the C^1 convergence, uniformly on the compacts N_R. We shall prove that $\lim_{p\to+\infty} F(W_p) = F(W)$.

For a function h_w let us denote $S_p h := h_{w_p} - h_w$.

Since the matrices A_i, $K \in C^1(\mathbb{R}^n)$ we get, with respect to the C^1 convergence:

$$\lim_{p\to+\infty} S_p A_i,\ S_p K,\ S_p K^{-1},\ S_p D_i := 0$$

We prove that (L_R is defined in Section 3.1.1)

$$\lim_{p\to+\infty} \sup_{(t,s,x)\,\in\, L_R} |S_p\phi^j(t,s,x)|_1 = 0$$

Indeed we have by (17):

$$|S_p\phi^j(t,s,x)| \leqslant mn\sigma_0 C \int_t^s |S_p\phi^j(\tau,s,x)|\,\mathrm{d}\tau + |t-s| \sup_{(t_0,x_0)\in N_R} |S_p H_j(t_0,x_0)|$$

then

$$|S_p\phi^j(t,s,x)| \leqslant T\mathrm{e}^{mn\sigma_0 CT} \sup_{(t_0,x_0)\in N_R} |S_p H_j(t_0,x_0)|$$

By this we obtain the C° convergence. By (4), $\phi^j_{w_p,t}$ converges uniformly to $\phi^j_{w,t}$ in the compacts L_R. In addition, $(\nabla_x H_{j,w_p})(\tau,\phi^j_{w_p}(\tau,s,x))$ converge in C° to $(\nabla_x H_{j,w})(\tau,\phi^j_w(\tau,s,x))$, hence in L_R we have

$$|S_p\phi^j_s(t,s,x)| \leqslant 3m\sigma_0 \int_t^s |(\nabla_x H^{\phi^j_{w_p}}_{j,w_p} - \nabla_x H^{\phi^j_w}_{j,w})(\tau,s,x))|\,\mathrm{d}\tau$$

$$+ |S_p H_j(s,x)| + mnC\sigma_0 \int_t^s |S_p\phi^j_s(\tau,s,x)|\,\mathrm{d}\tau$$

hence

$$|S_p\phi^j_s(t,s,x)| \leqslant \left(\sup_{(s,x)\in N_R} |S_p H_j(s,x)|\right) \mathrm{e}^{mnC\sigma_0 T}$$

$$+ \left(3Tm\sigma_0 \sup_{(\tau,s,x)\in L_R} |(\nabla_x H^{\phi^j_{w_p}}_{j,w_p} - \nabla_x H^{\phi^j_w}_{j,w})(\tau,s,x))|\right) \mathrm{e}^{mnC\sigma_0 T}$$

In the same way we can estimate $|S_p\phi^j_{x_h}(t,s,x)|$. Then we have the required convergence.

Let us put for a function $q_w(t,x)$:

$$q_{p,j}(\tau,t,x) := q_{w_p}(\tau,\phi^j_{w_p}(\tau,t,x)) - q_w(\tau,\phi^j_w(\tau,t,x))$$

$$q_{p,\rho,j}(\tau,t,x) := q_{w_p}(\tau,\phi^j_{w_p}(\tau,t,x))\rho_{w_p}(\tau,t,x) - q_w(\tau,\phi^j_w(\tau,t,x))\rho_w(\tau,t,x)$$

Let us consider

$$\beta_p := \sup_{(\tau,t,x)\in L_R} |[e_j((KU_0)_{p,j}(\tau,t,x))]_j|, \quad \psi_p := \sup_{(\tau,t,x)\in L_R} |[e_j(g_{p,j}(\tau,t,x))]_j|$$

$$\chi_p := \sup_{(\tau,t,x)\in L_R} \left|\left[e_j\left(\frac{\partial}{\partial\tau}K_{p,j}(\tau,t,x)\right)\right]_j\right|, \quad \mu_p := \sup_{j\leqslant n,\, h\leqslant m,\, (\tau,t,x)\in L_R} |S_p\phi^j_{x_h}(\tau,t,x)|$$

$$\lambda_p := \sup_{h\leqslant m,\, (\tau,t,x)\in L_R} \sum_{i=1}^m |[e_j((g_{x_i})_{p,\phi^{j,i}_{x_h},j}(\tau,t,x))]_j|, \quad \kappa_p := \sup_{(t,x)\in N_R} |S_p K^{-1}(t,x)|$$

$$\theta_p := \sup_{h\leqslant m,\, (\tau,t,x)\in L_R} \sum_{i=1}^m |[e_j((K_{x_i})_{p,\phi^{j,i}_{x_h},j}(\tau,t,x))]_j|$$

