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Some Remarks on Global Solutions to Nonlinear Dissipative Mildly Degenerate Kirchhoff Strings.

MARINA GHISI

ABSTRACT - We investigate the evolution problem

$$u_{tt} + \delta u_t - m\left(\int_\Omega |\nabla u|^2 \right) \Delta u + f(u) = 0\,,$$

$$u(0, x) = u_0(x)\,, \qquad u_t(0, x) = u_1(x)\,, \qquad x \in \Omega\,, t \geq 0$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\delta > 0$, and m is a locally Lipschitz continuous function, with $m(0) = 0$ and $m(r) > 0$ in a neighbourhood of 0. We prove that, if $\beta = \max\{1, [n/2]\}$, this problem has a unique global solution for positive times, provided that $(u_0, u_1) \in (H_0^\beta \cap H^{\beta+1})(\Omega) \times H_0^\beta(\Omega)$ and u_0, u_1, f satisfy suitable smallness assumptions and the non-degeneracy condition $u_0 \neq 0$ holds. We prove also that $(u(t), u_t(t), u_{tt}(t)) \to (0, 0, 0)$ in $(H_0^\beta \cap H^{\beta+1})(\Omega) \times H^\beta(\Omega) \times H^{\beta-1}(\Omega)$ as $t \to \infty$.

1. Introduction.

Let $\Omega \subseteq \mathbb{R}^n$ be an open domain, $H := L^2(\Omega)$, with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. Let us set $A := -\Delta$, with domain $D(A) := (H_0^1 \cap \cap H^2)(\Omega)$. We consider the Cauchy problem

$$\text{(1.1)} \qquad \begin{cases} u''(t) + \delta u'(t) + m(\|A^{1/2} u(t)\|^2) Au(t) + f(u(t)) = 0, & t \geq 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

(*) Indirizzo dell'A.: Università degli Studi di Pisa, Dipartimento di Matematica, via M. Buonarroti 2, 56127 Pisa, Italy.

where $\delta > 0$, $m : [0, +\infty[\to [0, +\infty[$ is a locally Lipschitz continuous function.

If Ω is an interval of the real line, this equation is a model for the damped small transversal vibrations of an elastic string with fixed endpoints.

The case $\delta = 0, f = 0$ (free vibrations) has long been studied: the interested reader can find appropriate references in the surveys of A. Arosio [1] and S. Spagnolo [15].

In the case $\delta = 0$, $f(u) = \pm |u|^\alpha u$ with large α and $m(r) \geq \nu > 0$, P. D'Ancona and S. Spagnolo [4] proved that if $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ are small, then problem (1.1) has a global solution.

The case $\delta > 0$ and $f = 0$ was considered by E. H. De Brito, Y. Yamada and K. Nishihara [2,14,3,10] if $m(r) \geq \nu > 0$ and by K. Nishihara and Y. Yamada [11] and in [6] if $m(r) \geq 0$. In [6] it was proved that if $(u_0, u_1) \in D(A) \times D(A^{1/2})$ are small enough and $m(\|A^{1/2} u_0\|^2) \neq 0$, there exists a unique global solution $u(t)$ of (1.1) and that $(u, u', u'') \to (u_\infty, 0, 0)$ in $D(A) \times D(A^{1/2}) \times H$ as $t \to +\infty$; moreover either $u_\infty = 0$ or $m(\|A^{1/2} u_\infty\|^2) = 0$.

Here we are interested in the case in which $f \neq 0$, i.e. we have a nonlinear perturbation effect (for example the presence of an external force).

The case $m(r) \geq \nu > 0$, $\delta > 0$, $f(u) = |u|^\alpha u$ has been considered by M. Hosoya and Y. Yamada [7] under the following condition:

$$(1.2) \qquad 0 \leq \alpha < \frac{2}{n-4} \quad \text{if} \quad n \geq 5, \qquad 0 \leq \alpha < +\infty \quad \text{if} \quad n \leq 4.$$

They proved that, if the initial data are small enough, problem (1.1) has a global solution and such a solution decays exponentially as $t \to +\infty$.

Degenerate equations ($m(r) \geq 0$) were considered by K. Ono [12] and in [5] when $n \leq 3$, $\delta > 0$ and $f(u)u \geq 0$. In [12] it was proved that if $m(r) = r^\gamma$, $f(u) \simeq |u|^\alpha u$ and the initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ are small enough, $u_0 \neq 0$, and:

$$(1.3) \qquad \alpha > 2\gamma - 1 \quad \text{if} \quad n = 1, 2, \qquad \alpha > 4\gamma - 2 \quad \text{if} \quad n = 3,$$

then problem (1.1) has a global solution, that decay as $t \to +\infty$.

In [5] the quoted result was extended to a general function $m(r)$ with $m(0) = 0$, $m(r) > 0$ in $]0, r_0]$ when, for some $\varepsilon > 0$:

either $f(y)y \geqslant 0$ and:

$$\max_{|y| \leqslant s} |f'(y)| \leqslant \begin{cases} Cm(s^{2+\varepsilon}) & \text{if } n = 1, 2, \\ Cm(s^4) & \text{if } n = 3, \end{cases} \tag{1.4}$$

or $f(0) = 0$, $f' \geqslant 0$ and:

$$\max_{|y| \leqslant s} |f'(y)| \leqslant \begin{cases} Cm(s^{2+\varepsilon})\, s^{-1+\varepsilon} & \text{if } n = 1, 2, \\ Cm(s^4)\, s^{-2+\varepsilon} & \text{if } n = 3. \end{cases} \tag{1.5}$$

Moreover K. Ono [13] proved that (1.1) has a global solution, if $f(u) \simeq \pm |u|^{\alpha} u$, $(u_0, u_1) \in D(A) \times D(A^{1/2})$ are small enough and at least one of the following conditions is verified:

1. $m(r) = r^{\gamma}$, $u_0 \neq 0$, $n \leqslant 3$, and

$$\alpha > 2\gamma \quad \text{if} \quad n = 1, 2 \qquad \alpha > 4\gamma - 2 \quad \text{if} \quad n = 3. \tag{1.6}$$

2. $m(r) \geqslant \nu > 0$, and satisfies (1.2) (see also R. Ikehata [8]).

