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Stability of simple modes of the Kirchhoff equation

Marina Ghisi[1] **and Massimo Gobbino**[2]

[1] Università degli Studi di Pisa, Dipartimento di Matematica, via M Buonarroti 2, 56127 Pisa, Italy
[2] Università degli Studi di Pisa, Dipartimento di Matematica Applicata 'Ulisse Dini', via Bonanno 25/B, 56126 Pisa, Italy

E-mail: ghisi@dm.unipi.it and m.gobbino@dma.unipi.it

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Abstract
It is well known that the Kirchhoff equation admits infinitely many simple modes, i.e. time-periodic solutions with only one Fourier component in the space variable(s). We prove that these simple modes are stable provided that their energy is small enough. Here stable means orbitally stable as solutions of the two-mode system obtained considering initial data with two Fourier components.

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1. Introduction

Let H be a real Hilbert space, with norm $|\cdot|$ and scalar product $\langle \cdot , \cdot \rangle$. Let A be a self-adjoint linear positive operator on H with dense domain $D(A)$ (i.e. $\langle Au, u\rangle > 0$ for all $u \in D(A)$). We consider the evolution problem

$$u''(t) + m(|A^{1/2}u(t)|^2)Au(t) = 0 \tag{1.1}$$

where $m : [0, +\infty) \to [0, +\infty)$ is a smooth function.

Equation (1.1) is an abstract setting of the hyperbolic partial differential equation (PDE) with a non-local nonlinearity of Kirchhoff type

$$u_{tt} - m\left(\int_\Omega |\nabla u|^2 \, dx\right) \Delta u = 0 \qquad \text{in } \Omega \times \mathbb{R} \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, ∇u is the gradient of u with respect to space variables and Δ is the Laplace operator.

If Ω is an interval of the real line, this equation is a model for the small transversal vibrations of an elastic string with fixed endpoints.

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In the case where H admits a complete orthogonal system made by eigenvectors of A (this is the case, for example, in the concrete situation of (1.2) if Ω is bounded), then (1.1) may be thought of as a system of ordinary differential equations (ODEs) with infinitely many unknowns, namely the components of u.

Many papers have been written on equations (1.1) and (1.2) after Kirchhoff's monograph [5]: the interested reader can find appropriate references in the surveys [1, 10]. We just recall that, at present, the existence of global solutions for all initial data in C^∞ or in Sobolev spaces is still an open problem.

In this paper we consider a particular class of periodic global solutions of (1.1). Let us assume that λ is an eigenvalue of A, and e_λ is a corresponding eigenvector, which we assume to be normalized such that $|e_\lambda| = 1$. If the initial data are multiples of e_λ, say

$$u(0) = w_0 e_\lambda \qquad u'(0) = w_1 e_\lambda$$

then the solution of (1.1) remains a multiple of e_λ for every $t \in \mathbb{R}$, i.e. we have that $u(t) = w(t)e_\lambda$, where $w(t)$ is the solution of the ODE

$$w''(t) + \lambda m(\lambda w^2(t))w(t) = 0 \qquad w(0) = w_0 \qquad w'(0) = w_1.$$

Such solutions are called *simple modes* of equation (1.1), and are known to be time periodic under very general assumptions on m.

In this paper we are interested in the stability of simple modes. This programme is, however, too optimistic for at least two reasons:

- how can one prove stability if global existence is still an open problem?
- stability problems are often already hard for systems with three unknowns, and we have seen that (1.1) has infinitely many degrees of freedom.

For these reasons we limit ourselves to considering two-mode solutions. To this end, let μ be another eigenvalue of A, and let e_μ be a corresponding eigenvector with $|e_\mu| = 1$. If the initial data of (1.1) belong to the two-dimensional subspace of H spanned by e_λ and e_μ, then the same holds true for the solution, which may be written in the form $w(t)e_\lambda + z(t)e_\mu$, where w and z solve the following system of ODEs:

$$\begin{aligned} &w''(t) + \lambda\, m(\lambda w^2(t) + \mu z^2(t))w(t) = 0 \\ &z''(t) + \mu\, m(\lambda w^2(t) + \mu z^2(t))z(t) = 0. \end{aligned} \tag{1.3}$$

It is clear that simple modes are particular solutions of this system, corresponding to initial data with $z(0) = z'(0) = 0$. What we actually study in this paper is the stability of simple modes as solutions of (1.3).

To this end, let us set

$$\nu := \frac{\mu}{\lambda} \qquad u(t) := \sqrt{\lambda}\, w\left(\frac{t}{\sqrt{\lambda}}\right) \qquad v(t) := \sqrt{\mu}\, z\left(\frac{t}{\sqrt{\lambda}}\right)$$

so that (1.3) is equivalent to

$$\begin{aligned} &u''(t) + m(u^2(t) + v^2(t))u(t) = 0 \\ &v''(t) + \nu\, m(u^2(t) + v^2(t))v(t) = 0. \end{aligned} \tag{1.4}$$

This system (as well as (1.3) and (1.1)) is a Hamiltonian, with conserved energy

$$H(u, u', v, v') := \frac{1}{2}\left\{[u']^2 + \frac{[v']^2}{\nu} + M(u^2 + v^2)\right\}$$

where

$$M(r) := \int_0^r m(s)\,\mathrm{d}s.$$

As far as we know, there are at least two other papers on this subject.

- Dickey [3] proved that simple modes are *linearly stable* provided that their energy is *small* enough. Roughly speaking, linearly stable means that $v(t) \equiv 0$ is a stable solution for the linearization of the second equation in (1.4). These results extend to systems with any *finite* number n of modes, because the $n-1$ equations resulting from linearization are uncoupled.
- A complementary result was proved by Cazenave and Weissler [2]. They showed that (under suitable assumptions on m) there exists a non-empty set $A \subseteq (1, +\infty)$ such that, if $\nu \in A$, then every simple mode of (1.4) with *large* enough energy is *unstable*. In the case where $m(r) = 1 + r$, and Ω is an interval of the real line, this result implies the instability of every simple mode of (1.2) with large enough energy.

In this paper we improve Dickey's result. Our main result is the following.

Theorem 1.1. *Let $\nu \neq 1$ be a positive real number. Let $m : [0, +\infty) \to [0, +\infty)$ be a smooth function such that $m(0) > 0$ and $m'(0) \neq 0$.*

Then there exists $\varepsilon_1 > 0$ such that, if $H(u_0, u_1, 0, 0) < \varepsilon_1$, then the simple mode of (1.4) with $u(0) = u_0$, $u'(0) = u_1$ is orbitally stable.

Roughly speaking, orbitally stable means that every solution $(u(t), v(t))$ of system (1.4) with initial data near $(u_0, u_1, 0, 0)$ remains close to the periodic orbit of the simple mode for every $t \in \mathbb{R}$.

The jump from linear to orbital stability is allowed by the application of KAM theory to the Poincaré map (see sections 2.2 and 2.3).

We conclude with a few comments on theorem 1.1.

- Assumption $m(0) > 0$ is necessary. Indeed, for $m(r) = r$ there exists $\nu > 1$ for which every simple mode of system (1.4) with positive energy is unstable (cf [2, theorem 4.1]).
- Assumption $m'(0) \neq 0$ is not essential. We need it in order to apply KAM theory using 'only' the first three terms in the Taylor expansion of the Poincaré map. Considering further terms, it should be possible to prove the same result assuming only that $m^{(k)}(0) \neq 0$ for some integer $k \geqslant 1$.
- In this paper we always assume $\nu \neq 1$, which corresponds to the case of two modes relative to distinct eigenvalues of the operator A. In the case $\nu = 1$, our expansions of section 4.2 are inconclusive, and further terms are also needed to prove (or disprove) linear stability.
- Smoothness of m is used only to give to the Poincaré map the smoothness required by KAM theory. To this end, $m \in C^5$ is enough. We do not know any counterexample for less regular ms.
- Due to the use of KAM theory, at the present there is no hope of extending our argument to systems with three or more modes. Indeed, for such systems, KAM theory says at most that, in a neighbourhood of the simple mode, 'a lot of solutions' stay on 'KAM tori around the periodic trajectory', but this does not prevent other solutions from exhibiting an unstable behaviour.
- Since system (1.4) is reversible (if $(u(t), v(t))$ is any solution, then $(u(-t), v(-t))$ is another solution), then a consequence of theorem 1.1 is the following: 'if the energy of a two-mode solution of (1.4) is small enough, then it is *not* possible that asymptotically all of this energy is absorbed by one of the two components'.

This paper is organized as follows. In section 2 we give definitions and preliminaries, in section 3 we state our results and outline the proofs, and in section 4 we give the details of the proofs.

2. Definitions and preliminaries

In this section we recall the notion of stability, and we describe how it can be proved using the Poincaré map and KAM theory. General references on these subjects are [4, 6–9].

Before we enter into the details, we fix some notation which will be used throughout this paper.

We assume that $m : [0, +\infty) \to [0, +\infty)$ is a function of class C^5, such that

$$m_0 := m(0) > 0 \qquad m_0' := m'(0) \neq 0.$$

Since the only essential thing is the behaviour of $m(r)$ for small r, we can assume without loss of generality that $\inf\{m(r): r \geqslant 0\} > 0$.

We denote by $M_{2\times 2}$ the set of 2×2 matrices. For each $A \in M_{2\times 2}$, a_{ij} is the element in the ith row and jth column, unless otherwise stated, and $\operatorname{Tr} A = a_{11} + a_{22}$ is the trace of A. For every $\omega \in \mathbb{R}$, R_ω denotes the rotation matrix

$$R_\omega = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$

2.1. Stability

In this section we recall some definitions of stability from the classical theory of Hamiltonian systems. For the sake of simplicity, we adapt definitions to the case of simple modes for the system (1.4).

To this end, we consider a simple mode $\bar{u}$ solution of the problem

$$\bar{u}''(t) + m(\bar{u}^2(t))\bar{u}(t) = 0 \qquad \bar{u}(0) = u_0 > 0 \qquad \bar{u}'(0) = 0. \tag{2.1}$$

We recall that $\bar{u}$ is a periodic function, and so we can assume $\bar{u}(0) > 0$ and $\bar{u}'(0) = 0$ without loss of generality.