By the convergence of the functions $\phi^j_{w_p}(\tau,s,x)$ we have

$$\lim_{p\to+\infty} \alpha_p := \beta_p + \psi_p + \chi_p + \lambda_p + \theta_p + \kappa_p + \mu_p = 0$$

Since $U_{w_p} = K^{-1}_{w_p} V_{w_p}$ and $\|U_{w_p}\|_\infty \leqslant \|U_0\| + CT$, using the definition of T we get

$$\|V_{w_p}\|_\infty \leqslant 2\sigma_0 \|U_0\|$$

hence by the Lipschitz continuity of U_{w_p}, we have

$$|S_pU(t,x)| \leqslant 2\sigma_0\|U_0\|\kappa_p + \sigma_0(\beta_p + T\psi_p + 2T\|U_0\|\chi_p) + 2\sigma_0^3TC^2nm \sup_{(\tau,t,x)\in L_R} \sum_{j=1}^n |S_p\phi^j(\tau,t,x)| + 2\sigma_0^3TCnm \sup_{(t_0,x_0)\in N_R} |S_pU(t_0,x_0)|$$

$$\leqslant \sigma_0(2\|U_0\|+1)\alpha_p + \sup_{(\tau,t,x)\in L_R} \sum_{j=1}^n |(S_p\phi^j(\tau,t,x)| + \frac{1}{16}\sup_{(t_0,x_0)\in N_R} |S_pU(t_0,x_0)|$$

hence

$$\sup_{(t_0,x_0)\in N_R} |(U_{w_p} - U_w)(t_0,x_0)| \leqslant 2\sigma_0(2\|U_0\|+1)\alpha_p + 2\sup_{(\tau,t,x)\in L_R} \sum_{j=1}^n |S_p\phi^j(\tau,t,x)| \to_{p\to+\infty} 0$$

Now, let us estimate in N_R

$$|(U_{w_p,x_h} - U_{w,x_h})(t,x)| \leqslant \kappa_p \sum_{i=1}^4 \|z_{i,h}\|_\infty + \sigma_0 \sum_{i=1}^4 |(z_{i,h,w_p} - z_{i,h,w})(t,x)|$$

Let us remark that thanks to the convergence of the $\phi^j_{w_p}(\tau,t,x)$ we obtain

$$\lim_{p\to+\infty} \gamma_p := \sup_{h\leqslant m,\,(x,t)\in N_R} \sum_{i=1,3} |(z_{i,h,w_p} - z_{i,h,w})(t,x)| = 0$$

Moreover, we get

$$|S_pU_{x_h}(t,x)| \leqslant \frac{C}{\sigma_0}\kappa_p + \sigma_0(\gamma_p + T\lambda_p) + 2T\sigma_0^3C^2nm^2\mu_p + 3T\sigma_0Cm\chi_p + 6\sigma_0^3Cm^2n \int_0^t \sup_{i=1,\dots,m} |[e_j(U_{x_i})_{p,j}(\tau,t,x)\,\mathrm{d}\tau]_j| + \sigma_0C(1+m\sigma_0)T\theta_p + 3\sigma_0^2Cmn \int_0^t \left|\left[e_j\left(\frac{\mathrm{d}}{\mathrm{d}\tau}U_{p,j}\right)(\tau,t,x)\,\mathrm{d}\tau\right]_j\right|$$

$$\leqslant C\alpha_p + \sigma_0\gamma_p + 9Cm^2n\sigma_0^3 \int_0^t \sup_{i=1,\dots,m} |[e_jS_pU_{x_i}(\tau,\phi^j_{w_p}(\tau,t,x))\,\mathrm{d}\tau]_j|$$

$$+ 9Cm^2n\sigma_0^3 \int_0^t \sup_{i=1,\dots,m} |[e_j((U_{w,x_i})^{\phi^j_{w_p}}(\tau,t,x) - (U_{w,x_i})^{\phi^j_w}(\tau,t,x))\,\mathrm{d}\tau]_j|$$

$$+ 3mn\sigma_0^2C \int_0^t |[e_jS_pU_t(\tau,\phi^j_{w_p}(\tau,t,x))\,\mathrm{d}\tau]_j|$$

$$+ 3mn\sigma_0^2C \int_0^t |[e_j((U_{w,t})^{\phi^j_{w_p}}(\tau,t,x) - (U_{w,t})^{\phi^j_w}(\tau,t,x))\,\mathrm{d}\tau]_j|$$

$$+ 3m^2n\sigma_0^2C^2 \int_0^t \sup_{i=1,\dots,m} |[e_j(D_i)_{p,j}(\tau,t,x)\,\mathrm{d}\tau]_j|$$