He use the modified potential well method and the general theory on the energy decay in Nakao [9]. Unfortunately this method does not seem to be extendible to the case of more general m.

Our purpose is to consider problem (1.1) where m is any non-negative locally Lipschitz continuous function, and $m(0) = 0$, $m(r) > 0$ in a neighbourhood of 0 and $f(u)u$ is not necessary positive.

Let us denote

$$\beta = \max\left\{1, \left[\frac{n}{2}\right]\right\} \qquad \text{and} \qquad B := A^{1/2}$$

where $[x]$ is the integer part of x.

We prove that there exists a unique global solution provided that $(u_0, u_1) \in D(B^{\beta+1}) \times D(B^{\beta})$ and u_0, u_1, f satisfy suitable smallness assumptions (cf. Theorem 2.2) and the non-degeneracy condition $u_0 \neq 0$ holds. Moreover we prove that $u(t) \to 0$ as $t \to \infty$. (cf. Theorem 2.3).

The differences with respect to the case considered in [5] are of two different types: first the term f is «not positive» and this compels us to modify the estimates (for example there exist no positive conserved energies...); here we must estimate with care some terms that in the case in [5] are negligible. Second we consider the case of all space dimensions, then we need more accurate estimates, in particular to reduce the requests at the minimum on the perturbation term.

Notations

In this paper, we denote by $a_{i,n}$ some constants such that

$$\begin{array}{ll}
\|u\| \leqslant a_{1,n}\|Bu\| & u \in D(B) \\
\|u\|_\infty \leqslant a_{2,n}\|Bu\|^\lambda \|B^{\beta+1}u\|^{1-\lambda} & u \in D(B^{\beta+1}) \\
\|u\|_{p_0} \leqslant a_{3,n}\|Bu\| & u \in D(B) \\
\|Bu\| \leqslant a_{4,n}\|B^{\beta+1}u\| & u \in D(B^{\beta+1})
\end{array}$$

where $p_0 = \dfrac{2n}{n-2}$ if $n \geqslant 3$ and

$$\lambda = \begin{cases} 1 & \text{if } n = 1\,, \\ 1 - \varepsilon & \text{if } n = 2\,, \\ 1 - \dfrac{n-2}{2\beta} & \text{if } n \geqslant 3\,. \end{cases}$$

2. Statement of the results.

In this section we state the main results of this paper. For sake of completeness, we recall the following local existence result.

THEOREM 2.1. *(Local existence) Let $\delta > 0$, let m be a locally Lipschitz continuous function, $f \in C^\beta(\mathbb{R})$, and let $(u_0, u_1) \in D(B^{\beta+1}) \times \times D(B^\beta)$ with $m(\|Bu_0\|^2) > 0$.*

Then there exists $T > 0$ such that problem (1.1) *has a unique solution*

$$u \in C^2([0, T]; D(B^{\beta-1})) \cap C^1([0, T]; D(B^\beta)) \cap C^0([0, T]; D(B^{\beta+1})).$$

Moreover, u can be uniquely continued to a maximal solution defined in an interval $[0, T_*[$, and at least one of the following statements holds:

(i) $T_* = \infty$;

(ii) $\limsup\limits_{t \to T_*^-} (\|B^\beta u'(t)\|^2 + \|B^{\beta+1}u(t)\|^2) = +\infty$;

(iii) $\liminf\limits_{t \to T_*^-} m(\|Bu(t)\|^2) = 0$.

The proof is standard and we can obtain it by following the outline of the one in [5] with the obvious changes in the notations. We can state the global existence's result.

THEOREM 2.2. *(Global existence) Let* $\delta > 0$, *and let* m *be a locally Lipschitz continuous function with* $m(0) = 0$ *and* $m(r) > 0$ *in* $]0, r_0]$ *for some* $r_0 > 0$. *Let us assume that* $f \in C^{\beta}(\mathbb{R})$ *verifies in a neighbourhood of* 0 *the following conditions:*

(i) *if* $n = 1, 2$: $f(0) = 0$ *and for some* $\varepsilon_1, \varepsilon_2 > 0$:

$$\max_{|y| \leqslant s} |f'(y)| \leqslant Cm(s^{2+\varepsilon_1})\, s^{\varepsilon_2} \tag{2.1}$$

(ii) *if* $n \geqslant 3$: $f(0) = f'(0) = \ldots = f^{(\beta-1)}(0) = 0$ *and there exists some* $\eta \geqslant 0$ *such that:*

$$\max_{|y| \leqslant s} \frac{|f^{\beta}(y)|}{|y|^{\eta}} \leqslant Cm(s^{2/\lambda})\, s^{-(\eta - \tilde{\eta} + \beta - \beta_0) + (\varepsilon - \tilde{\eta})/\lambda} \tag{2.2}$$

for some $\varepsilon > 0$, *where* $\tilde{\eta} = \min\left\{\eta, \dfrac{2}{n-2}\right\}$ *and* $\beta_0 := \max\{1, [\beta/2]\}$.
Moreover let us assume that the initial data $(u_0, u_1) \in D(B^{\beta+1}) \times \times D(B^{\beta})$ *are small enough and satisfy the non-degeneracy condition* $u_0 \neq 0$.

Then problem (1.1) *admits a unique global solution*

$$u \in C^2([0, +\infty[; D(B^{\beta-1})) \cap C^1([0, \infty[; D(B^{\beta})) \cap C^0([0, \infty[; D(B^{\beta+1})).$$

If $n \leqslant 3$, $m(r) = r^{\gamma}$ and $|f'(u)| \leqslant k|u|^{\alpha}$, by Theorem 2.2 we obtain the result in [12] (cf. (1.6)).