Now in the phase space $\mathbb{R}^4$ we consider the energy level

$$\mathcal{H}_{\bar{u}} := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \colon H(x_1, x_2, x_3, x_4) = H(u_0, 0, 0, 0)\right\}$$

and the orbit

$$\Gamma_{\bar{u}} := \left\{(\bar{u}(t), \bar{u}'(t), 0, 0) \colon t \in \mathbb{R}\right\}.$$

Definition 2.1. The simple mode $\bar{u}$ is said to be *orbitally stable* if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every solution $(u(t), v(t))$ of system (1.4), the following property holds: if the initial datum $(u(0), u'(0), v(0), v'(0))$ belongs to a δ neighbourhood of $(u_0, 0, 0, 0)$, then for every $t \in \mathbb{R}$ the point $(u(t), u'(t), v(t), v'(t))$ lies in an ε neighbourhood of $\Gamma_{\bar{u}}$.

Definition 2.2. The simple mode $\bar{u}$ is said to be *isoenergetically orbitally stable* if the condition of definition 2.1 is satisfied with the restriction that $(u(0), u'(0), v(0), v'(0)) \in \mathcal{H}_{\bar{u}}$.

Definition 2.3. The simple mode $\bar{u}$ is said to be *linearly stable* if $v(t) \equiv 0$ is a stable solution of the linear equation $v''(t) + \nu\, m(\bar{u}^2(t))v(t) = 0$, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|(v(0), v'(0))\| < \delta \quad \Longrightarrow \quad \|(v(t), v'(t))\| < \varepsilon \qquad \forall t \in \mathbb{R}.$$

This linear equation is the linearization of the second equation in (1.4).

We end with some remarks on the above definitions.

- Due to the *reversibility* of solutions of (1.4), using $t \geqslant 0$ is equivalent to using $t \in \mathbb{R}$ in the definitions of stability.
- Definition 2.1 says that any orbit starting near $(u_0, 0, 0, 0)$ remains close to $\Gamma_{\bar{u}}$ for all times. It is in general not true that, for every $t \in \mathbb{R}$, the corresponding points on the two trajectories are close to each other.
- As observed in [3], linear stability is equivalent to the following: 'let (u, v) be a solution of (1.4); if we represent v as a formal power series in $v(0)$, $v'(0)$ with time-dependent coefficients, then the linear terms are uniformly bounded for every $t \in \mathbb{R}$'.
- It is obvious that orbital stability implies isoenergetic orbital stability. In non-degenerate situations, isoenergetic orbital stability implies linear stability. Here 'non-degenerate situation' means that $(0, 0)$ is not a parabolic point for the associated Poincaré map (see sections 2.2 and 2.3 below). It is not essential to explain such a condition now; we just note that it is satisfied by small energy simple modes.

2.2. The Poincaré map

Let $\bar{u}$ be the simple mode solution of (2.1). Let us consider the open set $\mathcal{U} \subseteq \mathbb{R}^2$ defined by

$$\mathcal{U} := \left\{(x, y) \in \mathbb{R}^2 : H(0, 0, x, \sqrt{\nu m_0}\, y) < H(u_0, 0, 0, 0)\right\}. \tag{2.2}$$

For every $(x, y) \in \mathcal{U}$, let $\alpha(x, y) > 0$ be the unique positive number such that

$$H(\alpha(x, y), 0, x, \sqrt{\nu m_0}\, y) = H(u_0, 0, 0, 0).$$

Let $(u(t), v(t))$ be the solution of system (1.4) with initial data

$$u(0) = \alpha(x, y) \qquad u'(0) = 0 \qquad v(0) = x \qquad v'(0) = \sqrt{\nu m_0}\, y.$$

Finally, let $T := T(x, y)$ be the smallest $t > 0$ such that $u'(t) = 0$ and $u(t) > 0$. The existence of such a T is classical up to restricting $\mathcal{U}$; on the other hand, in our case, writing equation (2.1) in polar coordinates, the interested reader can verify that such a T exists for every $(x, y) \in \mathcal{U}$, and is $\leqslant 2\pi\mu^{-1}$, where $\mu := \min\{1, \inf\{m(r) : r \geqslant 0\}\}$.

The Poincaré map $P_{\bar{u}} : \mathcal{U} \to \mathcal{U}$, relative to the simple mode $\bar{u}$, is defined by

$$P_{\bar{u}}(x, y) := \left(v(T), (\nu m_0)^{-1/2} v'(T)\right).$$

We point out that both v and T depend on (x, y). The coefficient νm_0 has been introduced only to simplify calculations.

When $(x, y) = (0, 0)$, then $u(t) = \bar{u}(t)$ and $v(t) = 0$ for every $t \in \mathbb{R}$. It follows that $P_{\bar{u}}(0, 0) = (0, 0)$, i.e. $(0, 0)$ is a fixed point of the Poincaré map.

For the convenience of the reader, we give here a heuristic description of the Poincaré map.

Let us consider in the phase space $\mathbb{R}^4$ the periodic orbit $\bar{u}$, its initial datum $(u_0, 0, 0, 0)$, and its energy level $\mathcal{H}_{\bar{u}}$. The tangent vector to the periodic orbit in the initial point is, up to constants, the vector $(0, 1, 0, 0)$. If we intersect $\mathcal{H}_{\bar{u}}$ with the hyperplane in $\mathbb{R}^4$ through $(u_0, 0, 0, 0)$ and orthogonal to $(0, 1, 0, 0)$ (i.e. the hyperplane orthogonal to the orbit in its initial point), we obtain a set whose connected component containing $(u_0, 0, 0, 0)$ is

$$\mathcal{O}_{\bar{u}} := \left\{(\alpha(x, y), 0, x, \sqrt{\nu m_0}\, y) : (x, y) \in \mathcal{U}\right\}.$$

In this way $\mathcal{U}$ is in one-to-one correspondence with $\mathcal{O}_{\bar{u}}$, a subset of $\mathcal{H}_{\bar{u}}$ orthogonal to the orbit in its initial point.

Now for every point $Q \in \mathcal{O}_{\bar{u}}$, let us consider the solution of system (1.4) which lies in Q at $t = 0$. Such a solution meets $\mathcal{O}_{\bar{u}}$ once again at time $T(Q) > 0$ in a point $P(Q)$. The map $Q \to P(Q)$ is the Poincaré map read in $\mathcal{O}_{\bar{u}}$.

Definition 2.4. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an open set containing (0, 0), and let $P : \mathcal{U} \to \mathcal{U}$ be a map such that $P(0, 0) = (0, 0)$. The fixed point (0, 0) is said to be *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(x, y) \in \mathcal{U} \qquad \|(x, y)\| < \delta \quad \Longrightarrow \quad \|P^n(x, y)\| < \varepsilon \qquad \forall n \in \mathbb{N}$$

where P^n denotes the nth iteration of P.

It is heuristically clear that the stability of $\bar{u}$ as a periodic solution is related to the stability of (0, 0) as a fixed point of $P_{\bar{u}}$. These relations are explicitly stated in theorem 2.6 below.

2.3. KAM theory for planar maps

The stability of planar maps has been studied over the last 40 years. In this subsection we summarize the basic results we need in the following. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an open set containing (0, 0), and let $P : \mathcal{U} \to \mathcal{U}$. The theory of planar maps has been developed for very general maps P; however, we state the results under suitable assumptions which allow one to simplify some notation, and are trivially satisfied in our case. Therefore, let us assume that:

P1 $P \in C^5(\mathcal{U}, \mathcal{U})$ and $P(0, 0) = (0, 0)$;
P2 P is area-preserving;
P3 if $P(x, y) = (a, b)$, then $P(a, -b) = (x, -y)$;
P4 $P(-x, -y) = -P(x, y)$.

The first object to look at in order to study the stability of the fixed point (0, 0) is the differential of P at (0, 0), which we denote by L. It is well known that the canonical form of L is one of the following three:

- $$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$
  for some $\lambda \in \mathbb{R}$, $|\lambda| > 1$. In this case (0, 0) is said to be *hyperbolic* and it is *unstable*.
- $$\begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$$
  for some $a \neq 0$. In this case (0, 0) is said to be *parabolic*. The map L is unstable, but nothing can be said about P. However, we will not find this degenerate case in this paper.
- R_ω for some $\omega \in \mathbb{R}$. In this case (0, 0) is said to be *elliptic*. The map L is stable, but this is in general not enough to guarantee the stability of P.

Therefore, L gives only necessary conditions for stability (i.e. non-hyperbolicity). KAM theory provides sufficient conditions in the case of elliptic fixed points. To describe such conditions, it is better to write P in polar coordinates up to terms of order three. If we choose coordinates where L is written in the canonical form of a rotation, then, in the corresponding polar coordinates, P becomes

$$P \begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} \rho + a(\theta)\rho^3 \\ \theta - \omega + b(\theta)\rho^2 \end{pmatrix} + \mathrm{o}(\rho^3)$$

where ω is the same as in the linear term L, and $a(\theta)$ and $b(\theta)$ are trigonometric polynomials of degree four. The absence of even powers of ρ in the first component, and of odd powers of ρ in the second component, is due to (P4). Finally, we set

$$\gamma(P) := \frac{1}{2\pi}\int_0^{2\pi} b(\theta)\,\mathrm{d}\theta. \tag{2.3}$$

Then we have the following KAM result [4, 7, 9].

Theorem 2.5. *Let $P : \mathcal{U} \to \mathcal{U}$ be a planar map satisfying (P1)–(P4). Let $(0, 0)$ be an elliptic fixed point, and let ω and γ be defined as above.*

Let us assume that

(KAM 1) $\mathrm{e}^{ki\omega} \neq 1$ *for every* $k \in \{1, 2, 3, 4\}$*;*
(KAM 2) $\gamma(P) \neq 0$.

Then $(0, 0)$ is stable for P according to definition 2.4.

The following result relates stability, Poincaré maps and KAM theory [4, 7, 9]. It is the fundamental tool in our analysis.