Let us observe that, since U_w is a C^1 function:

$$\lim_{p\to+\infty} \xi_p := \int_0^t \sup_{i=1,\dots,m} |[e_j(U_{w,x_i})^{\phi^j_{w_p}}(\tau,t,x) - (U_{w,x_i})^{\phi^j_w}(\tau,t,x)\,\mathrm{d}\tau]_j|$$

$$+ \int_0^t |[e_j(U_{w,t})^{\phi^j_{w_p}}(\tau,t,x) - (U_{w,t})^{\phi^j_w}(\tau,t,x)\,\mathrm{d}\tau]_j| := 0$$

In addition, one has

$$\sup_{(t,x)\in N_R} |(U_{w_p,x_h} - U_{w,x_h})(t,x)| \leqslant C\alpha_p + \sigma_0\gamma_p + 9Cmn\sigma_0^2(1+\sigma_0 m)\xi_p$$

$$+ \frac{1}{8(1+m\sigma_0)^5} \sup_{i\leqslant m,\,(t,x)\in N_R} |S_p U_{x_i}(t,x)|$$

$$+ \sup_{i\leqslant m,\,j\leqslant n,\,(\tau,t,x)\in L_R} |(D_i)_{p,j}(\tau,t,x)|$$

$$+ \frac{1}{16(1+m\sigma_0)^5} \sup_{(t,x)\in N_R} |S_p U_t(t,x)|$$

Hence, recalling the form of $U_{w_p,t}$ in (2), we have

$$\sup_{(t,x)\in N_R} |(\nabla U_{w_p} - \nabla U_w)(t,x)| \leqslant Cm \sup_{i=1,\dots,m,\,(t,x)\in N_R} |S_p A_i(t,x)| + \sup_{(t,x)\in N_R} |S_p f(t,x)|$$

$$+ m(1+\sigma_0)[C\alpha_p + \sigma_0\gamma_p] + 2\sigma_0 C^2 \xi_p + \frac{3}{16} \sup_{(t,x)\in N_R} |\nabla(S_p U)(t,x)|$$

$$+ m(1+\sigma_0) \sup_{i\leqslant m,\,j\leqslant n,\,(\tau,t,x)\in L_R} |(D_i)_{p,j}(\tau,t,x)|$$

Therefore, immediately we get

$$\lim_{p\to+\infty} \sup_{(t,x)\in N_R} |(\nabla U_{w_p} - \nabla U_w)(t,x)| = 0$$

3.5. Uniqueness

Let us assume that U and V are solutions of (1), then $W := U - V$ is a solution, with zero initial data, of (2), where

$$B_i(t,x) = A_{i,U}(t,x); \quad G(t,x) = \sum_{i=1}^m (A_{i,U}(t,x) - A_{i,V}(t,x))V_{x_i} + f_U(t,x) - f_V(t,x)$$

Then we have

$$W(t,x) = K_U^{-1} \left[\int_0^t e_j \left(\frac{\partial}{\partial\tau} K_U^{\phi^j_U} W + K_U G \right)(\tau, \phi^j_U(\tau,t,x))\,\mathrm{d}\tau \right]_j$$

hence, since $A_i(\alpha)$, $f(\alpha)$ are Lipschitz continuous:

$$\begin{aligned}|W(t,x)| &\leqslant \sigma_0 nT(2mn\sigma_0^2 C\|W\|_\infty + \sigma_0^2 mC\|W\|_\infty + \sigma_0^2\|W\|_\infty)\\ &\leqslant 2\sigma_0^3 nTCm(n+1)\|W\|_\infty \leqslant \tfrac{1}{4}\|W\|_\infty\end{aligned}$$

It follow that $\|W\|_\infty = 0$, hence $U = V$.

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