Finally we have the following result.

THEOREM 2.3. *(Asymptotic behaviour) Let us assume that all the hypotheses of Theorem* 2.2 *are satisfied.*

Then we have that:

(i) $m(\|Bu(t)\|^2) > 0$ *for all* $t \geqslant 0$;

(ii) $(u(t), u'(t), u''(t)) \to (0, 0, 0)$ *in the space* $D(B^{\beta+1}) \times D(B^{\beta}) \times \times D(B^{\beta-1})$ *as* $t \to \infty$.

The proof of Theorem 2.3 relies on a result about the asymptotic behaviour of solutions of the linearization of (1.1) (see Lemma 3.1 for the precise statement).

3. Proofs.

3.1. *Proof of Theorem* 2.2.

Case $n = 1, 2$

We use the following notations:

$$\phi_\varepsilon(n) := \begin{cases} (a_{1,1}^\varepsilon a_{2,1})^{-2/\varepsilon} & n = 1 \\ (a_{2,2})^{-2/\varepsilon} & n = 2 . \end{cases}$$

$$\mu_f(s) := \max_{|y| \leq s} |f'(y)|, \ \sqrt{c} := C .$$

With these notations we can rewrite, without loss of generality, (2.1) as follows:

$$\mu_f(s^{1-\varepsilon_1}) \leq \sqrt{c}\, m(s^2)\, s^{\varepsilon_2} \qquad s \in [0, \sqrt{r_0}] \tag{3.1}$$

for some constants $0 < \varepsilon_1, \varepsilon_2 < 1$.

Let us set:

$$\sigma := \min \{ \phi_{\varepsilon_1}, r_0 a_{1,n}^{-2}, (2\sqrt{c} a_{1,n}^{2+\varepsilon_2})^{-2/\varepsilon_2} \},$$

$$M := \max_{|r| \leq r_0} m(r), \qquad L := \sup_{|r| \leq r_0} |m'(r)| .$$

Let us assume that, for a suitable $0 < \sigma_1 \leq \sigma$:

$$F_0 := F(0) + H_0 \frac{a_{1,n}^{2\varepsilon_2} c}{\delta} \sigma_1^{\varepsilon_2} < \sigma_1, \ LG_0 \sqrt{F_0} < \frac{\delta}{4}$$

where

$$H_0 := \frac{4}{\delta} a_{1,n}^2 M \sigma_1 + 2\langle u_0, u_1 \rangle + \delta \|u_0\|^2$$

$$F(0) := \frac{\|Bu_1\|^2}{m(\|Bu_0\|^2)} + \|B^2 u_0\|^2$$

$$G_0 := \max \left\{ \frac{\|u_1\|}{m(\|Bu_0\|^2)}, \ \frac{2}{\delta} (\sqrt{c r_0^{\varepsilon_2}} a_{1,n}^2 + 1) \sqrt{F_0} \right\}.$$

We prove that under these smallness assumptions the solution u of (1.1) is a global solution.

In the following let us denote

$$c(t) = m(\|Bu(t)\|^2).$$

Let us assume that $m \in C^1([0, +\infty[; \mathbb{R})$, and let $[0, T_*[$ be the maximal interval where the solution exists.

Step 1. Let us define:

$$F(t) := \frac{\|Bu'(t)\|^2}{c(t)} + \|B^2 u(t)\|^2 + \frac{\delta}{2}\int_0^t \frac{\|Bu'(s)\|^2}{c(s)}\,ds\,,$$

$$T := \sup\left\{\tau \in [0, T_*[: c(t) > 0,\ \left|\frac{c'(t)}{c(t)}\right| \leqslant \frac{\delta}{2},\ F(t) \leqslant \sigma_1\ \forall t \in [0, \tau]\right\}.$$

We show that $T = T_*$. Let us assume by contradiction that $T < T_*$. Since $|c'(t)| \leqslant \frac{\delta}{2} c(t)$ in $[0, T[$ we have that

$$0 < c(0)\, e^{-\delta T/2} \leqslant c(t) \leqslant c(0)\, e^{\delta T/2} \qquad t \in [0, T[. \tag{3.2}$$

Moreover, by $\|B^2 u(t)\|^2 \leqslant \sigma$ we obtain:

$$\|Bu(t)\|^2 \leqslant a_{1,n}^2 \|B^2 u(t)\|^2 \leqslant r_0 \qquad t \in [0, T[.$$

Since $c(\cdot)$, $c'(\cdot)$, and $F(t)$ are continuous functions, by the maximality of T we have that necessarily

$$\left|\frac{c'(T)}{c(T)}\right| = \frac{\delta}{2}\,; \tag{3.3}$$

or

$$F(T) = \sigma_1. \tag{3.4}$$

Step 2. Firstly, let us remark that, since:

$$\|u\|_\infty \leqslant \phi_{\varepsilon_1}^{-\varepsilon_1/2} \|Bu\|^{1-\varepsilon_1} \|B^2 u\|^{\varepsilon_1} \leqslant \|Bu\|^{1-\varepsilon_1} \tag{3.5}$$

then, using $f(0) = 0$:

$$\int_\Omega |f(u)\, u|\, dx = \int_\Omega |f'(\xi_u)\, u^2|\, dx \leqslant a_{1,n}^2 \mu_f(\|Bu\|^{1-\varepsilon_1}) \|Bu\|^2. \tag{3.6}$$