Theorem 2.6. *Let $\bar{u}$ be a simple mode of system (1.4), and let $P_{\bar{u}}$ be the associated Poincaré map. Then:*

- *$\bar{u}$ is* linearly *stable if and only if $(0, 0)$ is an elliptic fixed point of $P_{\bar{u}}$;*
- *$\bar{u}$ is* isoenergetically orbitally *stable if and only if $(0, 0)$ is a stable fixed point of the Poincaré map $P_{\bar{u}}$;*
- *if $(0, 0)$ is an elliptic fixed point of $P_{\bar{u}}$, and $P_{\bar{u}}$ satisfies (KAM 1) and (KAM 2), then $\bar{u}$ is* orbitally *stable.*

Thanks to theorems 2.5 and 2.6, the orbital stability of a periodic solution in the four-dimensional space can be proved by verifying that a planar map satisfies two algebraic conditions.

3. Statements

In this paper we consider the simple modes u_ε of system (1.4) which solve the problem

$$u_\varepsilon''(t) + m(u_\varepsilon^2(t))u_\varepsilon(t) = 0 \qquad u_\varepsilon(0) = \varepsilon > 0 \qquad u_\varepsilon'(0) = 0. \tag{3.1}$$

Let us recall once more that we can assume $u_\varepsilon(0) > 0$ and $u_\varepsilon'(0) = 0$ because $u_\varepsilon(t)$ is a periodic function.

We also remark that the smallness of ε is equivalent to the smallness of the energy of u_ε.

Let us denote by $P_\varepsilon : \mathcal{U}_\varepsilon \to \mathcal{U}_\varepsilon$ the Poincaré map associated with u_ε as in section 2.2, and by L_ε its differential at the fixed point $(0, 0)$.

In the next result we sum up the main properties of P_ε and L_ε.

Theorem 3.0. *For every $\varepsilon > 0$, let P_ε and L_ε be as above. Then*

(1) P_ε satisfies (P1)–(P4);
(2) $\det L_\varepsilon = 1$*;*
(3) if L_ε^{ij} are the entries of L_ε, then $L_\varepsilon^{11} = L_\varepsilon^{22}$.

We do not give a proof of such properties here, since they are well known in the literature. We only note that (P2) follows from the Hamiltonian character of the system, (P3) is a consequence of reversibility, while (P4) is a consequence of the following fact: if $(u(t), v(t))$ is a solution of (1.4), then $(-u(t), -v(t))$ is also a solution; finally (2) and (3) are once again consequences of reversibility (for details, see the proof of [2, lemma 3.3]).

The main result of this paper (theorem 1.1) is that u_ε is orbitally stable if ε is small enough. Thanks to theorems 2.6 and 2.5, the main result will be proved if we show that P_ε satisfies assumptions (KAM 1) and (KAM 2) of theorem 2.5.

Assumption (KAM 1) follows from statements (1)–(3) of the following result, where the behaviour of L_ε for small ε is considered.

Theorem 3.1. *Let $\nu \neq 1$ be a positive real number. For each $\varepsilon > 0$, let L_ε be the differential in $(0, 0)$ of the Poincaré map P_ε associated with the simple mode u_ε.*

Then there exist $\varepsilon_1 > 0$, $\omega : (0, \varepsilon_1) \to \mathbb{R}$, and $\delta : (0, \varepsilon_1) \to (0, +\infty)$ such that

(1) for every $\varepsilon \in (0, \varepsilon_1)$ the eigenvalues of L_ε are $\left\{e^{\pm i\omega(\varepsilon)}\right\}$;
(2) $\omega(\varepsilon) \to 2\pi\sqrt{\nu}$ as $\varepsilon \to 0^+$;
(3) $\omega(\varepsilon) \neq 2\pi\sqrt{\nu}$ for ε small enough;
(4) setting

$$D(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \delta(\varepsilon) \end{pmatrix}$$

we have that

$$[D(\varepsilon)]^{-1} L_\varepsilon D(\varepsilon) = R_{\omega(\varepsilon)}$$

(5) $\delta(\varepsilon) \to 1$ as $\varepsilon \to 0^+$.

Statements (2) and (3) prevent $e^{i\omega(\varepsilon)}$ from being a kth root of 1 for $k \in \{1, 2, 3, 4\}$ and ε small. Indeed, if $e^{2\pi\sqrt{\nu}ki} \neq 1$ for $k \in \{1, 2, 3, 4\}$, then by (2) the same holds true for $e^{\omega(\varepsilon)ki}$, provided that ε is small enough; if in contrast $e^{2\pi\sqrt{\nu}i}$ is a kth root of 1 for some $k \in \{1, 2, 3, 4\}$, then for ε small $e^{\omega(\varepsilon)i}$ is not because of (3). This shows, in particular, that $(0, 0)$ is an elliptic fixed point of L_ε for ε small.

Statements (4) and (5) say that L_ε can be written in the canonical form by a diagonal matrix $D(\varepsilon)$ which approaches the identity as $\varepsilon \to 0^+$. These last two properties of L_ε will be fundamental in the proof of theorem 3.3 below.

Remark 3.2. The differential L_ε has already been studied in the mathematical literature. In particular:

- Dickey [3] proved statements (1) and (2) and statement (3) in the case where $\nu = n^2$ for some integer $n > 1$. In such a way he proved the linear stability of u_ε for ε small.
- Cazenave and Weissler [2] proved (under suitable assumptions on m) that there exists a non-empty set $A \subseteq (0, +\infty)$ such that, if $\nu \in A$ and ε is *large* enough, then $(0, 0)$ is a *hyperbolic* fixed point of P_ε, and therefore the simple mode u_ε is *unstable*.

The following result implies that P_ε satisfies assumption (KAM 2) for ε small.

Theorem 3.3. *Let P_ε be the Poincaré map associated with the simple mode u_ε, and let $\gamma_\varepsilon := \gamma(P_\varepsilon)$ be as formula (2.3).*

Then

$$\lim_{\varepsilon \to 0^+} \gamma_\varepsilon = -\frac{m_0'}{m_0}\frac{\pi}{2}\sqrt{\nu}.$$

In particular, $\gamma_\varepsilon \neq 0$ for ε small enough.

We have therefore reduced the proof of theorem 1.1 to the proof of theorems 3.1 and 3.3.

3.1. Strategy of the proofs

For the convenience of the reader, we sketch in this subsection the guidelines of the proof of theorems 3.1 and 3.3. The full details are given in section 4.

The proof of theorem 3.1 is divided into two parts.

- *Part 1.* For ε fixed, we write $L_\varepsilon(x, y)$ in terms of x and y, following [2, section 2].
- *Part 2.* We consider the behaviour of L_ε as $\varepsilon \to 0^+$. We prove, in particular, that
$$L_\varepsilon = R_{\omega_0} + \varepsilon^2 B + \mathrm{o}(\varepsilon^2)$$
where $\omega_0 = 2\pi\sqrt{\nu}$, and B is a matrix which we compute in terms of ν, m_0, m_0'. At this point, theorem 3.1 follows from the properties of B by a general linear algebra result (proposition 4.1).

The strategy of the proof of theorem 3.3 is analogous, but we need the expansion of P_ε near (0, 0) up to terms of order three.

- *Part 1.* For ε fixed, we write P_ε in polar coordinates (since the constant γ_ε we need to compute has a simple expression in polar coordinates). We determine (as solutions of suitable Cauchy problems) functions α_1, α_3, β_0, β_2 such that, in a neighbourhood of (0, 0),
$$P_\varepsilon\begin{pmatrix}\rho\\ \theta\end{pmatrix} = \begin{pmatrix}\alpha_1(\varepsilon,\theta)\rho\\ \beta_0(\varepsilon,\theta)\end{pmatrix} + \begin{pmatrix}\alpha_3(\varepsilon,\theta)\rho^3\\ \beta_2(\varepsilon,\theta)\rho^2\end{pmatrix} + \mathrm{o}(\rho^3)$$
where the first summand is simply L_ε written in polar coordinates (when a rotation is not in canonical form, it is not so good in polar coordinates!). If we now change variables in order to put L_ε in the canonical form (using what in Cartesian coordinates is the change of variables given by the matrix $D(\varepsilon)$ of theorem 3.1), then P_ε takes the form (with an abuse of notation, we also denote by ρ, θ the new coordinates):
$$P_\varepsilon\begin{pmatrix}\rho\\ \theta\end{pmatrix} = \begin{pmatrix}\rho\\ \theta-\omega(\varepsilon)\end{pmatrix} + \begin{pmatrix}a(\varepsilon,\theta)\rho^3\\ b(\varepsilon,\theta)\rho^2\end{pmatrix} + \mathrm{o}(\rho^3).$$
In this representation the first summand looks more familiar (a clockwise rotation by $\omega(\varepsilon)$) and moreover (cf (2.3))
$$\gamma_\varepsilon = \frac{1}{2\pi}\int_0^{2\pi} b(\varepsilon,\theta)\,\mathrm{d}\theta.$$
- *Part 2.* We now consider the behaviour of γ_ε as $\varepsilon \to 0^+$. Unfortunately, we do not have a good expression for $b(\varepsilon,\theta)$. However, we prove that the limit of $b(\varepsilon,\theta)$ coincides with the limit of $\beta_2(\varepsilon,\theta)$, hence
$$\lim_{\varepsilon\to 0^+}\gamma_\varepsilon = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_0^{2\pi} b(\varepsilon,\theta)\,\mathrm{d}\theta = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_0^{2\pi}\beta_2(\varepsilon,\theta)\,\mathrm{d}\theta.$$
The last equality, proved formally in section 4.4.7, can be justified heuristically as follows: b is obtained from β_2 by changing coordinates via $D(\varepsilon)$. By statement (5) of theorem 3.1 we know that $D(\varepsilon)$ tends to the identity as $\varepsilon \to 0^+$, and so the asymptotic behaviour of b and β_2 is the same.
Therefore, in order to prove theorem 3.3 we only need to compute the limit of the function $\beta_2(\varepsilon,\theta)$ found in part 1, and then compute the average in $[0, 2\pi]$ of this limit.