Furthermore, by taking the scalar product of the equation (1.1) with u, and integrating on $[0, t]$ we obtain:

$$\int_0^t (c(s)\|Bu(s)\|^2 + \langle f(u(s)), u(s)\rangle)\, ds =$$

$$= \int_0^t \|u'(s)\|^2 ds + \langle u_0, u_1\rangle - \langle u(t), u'(t)\rangle + \frac{\delta}{2}\|u_0\|^2 - \frac{\delta}{2}\|u(t)\|^2$$

$$\leq a_{1,n}^2 M\left(\int_0^t \frac{\|Bu'(s)\|^2}{c(s)} ds + \frac{\|Bu'(t)\|^2}{2\delta c(t)}\right) + \langle u_0, u_1\rangle + \frac{\delta}{2}\|u_0\|^2$$

$$\leq \frac{2}{\delta} a_{1,n}^2 M\sigma_1 + \langle u_0, u_1\rangle + \frac{\delta}{2}\|u_0\|^2.$$

Hence, for $t \in [0, T[$, by (3.6):

$$\int_0^t c(s)\|Bu(s)\|^2 ds - \sqrt{c}\, a_{1,n}^2 \int_0^t c(s)\|Bu(s)\|^{2+\varepsilon_2} \leq \frac{1}{2} H_0. \tag{3.7}$$

Furthermore, since $\sqrt{c}\, a_{1,n}^{2+\varepsilon_2} \sigma_1^{\varepsilon_2/2} \leq 1/2$, then:

$$\int_0^t c(s)\|Bu(s)\|^2 ds \leq H_0 \tag{3.8}$$

Step 3. We prove that (3.4) is false. A standard calculation show that on $[0, T[$ we have:

$$F'(t) \leq -\left(\frac{3}{2}\delta + \frac{c'(t)}{c(t)}\right)\frac{\|Bu'(t)\|^2}{c(t)} + \frac{2}{c(t)}\|Bu'(t)\|\|f'(u(t))\, Bu(t)\|$$

$$\leq \frac{1}{\delta c(t)}\|f'(u(t))\, Bu(t)\|^2.$$

Using (3.1), and (3.5) we obtain:

$$\|f'(u(t))\, Bu(t)\|^2 \leq \mu_f(\|Bu(t)\|^{1-\varepsilon_1})^2 \|Bu(t)\|^2 \tag{3.9}$$

$$\leq cm(\|Bu(t)\|^2)^2 \|Bu(t)\|^{2+2\varepsilon_2};$$

hence, by (3.8), for all $t \in [0, T]$:

$$F(t) \leq F(0) + \frac{c}{\delta} a_{1,n}^{2\varepsilon_2} \sigma_1^{\varepsilon_2} \int_0^t c(s) \|Bu(s)\|^2 ds \leq F_0 < \sigma_1. \tag{3.10}$$

This contradicts (3.4).

Step 4. We prove that (3.3) is false. Let us define $G(t) := \frac{\|u'(t)\|}{c(t)}$. By a simple computation, on $[0, T[$ we obtain:

$$(G^2(t))' \leq -\delta G^2(t) + 2G(t)\|Bu(t)\| + 2G(t) \frac{\|f(u(t))\|}{c(t)}.$$

Moreover, since $f(0) = 0$, by (3.1) and (3.5) we have:

$$\int_\Omega f(u(t, x))^2 \leq a_{1,n}^2 \mu_f(\|Bu(t)\|^{1-\varepsilon_1})^2 \|Bu(t)\|^2 \tag{3.11}$$

$$\leq cr_0^{\varepsilon_2} a_{1,n}^4 m(\|Bu(t)\|^2)^2 \|B^2 u(t)\|^2.$$

By this fact:

$$(G^2(t))' \leq -G(t)(\delta G(t) - 2(1 + \sqrt{cr_0^{\varepsilon_2}} a_{1,n}^2) \sqrt{F_0}).$$

Hence, by a standard ODE's inequality we have:

$$G(T) \leq \max\left\{G(0), \frac{2(1 + \sqrt{cr_0^{\varepsilon_2}} a_{1,n}^2)}{\delta} \sqrt{F_0}\right\} = G_0. \tag{3.12}$$

By (3.10) - (3.12), we have then

$$\left|\frac{c'(T)}{c(T)}\right| = \left|\frac{2m'(|Bu(T)|^2)\langle u'(T), B^2 u(T)\rangle}{c(T)}\right|$$

$$\leq 2L \frac{|u'(T)|}{c(T)} |B^2 u(T)|$$

$$\leq 2LG_0 \sqrt{F_0} < \frac{\delta}{2}.$$

This contradicts (3.3).

Step 5. Let us assume by contradiction that $T_* < +\infty$. By (3.2) and (3.8) it follows that

$$\liminf_{t \to T_*^-} m(\|Bu(t)\|^2) \geq m(\|Bu_0\|^2)\, e^{-\delta T_*/2} > 0\,,$$

$$\limsup_{t \to T_*^-} (\|Bu'(t)\|^2 + \|B^2 u(t)\|^2) \leq \max\{1, c(0)\, e^{\delta T_*/2}\}\, F_0 < +\infty\,.$$

By the last statement of Theorem 2.1 this is a contradiction. This completes the proof if m' is continuous. If m is only locally Lipschitz continuous, thesis follows from a standard approximation argument. ■

Case $n \geq 3$

In the following we denote by $\overline{a}, \overline{b}, \overline{c} \ldots$ some constants independent from the initial data, which we use in the proof. Moreover let us define:

$$\mu_f(s) := \max_{|y| \leq s} \left| \frac{f^\beta(y)}{y^\eta} \right|$$

With these notations we can rewrite, without loss of generality, (2.2) as follows:

$$\mu_f(s^\lambda) \leq Cm(s^2)\, s^{\varepsilon - \tilde{\eta} - \lambda(\eta - \tilde{\eta} + \beta - \beta_0)} \qquad s \in [0, \sqrt{r_0}] \tag{3.13}$$

for some constant $0 < \varepsilon < 1$. We can also assume $r_0 \leq 1$.

Let us set:

$$\sigma := \min\{1, a_{2,n}^{2/(\lambda - 1)}, r_0\, a_{4,n}^{-2}, (4 a_{1,n}^2 \overline{a})^{-1/\varepsilon}\}\,,$$

$$M := \max_{|r| \leq r_0} m(r), \qquad L := \sup_{|r| \leq r_0} |m'(r)|\,.$$

Let us assume that, for a suitable $0 < \sigma_1 \leq \sigma$:

$$F_0 := F(0) + \frac{\overline{a}}{\delta} H_0 \sigma_1^\varepsilon < \sigma_1,\ LG_0\sqrt{F_0} < \frac{\delta}{4}$$

where

$$H_0 := \frac{4}{\delta} M\sigma_1 + 2\langle B^\beta u_0, B^\beta u_1\rangle + \delta \|B^\beta u_0\|^2$$

$$F(0) := \frac{\|B^\beta u_1\|^2}{m(\|Bu_0\|^2)} + \|B^{\beta+1} u_0\|^2$$

$$G_0 := \max\left\{ \frac{\|u_1\|}{m(\|Bu_0\|^2)}, \frac{\bar{b}}{\delta}(F_0^{(\lambda\beta_0+\varepsilon)/2} + \sqrt{F_0})\right\}.$$

We prove that under these smallness assumptions the solution u of (1.1) is a global solution.

In the following let us denote

$$c(t) = m(\|Bu(t)\|^2).$$

Let us assume that $m \in C^1([0, +\infty[; \mathbb{R})$, and let $[0, T_*[$ be the maximal interval where the solution exists.

Step 1. Let us define

$$F(t) := \frac{\|B^\beta u'(t)\|^2}{c(t)} + \|B^{\beta+1} u(t)\|^2 + \frac{\delta}{2}\int_0^t \frac{\|B^\beta u'(s)\|^2}{c(s)} ds .$$

Let us set

$$T := \sup\left\{\tau \in [0, T_*[: c(t) > 0, \left| \frac{c'(t)}{c(t)} \right| \leq \frac{\delta}{2}, F(t) \leq \sigma_1 \forall t \in [0, \tau]\right\}.$$

We show that $T = T_*$. Let us assume by contradiction that $T < T_*$. Since $|c'(t)| \leq \frac{\delta}{2} c(t)$ in $[0, T[$ we have that

$$0 < c(0)e^{-\delta T/2} \leq c(t) \leq c(0)e^{\delta T/2} \qquad t \in [0, T[. \tag{3.14}$$

Moreover, by $\|B^{\beta+1}u(t)\|^2 \leq \sigma$ we obtain:

$$\|Bu(t)\|^2 \leq a_{4,n}^2 \|B^{\beta+1} u(t)\|^2 \leq r_0 \qquad t \in [0, T[,$$

and

$$\|u(t)\|_\infty \leq \|Bu(t)\|^\lambda \qquad t \in [0, T[.$$

Since $c(\cdot)$, $c'(\cdot)$, and $F(t)$ are continuous functions, by the maximality of T

we have that necessarily

$$\left| \frac{c'(T)}{c(T)} \right| = \frac{\delta}{2}; \tag{3.15}$$

or

$$F(T) = \sigma_1. \tag{3.16}$$

Step 2. In this step we denote various constants depending only from n by c. Let us set

$$q = \frac{n}{n-2}, \, p = \begin{cases} +\infty & \text{if } \tilde{\eta} = 0 \\ \dfrac{n}{(n-2)\,\tilde{\eta}} & \text{if } \tilde{\eta} > 0 . \end{cases}$$

Let $\beta_1, \ldots, \beta_\nu > 0$ be integers such that $\beta_1 + \ldots + \beta_\nu = \beta$.

Let us suppose that k of the β_j are equal to 1. We can assume that they are the first k. Let us define for $j = k+1, \ldots, \nu$:

$$\frac{1}{p_j} = \frac{2}{n-2}\left(\frac{n}{2} - \beta - 1 + \beta_j\right).$$

Now let us assume that $k \geq 1$ and let us set, for $\nu \geq 2$:

$$\frac{1}{p_{0,\nu}} = \frac{1}{k}\left(1 - \frac{2}{n-2}\left(\frac{n}{2}(\nu - k) + \beta(1 - \nu + k) - \nu\right)\right).$$

Using the Sobolev inequalities we have then:

$$\|B^\beta u\|_{2q} \leq c\|B^{\beta+1}\|, \qquad \|B^{\beta_j} u\|_{2qp_j} \leq c\|B^{\beta+1}\| \quad j = k+1, \ldots, \nu .$$

Furthermore, since:

$$0 < \theta = \left(\frac{1}{2} - \frac{1}{2qp_{0,\nu}}\right)\frac{n}{\beta} = \frac{1-\nu}{k} + 1 + (\nu - 1)\frac{n-2}{2\beta k} < 1$$

we have:

$$\|Bu\|_{2qp_{0,\nu}} \leq c\|Bu\|^{1-\theta}\|B^{\beta+1}u\|^{\theta}.$$

By $k/p_{0,\nu} + 1/p_{k+1} + \ldots + 1/p_\nu = 1$ we can then also deduce:

$$\|B^{\beta_1}u \ldots B^{\beta_\nu}u\|_{2q}^2 \leq \begin{cases} c\|B^{1+\beta}u\|^{2\nu} & \text{if } k=0 \\ c\|Bu\|^{2(\nu-1)\lambda}\|B^{\beta+1}u\|^{2+2(\nu-1)(1-\lambda)} & \text{if } k \geq 1\,. \end{cases}$$

We are now able to estimate $\|B^\beta f(u)\|$.