Remark 3.4. We observe that considering the limit of P_ε as $\varepsilon \to 0^+$ makes no sense, since the open sets $\mathcal{U}_\varepsilon$ where P_ε is defined shrink to the point (0, 0). What makes sense is considering the limit of the linear and the cubic term in the expansion of P_ε (they are defined on the whole plane). This is exactly what we do in the proofs of theorems 3.1 and 3.3.

4. Proofs

In this section we prove theorems 3.1 and 3.3, according to the strategy described in section 3.1. In all the proofs, we need expansions of solutions of Cauchy problems depending on some small parameter. We will always work formally as follows. Assume that the Cauchy problem is

$$Z' = F(Z, \mu) \qquad Z(0) = \Phi(\mu) \tag{4.1}$$

where μ is the small parameter, and $Z(t) \in \mathbb{R}^k$ is the unknown. Then we look for an expansion like

$$Z(t) = Z_0(t) + Z_1(t)\mu + Z_2(t)\mu^2 + \cdots + Z_h(t)\mu^h + \mathrm{o}(\mu^h). \tag{4.2}$$

We replace Z in (4.1) with this expression, and using the Taylor formula, we also write $F(Z, \mu)$ and $\Phi(\mu)$ as polynomials of degree h in μ (in the first case the coefficients depend on $Z_0, Z_1, \ldots, Z_h$) plus $\mathrm{o}(\mu^h)$. Finally, considering the coefficients of $\mu^0, \mu^1, \ldots, \mu^h$, we find the Cauchy problems solved by $Z_0, Z_1, \ldots, Z_h$.

It is well known that, if F and Φ are smooth enough, then this procedure can be rigorously justified, and that (4.2) turns out to be uniform on bounded time intervals.

In order to avoid terms which are not useful in writing the expansion (4.2), we always omit from the beginning the terms which *a posteriori* would turn out to be zero.

4.1. Proof of theorem 3.1, part 1

In this first part of the proof, ε and the simple mode u_ε are considered fixed. Let P_ε be the Poincaré map associated with u_ε, and let L_ε be its differential in $(0, 0)$. Then the linear operator $L_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ can be characterized in the following way.

Given $(x, y) \in \mathbb{R}^2$, let $v_\varepsilon(t)$ be the solution of the linear problem

$$v_\varepsilon''(t) + \nu\, m(u_\varepsilon^2(t))v_\varepsilon(t) = 0 \qquad v_\varepsilon(0) = x \qquad v_\varepsilon'(0) = \sqrt{\nu m_0}\, y. \tag{4.3}$$

This problem is the linearization of the second equation of system (1.4). Then we have that

$$L_\varepsilon(x, y) := \big(v_\varepsilon(\tau_\varepsilon), (\nu m_0)^{-1/2} v_\varepsilon'(\tau_\varepsilon)\big)$$

where τ_ε is the period of u_ε. We point out that τ_ε depends only on ε, while v_ε depends on x, y and ε.

We do not give the proof of this characterization, since it is completely analogous to the proof of [2, proposition 2.1]. However, L_ε is the first term in the Taylor expansion of P_ε. In the proof of theorem 3.3, we find the first three terms in the Taylor expansion of P_ε, and then we focus our attention on the term of order three. The interested reader can verify that the term of order one found in section 4.4 is exactly L_ε.

4.2. Proof of theorem 3.1, part 2

We now consider the asymptotic behaviour of L_ε as $\varepsilon \to 0^+$. The fundamental tool is the following linear algebra result.

Proposition 4.1. *Let $\varepsilon_0 > 0$, and let $A : (0, \varepsilon_0) \to M_{2\times 2}$. Let us assume that*

(a) $\det A(\varepsilon) = 1$ *for every* $\varepsilon \in (0, \varepsilon_0)$*;*
(b) $a_{11}(\varepsilon) = a_{22}(\varepsilon)$ *for every* $\varepsilon \in (0, \varepsilon_0)$*;*

(c) *there exists* $\omega_0 \in \mathbb{R}$, *and* $B \in M_{2\times 2}$ *such that, for* $\varepsilon \to 0^+$,

$$A(\varepsilon) = R_{\omega_0} + \varepsilon^2 B + o(\varepsilon^2);$$

(d) ω_0 *and* B *satisfy one of the following:*

1. $\omega_0 \neq k\pi$ *for every* $k \in \mathbb{Z}$, *and* $\operatorname{Tr} B \neq 0$;
2. $\omega_0 = k\pi$ *for some* $k \in \mathbb{Z}$, $\operatorname{Tr} B = 0$, *and* $b_{12} = -b_{21} \neq 0$.

Then there exist $\varepsilon_1 \in (0, \varepsilon_0)$, $\omega : (0, \varepsilon_1) \to \mathbb{R}$, *and* $\delta : (0, \varepsilon_1) \to (0, +\infty)$ *such that*

(1) *for every* $\varepsilon \in (0, \varepsilon_1)$ *the eigenvalues of* $A(\varepsilon)$ *are* $\left\{e^{\pm i\omega(\varepsilon)}\right\}$;
(2) $\omega(\varepsilon) \to \omega_0$ *as* $\varepsilon \to 0^+$;
(3) $\omega(\varepsilon) \neq \omega_0$ *for* ε *small enough;*
(4) *setting*

$$D(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \delta(\varepsilon) \end{pmatrix}$$

we have that

$$[D(\varepsilon)]^{-1} A(\varepsilon) D(\varepsilon) = R_{\omega(\varepsilon)}$$

(5) $\delta(\varepsilon) \to 1$ *as* $\varepsilon \to 0^+$.

Proof. In order to prove statements (1)–(3) it is enough to show that

C1 $|\operatorname{Tr} A(\varepsilon)| < 2$ for ε small enough;
C2 $\operatorname{Tr} A(\varepsilon) \neq 2\cos\omega_0$ for ε small enough.

Indeed, by (a) the eigenvalues of $A(\varepsilon)$ are either $\left\{\lambda, \lambda^{-1}\right\}$ for some $\lambda \in \mathbb{R}\backslash\{0\}$, or $\left\{e^{\pm i\omega}\right\}$ for some $\omega \in \mathbb{R}$. However, in the first case we would have that

$$|\operatorname{Tr} A(\varepsilon)| = \left|\lambda + \lambda^{-1}\right| \geqslant 2$$

which contradicts (C1). This proves statement (1). At this point we have that

$$\operatorname{Tr} A(\varepsilon) = e^{i\omega(\varepsilon)} + e^{-i\omega(\varepsilon)} = 2\cos\omega(\varepsilon) \tag{4.4}$$

and since

$$\lim_{\varepsilon \to 0^+} \operatorname{Tr} A(\varepsilon) = \operatorname{Tr} R_{\omega_0} = 2\cos\omega_0$$

then it is clear that we can choose $\omega(\varepsilon)$ in such a way that (2) is satisfied.

Statement (3) follows trivially from (C2) and (4.4).

At this point statements (4) and (5) follow with

$$\delta(\varepsilon) = \frac{\sin\omega(\varepsilon)}{a_{12}(\varepsilon)} \tag{4.5}$$

provided that we prove that

C3 $a_{12}(\varepsilon) \neq 0$ for ε small enough;
C4 the limit of the right-hand side of (4.5) is 1.

Indeed, by (C3) and (C4), $\delta(\varepsilon)$ is well defined and $\neq 0$ for ε small, so that $D(\varepsilon)$ is invertible. With simple calculations it turns out that

$$[D(\varepsilon)]^{-1} A(\varepsilon) D(\varepsilon) = \begin{pmatrix} a_{11}(\varepsilon) & \sin\omega(\varepsilon) \\ \dfrac{a_{12}(\varepsilon)a_{21}(\varepsilon)}{\sin\omega(\varepsilon)} & a_{22}(\varepsilon) \end{pmatrix}. \tag{4.6}$$

Now from (4.4) and assumption (b) we have that $a_{11}(\varepsilon) = a_{22}(\varepsilon) = \cos\omega(\varepsilon)$. Moreover, from (a) it follows that

$$1 = \det A(\varepsilon) = a_{11}(\varepsilon)a_{22}(\varepsilon) - a_{12}(\varepsilon)a_{21}(\varepsilon) = \cos^2\omega(\varepsilon) - a_{12}(\varepsilon)a_{21}(\varepsilon)$$

hence

$$a_{12}(\varepsilon)a_{21}(\varepsilon) = \cos^2\omega(\varepsilon) - 1 = -\sin^2\omega(\varepsilon).$$

This proves that the right-hand side of (4.6) is the rotation matrix $R_{\omega(\varepsilon)}$. Finally, statement (5) is exactly (C4).

In order to prove (C1)–(C4) we distinguish two cases.

Case 1. $\omega_0 \neq k\pi$ for every $k \in \mathbb{Z}$. In this case we have that

$$\lim_{\varepsilon\to 0^+} |\mathrm{Tr}\, A(\varepsilon)| = \left|\mathrm{Tr}\, R_{\omega_0}\right| = |2\cos\omega_0| < 2.$$

This proves (C1). Moreover, from (c) we have that

$$\mathrm{Tr}\, A(\varepsilon) = \mathrm{Tr}\, R_{\omega_0} + \varepsilon^2\, \mathrm{Tr}\, B + \mathrm{o}(\varepsilon^2)$$

so that (C2) follows from assumption (d-1).

Using (c) once again we see that $a_{12}(\varepsilon) \to \sin\omega_0 \neq 0$, and this proves (C3). Finally, (C4) is satisfied since both the numerator and the denominator in (4.5) tend to $\sin\omega_0 \neq 0$.