Since, for all $b = (b_1, \ldots, b_n)$ we have:

$$\partial^b f(u) = \sum_{\nu=1}^{|b|} \sum_{\substack{B_1 + \ldots + B_\nu = b \\ |B_i| > 0}} c_{b, B_1, \ldots, B_\nu} f^{(\nu)}(u)\, \partial^{B_1}u \ldots \partial^{B_\nu}u$$

then:

$$\|B^\beta f(u)\|^2 \leq c \sum_{\nu=1}^{\beta} \sum_{\substack{\beta_1 + \ldots + \beta_\nu = \beta, \\ \beta_i > 0}} \|f^{(\nu)}(u)\, B^{\beta_1}u \ldots B^{\beta_\nu}u\|^2$$

$$\leq c\mu_f(\|Bu\|^\lambda)^2 \sum_{\nu=1}^{\beta} \sum_{\substack{\beta_1 + \ldots + \beta_\nu = \beta \\ \beta_i > 0}} \|u^{\eta+\beta-\nu} B^{\beta_1}u \ldots B^{\beta_\nu}u\|^2$$

$$\leq c\mu_f(\|Bu\|^\lambda)^2 \|u\|_{2n/(n-2)}^{2\tilde{\eta}} \times$$

$$\times \sum_{\nu=1}^{\beta} \sum_{\substack{\beta_1 + \ldots + \beta_\nu = \beta \\ \beta_i > 0}} \|u\|_\infty^{2(\eta - \tilde{\eta} + \beta - \nu)} \|B^{\beta_1}u \ldots B^{\beta_\nu}u\|_{2q}^2$$

Now let us remark that if $\beta \geq 2$ and $\nu \leq \beta_0$ then there exists $\beta_1 + \ldots + \beta_\nu = \beta$ with $\beta_i \geq 2$ for all $i = 1, \ldots, \nu$. Moreover if $\nu > \beta_0$ and $\beta_1 + \ldots + \beta_\nu = \beta$ then at least one of the β_i is equal to 1. Furthermore:

$$2\lambda(\beta_0 - \nu) + 2(\nu - 1)\lambda = 2\lambda(\beta_0 - 1) \geq 0\,.$$

Hence, using $\|Bu\| \leq a_{4,n}\|B^{1+\beta}u\| \leq 1$ we obtain:

$$\text{(3.17)} \quad \|B^\beta f(u)\|^2 \leq cc(t)^2 \|Bu\|^{2\varepsilon} \|B^{\beta+1}u\|^2 \sum_{\nu=1}^{\beta} \|B^{1+\beta}u\|^{2\lambda(\beta_0-\nu)+2(\nu-1)}$$

$$\leq \bar{a}c^2(t)\|Bu\|^{2\varepsilon}\|B^{\beta+1}u\|^2$$

$$\leq \bar{a}c^2(t)\,\sigma_1^\varepsilon \|B^{\beta+1}u\|^2 \leq \frac{1}{4a_{1,n}^2} c^2(t)\|B^{\beta+1}u\|^2.$$

Step 3. By applying to the equation (1.1) the operator $B^{\beta-1}$, taking the scalar product of the obtained equation with $B^{\beta+1}$, and integrating on $[0, T]$, we obtain:

$$\int_0^T c(t)\|B^{\beta+1}u(t)\|^2\,dt + \int_0^T \langle B^\beta f(u(t)), B^\beta u(t)\rangle\,dt =$$

$$= \int_0^T \|B^\beta u'(t)\|^2\,dt - \langle B^\beta u'(T), B^\beta u(T)\rangle - \frac{\delta}{2}\|B^\beta u(T)\|^2 +$$

$$+ \langle B^\beta u_1, B^\beta u_0\rangle + \frac{\delta}{2}\|B^\beta u_0\|^2$$

$$\leq M\left(\int_0^T \frac{\|B^\beta u'(t)\|^2}{c(t)}\,dt + \frac{1}{2\delta}\,\frac{\|B^\beta u'(T)\|^2}{c(T)}\right) + \frac{H_0}{2} - \frac{2}{\delta}M\sigma_1 \leq \frac{H_0}{2}\,.$$

Hence, using (3.17):

$$\int_0^T c(t)\|B^{\beta+1}u(t)\|^2\,dt \leq H_0. \tag{3.18}$$

Step 4. We prove that (3.16) is false. By a simple calculation using (3.17) in $[0, T[$ we have:

$$F'(t) \leq \frac{1}{\delta}\,\frac{\|B^\beta f(u(t))\|^2}{c(t)} \leq \frac{\bar{a}}{\delta}\sigma_1^\varepsilon c(t)\|B^{\beta+1}u(t)\|^2$$

hence, by (3.18):

$$F(t) \leq F(0) + \frac{\bar{a}}{\delta}\sigma_1^\varepsilon H_0 = F_0 < \sigma_1. \tag{3.19}$$

Step 5. We prove that (3.15) is false. Let us firstly remark that, since $f(u) = \frac{f^{(\beta)}(\xi_u)}{\beta!}u^\beta$, hence:

$$\|f(u(t))\|^2 \leq \begin{cases} c_0\,\dfrac{\mu_f(\|Bu(t)\|^\lambda)^2}{\beta!}\|u\|_\infty^{2\beta} & \text{if } \eta = 0 \\[2ex] \dfrac{\mu_f(\|Bu(t)\|^\lambda)^2}{\beta!}\|u\|_\infty^{2(\beta+\eta-\tilde{\eta})}\|u\|_{2\frac{\tilde{\eta}}{\eta}}^{2\tilde{\eta}} & \text{otherwise}\,. \end{cases}$$

Therefore by (3.13):

$$\|f(u(t))\|^2 \leq \bar{c}^2 c^2(t)\|B^{\beta+1}u(t)\|^{2(\varepsilon+\lambda\beta_0)}. \tag{3.20}$$

We can now easy estimate $G(t) := \dfrac{\|u'(t)\|}{c(t)}$ as follows:

$$(G^2(t))' \leq -\delta G^2(t) + 2G\left(\|B^2 u(t)\| + \frac{\|f(u(t))\|}{c(t))}\right)$$

$$\leq -G(t)(\delta G(t) - \bar{b}(\sqrt{F_0} + F_0^{(\varepsilon + \lambda\beta_0)/2}))\,,$$

hence, by a standard ODE's inequality we obtain $G(t) \leq G_0$. Then as in proof of case $n = 1, 2$, step 4:

$$\left|\frac{c'(T)}{c(T)}\right| \leq 2LG_0\sqrt{F_0} < \frac{\delta}{2}\,.$$

Step 5. We can conclude as in step 5 of proof of case $n = 1, 2$. ■

3.2. *Asymptotic behaviour.*

In order to study the asymptotic behaviour of the solutions of (1.1), we consider the linearized problem

$$\text{(3.21)} \qquad \begin{cases} v''(t) + \delta v'(t) + c(t)\,B^2 v(t) + f(t, x) = 0\,, & t \geq 0\,, \\ v(0) = v_0, \qquad v'(0) = v_1. \end{cases}$$

In the following lemma we examine the asymptotic behaviour of the solutions of (3.21).