Case 2. $\omega_0 = k\pi$ for some $k \in \mathbb{Z}$. Let us assume that k is even, hence R_{ω_0} is the identity (a similar argument works when k is odd). Let us look at expansion (c). By (b) it follows that $b_{11} = b_{22}$, hence by assumption (d-2) $b_{11} = b_{22} = 0$. Now let $r_{11}, r_{12}, r_{21} : (0, \varepsilon_0) \to \mathbb{R}$ be functions (infinitesimal as $\varepsilon \to 0^+$) such that

$$A(\varepsilon) = \begin{pmatrix} 1 + \varepsilon^2 r_{11}(\varepsilon) & b_{12}\varepsilon^2 + \varepsilon^2 r_{12}(\varepsilon) \\ b_{21}\varepsilon^2 + \varepsilon^2 r_{21}(\varepsilon) & 1 + \varepsilon^2 r_{11}(\varepsilon) \end{pmatrix}.$$

From (a) it follows that

$$1 + 2\varepsilon^2 r_{11}(\varepsilon) - \varepsilon^4 b_{12}b_{21} + \mathrm{o}(\varepsilon^4) = 1$$

so that

$$\frac{2r_{11}(\varepsilon)}{\varepsilon^2} - b_{12}b_{21} + \frac{\mathrm{o}(\varepsilon^4)}{\varepsilon^4} = 0.$$

By (d-2) we therefore have that

$$\lim_{\varepsilon\to 0^+} \frac{r_{11}(\varepsilon)}{\varepsilon^2} = \frac{b_{12}b_{21}}{2} = -\frac{b_{12}^2}{2} < 0. \tag{4.7}$$

This proves that $r_{11}(\varepsilon) < 0$ for ε small enough. Since $\mathrm{Tr}\, A(\varepsilon) = 2 + 2\varepsilon^2 r_{11}(\varepsilon)$, both (C1) and (C2) are proved. Since $b_{12} \neq 0$, and $a_{12}(\varepsilon) = \varepsilon^2(b_{12} + r_{12}(\varepsilon))$, then (C3) is also proved.

In order to prove (C4) we first note that in this case we can choose $\omega(\varepsilon)$ in such a way that $\sin\omega(\varepsilon)$ has the same sign as b_{12}. Then recalling that $a_{11}(\varepsilon) = \cos\omega(\varepsilon)$, we have that

$$\lim_{\varepsilon\to 0^+} \frac{\sin\omega(\varepsilon)}{a_{12}(\varepsilon)} = \lim_{\varepsilon\to 0^+} \frac{\sqrt{1-a_{11}^2(\varepsilon)}}{|a_{12}(\varepsilon)|} = \lim_{\varepsilon\to 0^+} \frac{\sqrt{1-(1+\varepsilon^2 r_{11}(\varepsilon))^2}}{\varepsilon^2\,|b_{12}+r_{12}(\varepsilon)|}$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{|b_{12}+r_{12}(\varepsilon)|}\sqrt{-2\frac{r_{11}(\varepsilon)}{\varepsilon^2}+\frac{\mathrm{o}(\varepsilon^4)}{\varepsilon^4}} = \frac{\sqrt{b_{12}^2}}{|b_{12}|} = 1.$$

This proves (C4) in the second case. □

In the following we prove that L_ε satisfies the assumptions of proposition 4.1. To this end, we consider the asymptotic behaviour of all the quantities involved in the definition of L_ε.

4.2.1. Asymptotic behaviour of τ_ε. Let τ_ε be the period of the simple mode u_ε. Then, as $\varepsilon \to 0^+$,

$$\tau_\varepsilon = \frac{2\pi}{\sqrt{m_0}} - \frac{3\pi}{4}\frac{m_0'}{(m_0)^{3/2}}\varepsilon^2 + \mathrm{o}(\varepsilon^2). \tag{4.8}$$

In order to compute τ_ε we recall that for a periodic solution which satisfies an energy equality

$$\left[u_\varepsilon'(t)\right]^2 + M(u_\varepsilon^2(t)) = M(\varepsilon^2)$$

the period is given by

$$\tau_\varepsilon = 4\int_0^\varepsilon \frac{\mathrm{d}x}{\sqrt{M(\varepsilon^2)-M(x^2)}} = 4\int_0^1 \frac{\varepsilon\,\mathrm{d}y}{\sqrt{M(\varepsilon^2)-M(\varepsilon^2 y^2)}}. \tag{4.9}$$

Computing the Taylor expansion of this integral is just an exercise in calculus, so we only sketch the main points.

Since $M(r) = m_0 r + m_0' r^2/2 + \mathrm{o}(r^2)$, then

$$M(\varepsilon^2) - M(\varepsilon^2 y^2) = m_0(1-y^2)\varepsilon^2\left\{1 + \frac{m_0'}{2m_0}(1+y^2)\varepsilon^2 + \mathrm{o}(\varepsilon^2)\right\}$$

hence

$$\frac{\varepsilon}{\sqrt{M(\varepsilon^2)-M(\varepsilon^2y^2)}} = \frac{1}{\sqrt{m_0}}\frac{1}{\sqrt{1-y^2}}\left\{1+\frac{m_0'}{2m_0}(1+y^2)\varepsilon^2+\mathrm{o}(\varepsilon^2)\right\}^{-1/2}$$

$$= \frac{1}{\sqrt{m_0}}\frac{1}{\sqrt{1-y^2}} - \frac{m_0'}{4(m_0)^{3/2}}\frac{1+y^2}{\sqrt{1-y^2}}\varepsilon^2 + \mathrm{o}(\varepsilon^2).$$

Substituting this expression in (4.9), and computing the integrals, we obtain (4.8).

4.2.2. Asymptotic behaviour of u_ε and $m(u_\varepsilon^2)$. As $\varepsilon \to 0^+$ we have that

$$u_\varepsilon(t) = \varepsilon\cos\left(\sqrt{m_0}\,t\right) + \mathrm{o}(\varepsilon) \tag{4.10}$$

$$m(u_\varepsilon^2(t)) = m_0 + m_0'\cos^2\left(\sqrt{m_0}\,t\right)\varepsilon^2 + \mathrm{o}(\varepsilon^2) \tag{4.11}$$

uniformly on every bounded time interval.

Indeed, setting $u_\varepsilon(t) = \varepsilon u_1(t) + \mathrm{o}(\varepsilon)$ in (3.1) we have

$$\varepsilon u_1'' + m(\mathrm{o}(\varepsilon))\varepsilon u_1 + \mathrm{o}(\varepsilon) = 0 \qquad \varepsilon u_1(0) + \mathrm{o}(\varepsilon) = \varepsilon \qquad \varepsilon u_1'(0) + \mathrm{o}(\varepsilon) = 0.$$

Since $m(\mathrm{o}(\varepsilon)) = m_0 + \mathrm{o}(\varepsilon)$, considering the coefficients of ε we find that u_1 solves

$$u_1'' + m_0 u_1 = 0 \qquad u_1(0) = 1 \qquad u_1'(0) = 0$$

hence $u_1(t) = \cos(\sqrt{m_0}\, t)$.

Expansion (4.11) follows from (4.10) and the Taylor expansion of m.

4.2.3. Polar coordinates for $v_\varepsilon(t)$. Let us write (4.3) as a first-order system. To this end we set

$$x_\varepsilon(t) = v_\varepsilon(t) \qquad y_\varepsilon(t) = (\nu m_0)^{-1/2} v_\varepsilon'(t)$$

so that (4.3) becomes

$$x_\varepsilon'(t) = \sqrt{\nu m_0}\; y_\varepsilon(t) \qquad y_\varepsilon'(t) = -(\nu m_0)^{-1/2}\, \nu\, m(u_\varepsilon^2(t))\, x_\varepsilon(t)$$

with initial data

$$x_\varepsilon(0) = x \qquad y_\varepsilon(0) = y.$$

If $(x, y) \neq (0, 0)$, then $(x_\varepsilon(t), y_\varepsilon(t)) \neq (0, 0)$ for every $t \in \mathbb{R}$. We can therefore study this system, introducing polar coordinates $\rho_\varepsilon(t), \theta_\varepsilon(t)$ such that

$$x_\varepsilon(t) = \rho_\varepsilon(t) \cos\theta_\varepsilon(t) \qquad y_\varepsilon(t) = \rho_\varepsilon(t) \sin\theta_\varepsilon(t).$$

In a standard way it turns out that ρ_ε and θ_ε solve the following system:

$$\rho_\varepsilon' = \sqrt{\nu m_0}\, \rho_\varepsilon \sin\theta_\varepsilon \cos\theta_\varepsilon \left\{1 - \frac{m(u_\varepsilon^2)}{m_0}\right\} \tag{4.12}$$

$$\theta_\varepsilon' = -\sqrt{\nu m_0} \left\{\sin^2\theta_\varepsilon + \frac{m(u_\varepsilon^2)}{m_0} \cos^2\theta_\varepsilon\right\} \tag{4.13}$$

with initial data

$$\rho_\varepsilon(0) = \rho \qquad \theta_\varepsilon(0) = \theta$$

such that $x = \rho\cos\theta$, $y = \rho\sin\theta$.

4.2.4. Asymptotic behaviour of ρ_ε and θ_ε. We look for functions $\rho_0, \rho_2, \theta_0, \theta_2$ such that

$$\rho_\varepsilon(t) = \rho_0(t) + \rho_2(t)\varepsilon^2 + \mathrm{o}(\varepsilon^2) \tag{4.14}$$

$$\theta_\varepsilon(t) = \theta_0(t) + \theta_2(t)\varepsilon^2 + \mathrm{o}(\varepsilon^2) \tag{4.15}$$

as $\varepsilon \to 0^+$, uniformly on every bounded time interval.