LEMMA 3.1. *Let* $\delta > 0$. *Let* $c : [0, +\infty[\to]0, +\infty[$ *be a Lipschitz continuous bounded function such that*

$$\left|\frac{c'(t)}{c(t)}\right| \leq \frac{\delta}{2} \qquad \textit{for a.e.} \quad t \geq 0\,.$$

Let $f : [0, +\infty[\times \Omega \to \mathbb{R}$ *be a continuous function such that* $f(t, \cdot) \in D(B^\beta)$ *for all* $t \geq 0$ *and*

$$\int_0^{+\infty} \frac{1}{c(s)}\|B^\beta f(s)\|^2 ds < +\infty\,, \qquad \sup_{t \geq 0} \frac{\|f(t)\|}{c(t)} < +\infty\,.$$

Let v *be the unique global solution of* (3.21) *with* $(v_0, v_1) \in D(B^{\beta+1}) \times D(B^\beta)$.

Then there exists $v_\infty \in D(B^{\beta+1})$ *such that*

$$v(t) \to v_\infty \quad \text{in } D(B^{\beta+1}), \tag{3.22}$$

$$v'(t) \to 0 \quad \text{in } D(B^{\beta}), \tag{3.23}$$

as $t \to \infty$. *Furthermore, if* $v_\infty \neq 0$, *then necessarily* $c(t) \to 0$ *as* $t \to \infty$.

PROOF OF LEMMA 3.1. We only give a sketch of the proof, we refer to [5] for the details.

Step 1. Let us consider the function

$$H(t) := \frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1} v(t)\|^2 - \frac{1}{\delta}\int_0^t \frac{1}{c(s)}\|B^\beta f(s)\|^2 \, ds .$$

A simple computation shows that

$$H'(t) \leq -\frac{\delta}{2}\frac{\|B^\beta v'(t)\|^2}{c(t)} . \tag{3.24}$$

By this fact we obtain:

1. for all $t \geq 0$:

$$\frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1} v(t)\|^2 + \frac{\delta}{2}\int_0^t \frac{\|B^\beta v'(s)\|^2}{c(s)} ds \leq$$

$$\leq \frac{\|B^\beta v_1\|^2}{c(0)} + \|B^{\beta+1} v_0\|^2 + \int_0^{+\infty} \frac{1}{\delta c(s)}\|B^\beta f(s,\cdot)\|^2 \, ds =: \gamma_0 .$$

2. Since the function $c(\cdot)$ is bounded then:

$$\int_0^{+\infty} \|B^\beta v'(t)\|^2 \, dt < +\infty \tag{3.25}$$

3. The function H is non-increasing, hence there exists:

$$F_\infty := \lim_{t\to\infty} \frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1} v(t)\|^2 .$$

If $F_\infty = 0$, then (3.22) holds true with $v_\infty = 0$. Since the function c is bounded, then also (3.23) follows from $F_\infty = 0$.

Therefore from now on we assume that $F_\infty > 0$.

Step 2. We show that

$$\int_0^\infty c(t)\|B^{\beta+1}v(t)\|^2\,dt < +\infty. \tag{3.26}$$

Indeed, applying the operator $B^{\beta-1}$ to the equation (3.21) and taking its scalar product with $B^{\beta+1}v$ and integrating on $[0, T]$, it follows that

$$\int_0^T c(t)\|B^{\beta+1}v(t)\|^2\,dt \leqslant \langle B^\beta v_1, B^\beta v_0\rangle + \left(\frac{2}{\delta}\|c\|_\infty + \frac{\delta a_{1,n}^2}{2}\right)\gamma_0 +$$

$$+\frac{1}{2}\int_0^T c(t)\|B^{\beta+1}u(t)\|^2\,dt + \frac{\delta}{2}\|B^\beta v_0\|^2.$$

Hence

$$\int_0^T c(t)\|B^{\beta+1}v(t)\|^2\,dt \leqslant 2\langle B^\beta v_1, B^\beta v_0\rangle + \delta\|B^\beta v_0\|^2 + 2\left(\frac{2}{\delta}\|c\|_\infty + \frac{\delta a_{1,n}^2}{2}\right)\gamma_0.$$

Passing to the limit as $T\to\infty$, we obtain (3.26).

Step 3. From (3.25) and (3.26) it follows that

$$\int_0^\infty c(t)\left(\frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1}v(t)\|^2\right)dt < +\infty.$$

Since, for $t \geqslant \overline{T}$:

$$\frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1}v(t)\|^2 \geqslant \frac{F_\infty}{2} > 0,$$

then also

$$\int_0^\infty c(t)\,dt < +\infty. \tag{3.27}$$

Since $c(\cdot)$ is Lipschitz continuous, it follows that $c(t)\to 0$ as $t\to\infty$. Since $\|B^\beta v'(t)\|^2 \leqslant c(t)\gamma_0$, then (3.23) is proved.