Indeed, from (4.11) we have that

$$1 - \frac{m(u_\varepsilon^2)}{m_0} = -\frac{m_0'}{m_0} \cos^2\left(\sqrt{m_0}\, t\right) \varepsilon^2 + \mathrm{o}(\varepsilon^2).$$

Using this expression, (4.14) and (4.15), in equation (4.12) we obtain that, up to $\mathrm{o}(\varepsilon^2)$,

$$\rho_0'(t) + \rho_2'(t)\varepsilon^2 = -\frac{m_0'}{m_0} \sqrt{\nu m_0} \cos^2\left(\sqrt{m_0}\, t\right) \rho_0 \sin\theta_0 \cos\theta_0\; \varepsilon^2.$$

Working in an analogous way with (4.13) we have that, up to $o(\varepsilon^2)$,

$$\theta_0'(t) + \theta_2'(t)\varepsilon^2 = -\sqrt{\nu m_0}\left\{1 + \frac{m_0'}{m_0}\cos^2\left(\sqrt{m_0}\,t\right)\cos^2\theta_0\ \varepsilon^2\right\}.$$

Considering the terms without ε in these two equations, we find that ρ_0 and θ_0 solve the following problems:

$$\rho_0' = 0 \qquad \rho_0(0) = \rho \tag{4.16}$$

$$\theta_0' = -\sqrt{\nu m_0} \qquad \theta_0(0) = \theta \tag{4.17}$$

while, considering the terms in ε^2, we find that ρ_2 and θ_2 solve the following problems:

$$\rho_2' = -\frac{m_0'}{m_0}\sqrt{\nu m_0}\cos^2(\sqrt{m_0}\,t)\rho_0\sin\theta_0\cos\theta_0 \qquad \rho_2(0) = 0 \tag{4.18}$$

$$\theta_2' = -\frac{m_0'}{m_0}\sqrt{\nu m_0}\cos^2(\sqrt{m_0}\,t)\cos^2\theta_0 \qquad \theta_2(0) = 0. \tag{4.19}$$

From (4.16) and (4.17) we easily obtain that

$$\rho_0(t) = \rho \qquad \theta_0(t) = \theta - \sqrt{\nu m_0}\,t. \tag{4.20}$$

Inserting (4.20) in the right-hand sides of (4.18) and (4.19), after some integrations we obtain that

$$\begin{aligned}\rho_2(t) = -\frac{m_0'}{m_0}\frac{\sqrt{\nu}}{16}\rho\Bigg\{&\frac{2}{\sqrt{\nu}}\cos\left(2\theta - 2\sqrt{\nu m_0}\,t\right) - 2\frac{2\nu-1}{\sqrt{\nu}(\nu-1)}\cos 2\theta\\ &+\frac{1}{\sqrt{\nu}-1}\cos\left(2\theta - 2\sqrt{\nu m_0}\,t + 2\sqrt{m_0}\,t\right)\\ &+\frac{1}{\sqrt{\nu}+1}\cos\left(2\theta - 2\sqrt{\nu m_0}\,t - 2\sqrt{m_0}\,t\right)\Bigg\}\end{aligned} \tag{4.21}$$

$$\begin{aligned}\theta_2(t) = -\frac{m_0'}{m_0}\frac{\sqrt{\nu}}{16}\Bigg\{&4\sqrt{m_0}\,t + 2\sin\left(2\sqrt{m_0}\,t\right) - \frac{2}{\sqrt{\nu}}\sin\left(2\theta - 2\sqrt{\nu m_0}\,t\right)\\ &-\frac{1}{\sqrt{\nu}-1}\sin\left(2\theta - 2\sqrt{\nu m_0}\,t + 2\sqrt{m_0}\,t\right) + 2\frac{2\nu-1}{\sqrt{\nu}(\nu-1)}\sin 2\theta\\ &-\frac{1}{\sqrt{\nu}+1}\sin\left(2\theta - 2\sqrt{\nu m_0}\,t - 2\sqrt{m_0}\,t\right)\Bigg\}.\end{aligned} \tag{4.22}$$

4.2.5. Asymptotic behaviour of L_ε. Setting $t = \tau_\varepsilon$ in (4.20)–(4.22) and using (4.8), after some calculations we obtain that, as $\varepsilon \to 0^+$,

$$\rho_0(\tau_\varepsilon) = \rho \qquad \theta_0(\tau_\varepsilon) = \theta - 2\pi\sqrt{\nu} + \frac{3\pi}{4}\frac{m_0'}{m_0}\sqrt{\nu}\,\varepsilon^2 + o(\varepsilon^2) \tag{4.23}$$

$$\rho_2(\tau_\varepsilon) = -\frac{m_0'}{8m_0}\frac{2\nu-1}{\nu-1}\rho\left\{\cos(4\pi\sqrt{\nu} - 2\theta) - \cos 2\theta\right\} + o(1) \tag{4.24}$$

$$\theta_2(\tau_\varepsilon) = -\frac{m_0'}{8m_0}\sqrt{\nu}\left\{4\pi + \frac{2\nu-1}{\sqrt{\nu}(\nu-1)}\left[\sin 2\theta - \sin(2\theta - 4\pi\sqrt{\nu})\right]\right\} + o(1). \tag{4.25}$$

We are now ready to study the asymptotic behaviour of $v_\varepsilon(\tau_\varepsilon)$. Indeed, by (4.14) and (4.15), we have that

$$\begin{aligned} v_\varepsilon(\tau_\varepsilon) &= \rho_\varepsilon(\tau_\varepsilon)\cos\theta_\varepsilon(\tau_\varepsilon) = \left[\rho_0(\tau_\varepsilon) + \varepsilon^2\rho_2(\tau_\varepsilon)\right]\cos\left[\theta_0(\tau_\varepsilon) + \varepsilon^2\theta_2(\tau_\varepsilon)\right] + \mathrm{o}(\varepsilon^2) \\ &= \rho_0(\tau_\varepsilon)\cos\theta_0(\tau_\varepsilon) + \{\rho_2(\tau_\varepsilon)\cos\theta_0(\tau_\varepsilon) - \rho_0(\tau_\varepsilon)\theta_2(\tau_\varepsilon)\sin\theta_0(\tau_\varepsilon)\}\,\varepsilon^2 + \mathrm{o}(\varepsilon^2) \end{aligned}$$

so that, using (4.23)–(4.25) and some trigonometry, we finally obtain that

$$\begin{aligned} v_\varepsilon(\tau_\varepsilon) &= \rho\cos(\theta - 2\pi\sqrt{\nu}) + \frac{m_0'}{4m_0}\rho \\ &\quad\times\left\{-\pi\sqrt{\nu}\sin(\theta - 2\pi\sqrt{\nu}) - \frac{2\nu-1}{\nu-1}\sin\theta\sin\left(2\pi\sqrt{\nu}\right)\right\}\varepsilon^2 + \mathrm{o}(\varepsilon^2). \end{aligned} \tag{4.26}$$

With analogous computations:

$$\begin{aligned} \frac{1}{\sqrt{\nu m_0}}v_\varepsilon'(\tau_\varepsilon) &= \rho\sin(\theta - 2\pi\sqrt{\nu}) - \frac{m_0'}{4m_0}\rho \\ &\quad\times\left\{-\pi\sqrt{\nu}\cos(\theta - 2\pi\sqrt{\nu}) + \frac{2\nu-1}{\nu-1}\cos\theta\sin\left(2\pi\sqrt{\nu}\right)\right\}\varepsilon^2 + \mathrm{o}(\varepsilon^2). \end{aligned} \tag{4.27}$$

Now let us denote by L_ε^{ij} the entries of the matrix L_ε. Then $(L_\varepsilon^{11}, L_\varepsilon^{21}) = \left(v_\varepsilon(\tau_\varepsilon), (\nu m_0)^{-1/2}v_\varepsilon'(\tau_\varepsilon)\right)$, where v_ε has initial data $x = 1$, $y = 0$, corresponding to $\rho = 1$, $\theta = 0$. From (4.26) and (4.27) we obtain that

$$L_\varepsilon^{11} = \cos(2\pi\sqrt{\nu}) + \frac{m_0'}{4m_0}\pi\sqrt{\nu}\sin(2\pi\sqrt{\nu})\varepsilon^2 + \mathrm{o}(\varepsilon^2)$$

$$L_\varepsilon^{21} = -\sin(2\pi\sqrt{\nu}) + \frac{m_0'}{4m_0}\left\{\pi\sqrt{\nu}\cos(2\pi\sqrt{\nu}) - \frac{2\nu-1}{\nu-1}\sin(2\pi\sqrt{\nu})\right\}\varepsilon^2 + \mathrm{o}(\varepsilon^2).$$

Making the same computations with initial data $x = 0$, $y = 1$, corresponding to $\rho = 1$, $\theta = \pi/2$, we find that $L_\varepsilon^{22} = L_\varepsilon^{11}$, and

$$L_\varepsilon^{12} = \sin(2\pi\sqrt{\nu}) + \frac{m_0'}{4m_0}\left\{-\pi\sqrt{\nu}\cos(2\pi\sqrt{\nu}) - \frac{2\nu-1}{\nu-1}\sin(2\pi\sqrt{\nu})\right\}\varepsilon^2 + \mathrm{o}(\varepsilon^2).$$

We have thus proved that

$$L_\varepsilon = R_{\omega_0} + \varepsilon^2 B + \mathrm{o}(\varepsilon^2)$$

where $\omega_0 = 2\pi\sqrt{\nu}$, and B is a matrix whose entries are

$$\begin{aligned} b_{11} = b_{22} &= \frac{m_0'}{4m_0}\pi\sqrt{\nu}\sin(2\pi\sqrt{\nu}) \\ b_{12} &= \frac{m_0'}{4m_0}\left\{-\pi\sqrt{\nu}\cos(2\pi\sqrt{\nu}) - \frac{2\nu-1}{\nu-1}\sin(2\pi\sqrt{\nu})\right\} \\ b_{21} &= \frac{m_0'}{4m_0}\left\{\pi\sqrt{\nu}\cos(2\pi\sqrt{\nu}) - \frac{2\nu-1}{\nu-1}\sin(2\pi\sqrt{\nu})\right\}. \end{aligned}$$

4.2.6. Properties of B. If $\omega_0 \neq k\pi$ for every $k \in \mathbb{Z}$ (i.e. $2\sqrt{\nu} \notin \mathbb{N}$) then $\mathrm{Tr}\,B \neq 0$. If $\omega_0 = k\pi$ for some $k \in \mathbb{Z}$, then

$$B = \pm\frac{m_0'}{m_0}\frac{\pi\sqrt{\nu}}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and therefore $\mathrm{Tr}\,B = 0$ and $b_{12} = -b_{21} \neq 0$.

In any case B satisfies assumption (d) of proposition 4.1.

4.2.7. Conclusion. From theorem 3.0 and the results of sections 4.2.5 and 4.2.6, we have that L_ε satisfies all the assumptions of proposition 4.1. Therefore, statements (1)–(5) of theorem 3.1 follow from the corresponding statements of proposition 4.1.