Step 4. We show that (3.22) holds true with the additional assumptions that $(v_0, v_1) \in D(B^{\beta+3}) \times D(B^{\beta+2})$, $f(t, \cdot) \in D(B^{\beta+2})$ for every t and

$$(3.28) \qquad \int_0^{+\infty} \frac{\|B^{\beta+2} f(t)\|}{c(t)} dt < +\infty, \qquad \sup_{t \geq 0} \frac{\|B^{\beta+1} f(t)\|}{c(t)} < +\infty.$$

To this end, let us introduce the function

$$\widehat{H}(t) := \frac{\|B^{\beta+2} v'(t)\|^2}{c(t)} + \|B^{\beta+3} v(t)\|^2 - \frac{1}{\delta} \int_0^t \frac{1}{c(s)} \|B^{\beta+2} f(s)\|^2 ds.$$

As in Step 1, it is possible to prove that $\widehat{H}$ is non-increasing, and that for every $t \geq 0$:

$$\|B^{\beta+3} v(t)\|^2 \leq \widehat{\gamma}_0.$$

Now let us consider the function $\widehat{G}(t) := \dfrac{\|B^{\beta+1} v'(t)\|}{c(t)}$. By a standard ODE's inequality, it follows that

$$\widehat{G}(t) \leq \max\left\{\widehat{G}(0), \frac{2}{\delta}\left(\sqrt{\widehat{\gamma}_0} + \sup_{t \geq 0} \frac{\|B^{\beta+1} f(t)\|}{c(t)}\right)\right\}.$$

By (3.27), this implies that

$$\int_0^{\infty} \|B^{\beta+1} v'(t)\| dt < +\infty$$

and therefore $B^{\beta+1} v(t)$ has a limit as $t \to \infty$.

Step 5. We show that (3.22) hold true for every initial data $(v_0, v_1) \in D(B^{\beta+1}) \times D(B^{\beta})$.

To this end, let us consider a sequence $\{(v_{0n}, v_{1n})\} \subseteq D(B^{\beta+3}) \times D(B^{\beta+2})$ converging to (v_0, v_1) in $D(B^{\beta+1}) \times D(B^{\beta})$ and f_n as in step 4, with:

$$\int_0^{+\infty} \frac{1}{c(t)} \|B^{\beta}(f(t) - f_n(t))\|^2 dt \to 0 \qquad \text{as} \qquad n \to +\infty.$$

Let $\{v_n\}$ be the corresponding solutions of (3.21), and let us set $w_n :=$

$:= v - v_n$. Since w_n is a solution of (3.21), with $f - f_n$ in place of f, we have that

$$\frac{\|B^{\beta} w_n'(t)\|^2}{c(t)} + \|B^{\beta+1} w_n(t)\|^2 \leqslant \frac{\|B^{\beta}(v_{1,n} - v_1)\|^2}{c(0)} + \|B^{\beta+1}(v_{0,n} - v_0)\|^2 + $$

$$+ \frac{1}{\delta} \int_0^{+\infty} \frac{1}{c(t)} \|B^{\beta}(f(t) - f_n(t))\|^2 \, dt .$$

This proves that $\{B^{\beta+1} v_n\} \to B^{\beta+1} v$ uniformly in $[0, +\infty[$. Since $B^{\beta+1} v_n(t)$ has a limit as $t \to \infty$ for every $n \in \mathbb{N}$ (see Step 4), then necessarily $B^{\beta+1} v(t)$ has a limit as $t \to \infty$.

This completes the proof of (3.22). ■

Proof of Theorem 2.3. We use the same notations as in the proof of Theorem 2.2 case $n = 1, 2$ (resp. case $n \geqslant 3$). Let us firstly remark that u is the solution of (3.21) with

$$c(t) = m(\|Bu(t)\|^2), \qquad (v_0, v_1) = (u_0, u_1), \qquad f(t, x) = f(u(t, x)).$$

In Step 1 of the proof of Theorem 2.2 case $n = 1, 2$ (resp. Step 1 of case $n \geqslant 3$), we showed that $c(t) > 0$ for every $t \geqslant 0$ (this proves statement (i)), and

$$\left| \frac{c'(t)}{c(t)} \right| \leqslant \frac{\delta}{2} \qquad \forall t \geqslant 0 .$$

Moreover in this step we proved also that $\|Bu\| \leqslant r_0$, hence $c(\cdot)$ is bounded. Since m is locally Lipschitz continuous, and $\|B^{\beta} u'(t)\|^2 \leqslant$ $\leqslant F(t) c(t) \leqslant F_0 c(t)$ (see (3.10) (resp. (3.19))), then it turns out that $c(\cdot)$ is globally Lipschitz continuous. Finally, by (3.8) (3.9), (3.11) (resp. (3.17), (3.18), (3.20)):

$$\int_0^{+\infty} \frac{\|B^{\beta} f(u(t))\|^2}{c(t)} \, dt \leqslant \begin{cases} \tilde{c} \displaystyle\int_0^{+\infty} c(t) \|Bu(t)\|^2 < +\infty & \text{if } n = 1, 2 \\ \tilde{c} \displaystyle\int_0^{+\infty} c(t) \|B^{\beta+1} u(t)\|^2 < +\infty & \text{if } n \geqslant 3 \end{cases}$$

$$\frac{\|f(u(t))\|^2}{c^2(t)} \leqslant \tilde{c}\|B^{\beta+1}u(t)\|^2 < c_0$$

for some c_0 independent from t.

By Lemma 3.1, there exists $u_\infty \in D(B^{\beta+1})$ such that $u \to u_\infty$ in $D(B^{\beta+1})$ and $u' \to 0$ in $D(B^\beta)$. Let us assume that $u_\infty \neq 0$, then by the last statement of Lemma 3.1 we have that $c(t) \to 0$ as $t \to \infty$, hence

$$0 = \lim_{t\to\infty} m(\|Bu(t)\|^2) = m(\|Bu_\infty\|^2).$$

Since $\|Bu_\infty\|^2 \leqslant r_0$, hence must be $u_\infty = 0$. Furthermore, by applying $B^{\beta-1}$ to the equation (1.1), $u'' \to 0$ in $D(B^{\beta-1})$. ■

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