4.3. Proof of theorem 3.3, part 1

In this first part of the proof, ε will be fixed. We compute the first three terms in the Taylor expansion in a neighbourhood of $(0, 0)$ of the Poincaré map P_ε associated with the simple mode u_ε given in (3.1). In order to fix notation, we write once again the definition of P_ε, following section 2.2.

Let $\mathcal{U}_\varepsilon \subseteq \mathbb{R}^2$ be defined in analogy with (2.2). Given $(x, y) \in \mathcal{U}_\varepsilon$ we consider the solution of the system,

$$U'' + m(U^2 + V^2)U = 0 \qquad U(0) = \alpha \qquad U'(0) = 0 \tag{4.28}$$

$$V'' + \nu m(U^2 + V^2)V = 0 \qquad V(0) = x \qquad V'(0) = \sqrt{\nu m_0}\, y \tag{4.29}$$

where α is the positive solution of

$$m_0 y^2 + M(\alpha^2 + x^2) = M(\varepsilon^2). \tag{4.30}$$

Let T be the smallest $t > 0$ such that $U'(t) = 0$ and $U(t) > 0$. Then

$$P_\varepsilon(x, y) := \big(V(T), (\nu m_0)^{-1/2} V'(T)\big).$$

Since we plan to use polar coordinates we assume in the following that $x = \rho\cos\theta$, $y = \rho\sin\theta$.

Formally this definition is very similar to the definition of L_ε. However, the situation is much more complicated here, because α depends on ε, ρ, θ, hence U, V and T also depend on ε, ρ, θ.

We use capital letters to avoid confusion with the corresponding functions used in the study of the linear term. We also write $U(\varepsilon, \rho, \theta, t)$, $\alpha(\varepsilon, \rho, \theta)$, and so on, to recall the dependence on all of these variables. The symbol $'$ will always denote differentiation with respect to the time variable t.

In this first part of the proof we consider the asymptotic behaviour of these functions as $\rho \to 0^+$ (ε fixed). All the terms $o(\rho^k)$ we introduce are uniform on $\theta \in [0, 2\pi]$, and on t belonging to any bounded time interval (when the functions we develop depend on t).

4.3.1. Asymptotic behaviour of α. We prove that, as $\rho \to 0^+$,

$$\alpha(\varepsilon, \rho, \theta) = \varepsilon - \left\{\frac{\cos^2\theta}{2\varepsilon} + \frac{m_0}{2\varepsilon m(\varepsilon^2)}\sin^2\theta\right\}\rho^2 + o(\rho^2). \tag{4.31}$$

Indeed, if $\rho = 0$ then $\alpha = \varepsilon$. So we look for an expansion like

$$\alpha(\varepsilon, \rho, \theta) = \varepsilon + \alpha_2(\varepsilon, \theta)\rho^2 + o(\rho^2).$$

Inserting this expression in (4.30) and using the Taylor formula for $M(r)$ in $r = \varepsilon^2$, we obtain that

$$\begin{aligned} M(\varepsilon^2) &= m_0\rho^2\sin^2\theta + M(\varepsilon^2 + 2\alpha_2(\varepsilon, \theta)\varepsilon\rho^2 + \rho^2\cos^2\theta + o(\rho^2)) \\ &= m_0\rho^2\sin^2\theta + M(\varepsilon^2) + M'(\varepsilon^2)\left\{2\alpha_2(\varepsilon, \theta)\varepsilon + \cos^2\theta\right\}\rho^2 + o(\rho^2) \end{aligned}$$

from which (4.31) follows, recalling that m is the derivative of M.

4.3.2. Polar coordinates for V. We argue as in section 4.2.3. Setting

$$X(\varepsilon,\rho,\theta,t)=V(\varepsilon,\rho,\theta,t) \qquad Y(\varepsilon,\rho,\theta,t)=(\nu m_0)^{-1/2}V'(\varepsilon,\rho,\theta,t)$$

and using polar coordinates $R(\varepsilon,\rho,\theta,t)$, $\Theta(\varepsilon,\rho,\theta,t)$ such that $X=R\cos\Theta$, $Y=R\sin\Theta$, it turns out that R and Θ solve the following system:

$$R'=\sqrt{\nu m_0}\,R\,\sin\Theta\cos\Theta\left\{1-\frac{m(U^2+R^2\cos^2\Theta)}{m_0}\right\} \tag{4.32}$$

$$\Theta'=-\sqrt{\nu m_0}\left\{\sin^2\Theta+\frac{m(U^2+R^2\cos^2\Theta)}{m_0}\cos^2\Theta\right\} \tag{4.33}$$

with initial data

$$R(\varepsilon,\rho,\theta,0)=\rho \qquad \Theta(\varepsilon,\rho,\theta,0)=\theta.$$

4.3.3. Asymptotic behaviour of U, R, Θ. We look for functions U_0, U_2, R_1, R_3, Θ_0, Θ_2 such that, as $\rho\to 0^+$,

$$U(\varepsilon,\rho,\theta,t)=U_0(\varepsilon,\theta,t)+U_2(\varepsilon,\theta,t)\rho^2+\mathrm{o}(\rho^3) \tag{4.34}$$

$$R(\varepsilon,\rho,\theta,t)=R_1(\varepsilon,\theta,t)\rho+R_3(\varepsilon,\theta,t)\rho^3+\mathrm{o}(\rho^3) \tag{4.35}$$

$$\Theta(\varepsilon,\rho,\theta,t)=\Theta_0(\varepsilon,\theta,t)+\Theta_2(\varepsilon,\theta,t)\rho^2+\mathrm{o}(\rho^3). \tag{4.36}$$

Using these expansions, we have that

$$\begin{aligned} m(U^2+V^2)&=m\left(U_0^2+(2U_0U_2+R_1^2\cos^2\Theta_0)\rho^2+\mathrm{o}(\rho^3)\right)\\ &=m(U_0^2)+m'(U_0^2)(2U_0U_2+R_1^2\cos^2\Theta_0)\rho^2+\mathrm{o}(\rho^3). \end{aligned} \tag{4.37}$$

Setting (4.34), (4.37) and (4.31) in equation (4.28), and looking at the terms without ρ, we find that U_0 solves

$$U_0''+m(U_0^2)U_0=0 \qquad U_0(\varepsilon,\theta,0)=\varepsilon \qquad U_0'(\varepsilon,\theta,0)=0 \tag{4.38}$$

while, looking at the terms in ρ^2, we find that U_2 solves

$$U_2''+m(U_0^2)U_2+m'(U_0^2)\left(2U_0U_2+R_1^2\cos^2\Theta_0\right)U_0=0 \tag{4.39}$$

with initial data

$$U_2(\varepsilon,\theta,0)=-\left\{\frac{\cos^2\theta}{2\varepsilon}+\frac{m_0}{2\varepsilon\, m(\varepsilon^2)}\sin^2\theta\right\} \qquad U_2'(\varepsilon,\theta,0)=0. \tag{4.40}$$

From (4.38) we can see that U_0 is just the simple mode u_ε. In particular, it is independent of θ, and so from now on we write $U_0(\varepsilon,t)$ or $u_\varepsilon(t)$, instead of $U_0(\varepsilon,\theta,t)$.

Setting (4.35)–(4.37) in equation (4.32), and looking at the terms in ρ, we find that R_1 solves

$$R_1'=\sqrt{\nu m_0}\left\{1-\frac{m(U_0^2)}{m_0}\right\}R_1\cos\Theta_0\sin\Theta_0 \qquad R_1(\varepsilon,\theta,0)=1. \tag{4.41}$$

In an analogous way, looking at the terms without ρ in (4.33), we find that Θ_0 solves

$$\Theta_0'=-\sqrt{\nu m_0}\left\{\sin^2\Theta_0+\frac{m(U_0^2)}{m_0}\cos^2\Theta_0\right\} \qquad \Theta_0(\varepsilon,\theta,0)=\theta. \tag{4.42}$$

Finally, using in equation (4.32) expansions (4.35)–(4.37), and recalling that by Taylor's formula

$$\sin^2\Theta = \sin^2\Theta_0 + 2\rho^2\Theta_2\cos\Theta_0\sin\Theta_0 + \mathrm{o}(\rho^2)$$
$$\cos^2\Theta = \cos^2\Theta_0 - 2\rho^2\Theta_2\cos\Theta_0\sin\Theta_0 + \mathrm{o}(\rho^2)$$

looking at the terms in ρ^2, we find that Θ_2 solves

$$\Theta_2' = -\sqrt{\nu m_0}\left\{2\Theta_2\cos\Theta_0\sin\Theta_0\left[1-\frac{m(U_0^2)}{m_0}\right]\right.$$
$$\left.+\frac{m'(U_0^2)}{m_0}\cos^2\Theta_0\left[2U_0U_2 + R_1^2\cos^2\Theta_0\right]\right\} \qquad \Theta_2(\varepsilon,\theta,0) = 0. \tag{4.43}$$

We do not write the equation for R_3 because we do not need it in the following.

4.3.4. Asymptotic behaviour of T. We prove that, as $\rho \to 0^+$,

$$T(\varepsilon,\rho,\theta) = \tau_\varepsilon + T_2(\varepsilon,\theta)\rho^2 + \mathrm{o}(\rho^3) \tag{4.44}$$

where τ_ε is the period of the simple mode u_ε, and

$$T_2(\varepsilon,\theta) = \frac{U_2'(\varepsilon,\theta,\tau_\varepsilon)}{\varepsilon\, m(\varepsilon^2)}. \tag{4.45}$$

It is natural to look for an expansion such as (4.44) since for $\rho = 0$, U is exactly the simple mode u_ε. Replacing the expansions of U and T in the condition $U'(T) = 0$ we obtain that

$$\begin{aligned}0 &= U'(\varepsilon,\rho,\theta,T(\varepsilon,\rho,\theta))\\ &= U_0'(\varepsilon,T(\varepsilon,\rho,\theta)) + \rho^2 U_2'(\varepsilon,\theta,T(\varepsilon,\rho,\theta)) + \mathrm{o}(\rho^3)\\ &= U_0'(\varepsilon,\tau_\varepsilon) + \rho^2\left\{U_0''(\varepsilon,\tau_\varepsilon)T_2(\varepsilon,\theta) + U_2'(\varepsilon,\theta,\tau_\varepsilon)\right\} + \mathrm{o}(\rho^3).\end{aligned} \tag{4.46}$$

Since U_0 is the simple mode u_ε, and τ_ε is its period, then $U_0'(\varepsilon,\tau_\varepsilon) = U_0'(\varepsilon,0) = 0$, so that the first summand is zero. Moreover, by equation (4.38)

$$U_0''(\varepsilon,\tau_\varepsilon) = U_0''(\varepsilon,0) = -m\left(U_0^2(\varepsilon,0)\right)U_0(\varepsilon,0) = -\varepsilon\, m(\varepsilon^2).$$

Setting the coefficient of ρ^2 in (4.46) equal to zero, and using the last equality, we obtain (4.45). It is easy to see that with this choice the condition $U(T) > 0$ is also satisfied for ρ small.

4.3.5. Asymptotic behaviour of the Poincaré map. Using the expansions we have found in sections 4.3.3 and 4.3.4, we obtain that

$$\begin{aligned}\Theta(\varepsilon,\rho,\theta,T(\varepsilon,\rho,\theta)) &= \Theta_0(\varepsilon,\theta,T(\varepsilon,\rho,\theta)) + \Theta_2(\varepsilon,\theta,T(\varepsilon,\rho,\theta))\rho^2 + \mathrm{o}(\rho^3)\\ &= \Theta_0(\varepsilon,\theta,\tau_\varepsilon) + \left\{\Theta_0'(\varepsilon,\theta,\tau_\varepsilon)T_2(\varepsilon,\theta) + \Theta_2(\varepsilon,\theta,\tau_\varepsilon)\right\}\rho^2 + \mathrm{o}(\rho^3)\end{aligned}$$

and similarly

$$R(\varepsilon,\rho,\theta,T(\varepsilon,\rho,\theta)) = R_1(\varepsilon,\theta,\tau_\varepsilon)\rho + \left\{R_1'(\varepsilon,\theta,\tau_\varepsilon)T_2(\varepsilon,\theta) + R_3(\varepsilon,\theta,\tau_\varepsilon)\right\}\rho^3 + \mathrm{o}(\rho^3).$$

Therefore, in polar coordinates the Poincaré map is

$$P_\varepsilon\begin{pmatrix}\rho\\ \theta\end{pmatrix} = \begin{pmatrix}R(\varepsilon,\rho,\theta,T(\varepsilon,\rho,\theta))\\ \Theta(\varepsilon,\rho,\theta,T(\varepsilon,\rho,\theta))\end{pmatrix} = \begin{pmatrix}\alpha_1(\varepsilon,\theta)\rho\\ \beta_0(\varepsilon,\theta)\end{pmatrix} + \begin{pmatrix}\alpha_3(\varepsilon,\theta)\rho^3\\ \beta_2(\varepsilon,\theta)\rho^2\end{pmatrix} + \mathrm{o}(\rho^3)$$

where

$$\begin{aligned}\alpha_1(\varepsilon,\theta) &= R_1(\varepsilon,\theta,\tau_\varepsilon)\\ \alpha_3(\varepsilon,\theta) &= R_1'(\varepsilon,\theta,\tau_\varepsilon)T_2(\varepsilon,\theta)+R_3(\varepsilon,\theta,\tau_\varepsilon)\\ \beta_0(\varepsilon,\theta) &= \Theta_0(\varepsilon,\theta,\tau_\varepsilon)\\ \beta_2(\varepsilon,\theta) &= \Theta_0'(\varepsilon,\theta,\tau_\varepsilon)T_2(\varepsilon,\theta)+\Theta_2(\varepsilon,\theta,\tau_\varepsilon).\end{aligned} \tag{4.47}$$

4.4. *Proof of theorem 3.3, part 2*

In this second part of the proof we consider the asymptotic behaviour as $\varepsilon \to 0^+$ of the functions introduced in the first part. We are interested, in particular, in the limit of β_2, but this limit involves the limits of Θ_0, Θ_2, T_2, which in turn require the limit of R_1 and U_2. In the following, all the terms $\mathrm{o}(\varepsilon^k)$ are uniform on $\theta \in [0, 2\pi]$, and on t belonging to any bounded time interval when needed.

4.4.1. Asymptotic behaviour of τ_ε, U_0, $m(U_0^2)$. Since $U_0(\varepsilon, t) = u_\varepsilon(t)$ is the simple mode we are considering, and τ_ε is the period of u_ε, from (4.8), (4.10) and (4.11) we just recall that

$$\tau_\varepsilon = \frac{2\pi}{\sqrt{m_0}} - \frac{3\pi}{4}\frac{m_0'}{(m_0)^{3/2}}\varepsilon^2 + \mathrm{o}(\varepsilon^2) \tag{4.48}$$

$$U_0(\varepsilon,t) = \varepsilon\cos\left(\sqrt{m_0}\,t\right) + \mathrm{o}(\varepsilon) \tag{4.49}$$

$$m(U_0^2(\varepsilon,t)) = m_0 + m_0'\cos^2\left(\sqrt{m_0}\,t\right)\varepsilon^2 + \mathrm{o}(\varepsilon^2). \tag{4.50}$$

4.4.2. Asymptotic behaviour of R_1 and Θ_0. We have that, as $\varepsilon \to 0^+$,

$$R_1(\varepsilon,\theta,t) = 1 + \mathrm{o}(1) \tag{4.51}$$

$$\Theta_0(\varepsilon,\theta,t) = \theta - \sqrt{\nu m_0}\,t + \mathrm{o}(1) \tag{4.52}$$

$$\Theta_0'(\varepsilon,\theta,t) = -\sqrt{\nu m_0} + \mathrm{o}(1). \tag{4.53}$$

Indeed, passing to the limit in problems (4.41) and (4.42), it turns out that $R_1(\varepsilon,\theta,t)$ and $\Theta_0(\varepsilon,\theta,t)$ converge with their derivatives to functions $R_{1l}(\theta,t)$ and $\Theta_{0l}(\theta,t)$, which solves the following problems:

$$\begin{aligned} R_{1l}' &= 0 \qquad & R_{1l}(\theta,0) &= 1\\ \Theta_{0l}' &= -\sqrt{\nu m_0} \qquad & \Theta_{0l}(\theta,0) &= \theta \end{aligned}$$

so that (4.51)–(4.53) easily follow.

4.4.3. Asymptotic behaviour of U_2. We prove that, as $\varepsilon \to 0^+$,

$$U_2(\varepsilon,\theta,t) = -\frac{1}{2\varepsilon}\cos(\sqrt{m_0}\,t) + W_2(\theta,t)\varepsilon + \mathrm{o}(\varepsilon) \tag{4.54}$$

where $W_2(\theta, t)$ is equal to

$$\frac{m_0'}{64 m_0}\left\{19\cos(\sqrt{m_0}\,t) + 20\sqrt{m_0}\,t\sin(\sqrt{m_0}\,t) - 3\cos(3\sqrt{m_0}\,t)\right\} + \Psi(\theta,t) \tag{4.55}$$

and

$$\int_0^{2\pi}\Psi(\theta,t)\,\mathrm{d}\theta = \int_0^{2\pi}\Psi'(\theta,t)\,\mathrm{d}\theta = 0. \tag{4.56}$$

To this end, we look for an expansion like

$$U_2(\varepsilon,\theta,t)=\frac{1}{\varepsilon}W_1(\theta,t)+W_2(\theta,t)\varepsilon+\mathrm{o}(\varepsilon) \tag{4.57}$$

where the term in ε^{-1} is required by the initial datum (see (4.40))

$$\begin{aligned}U_2(\varepsilon,\theta,0)&=-\frac{1}{2\varepsilon}\left\{\cos^2\theta+\frac{m_0}{m_0+m_0'\,\varepsilon^2+\mathrm{o}(\varepsilon^2)}\sin^2\theta\right\}\\&=-\frac{1}{2\varepsilon}+\varepsilon\,\frac{m_0'}{2m_0}\sin^2\theta+\mathrm{o}(\varepsilon).\end{aligned}$$

Using (4.49), (4.51), (4.52) and (4.57), we see that

$$\begin{aligned}\left(2U_0U_2+R_1^2\cos^2\Theta_0\right)U_0&=\varepsilon\left\{2\cos\left(\sqrt{m_0}\,t\right)W_1(\theta,t)\right.\\&\quad\left.+\cos^2\left(\theta-\sqrt{\nu m_0}\,t\right)\right\}\cos\left(\sqrt{m_0}\,t\right)+\mathrm{o}(\varepsilon).\end{aligned}$$

Using this expression and (4.50) in equation (4.39), and considering the terms in ε^{-1}, we find that W_1 solves the following problem:

$$W_1''+m_0W_1=0\qquad W_1(\theta,0)=-\tfrac{1}{2}\qquad W_1'(\theta,0)=0 \tag{4.58}$$

while, considering the terms in ε, we find that W_2 solves

$$W_2''+m_0W_2+m_0'\left\{3\cos^2(\sqrt{m_0}\,t)W_1+\cos^2(\theta-\sqrt{\nu m_0}\,t)\cos(\sqrt{m_0}\,t)\right\}=0 \tag{4.59}$$

with initial data

$$W_2(\theta,0)=\frac{m_0'}{2m_0}\sin^2\theta\qquad W_2'(\theta,0)=0. \tag{4.60}$$

From (4.58) we have immediately that

$$W_1(\theta,t)=-\tfrac{1}{2}\cos(\sqrt{m_0}\,t).$$

Now it is possible to find W_2 explicitly by integrating the Cauchy problem (4.59) and (4.60), but this leads to cumbersome calculations. We prefer to set

$$\overline{W}_2(t):=\frac{1}{2\pi}\int_0^{2\pi}W_2(\theta,t)\,\mathrm{d}\theta$$

so that

$$W_2(\theta,t)=\overline{W}_2(t)+\Psi(\theta,t)$$

where Ψ satisfies (4.56). In order to compute $\overline{W}_2(t)$ we take the average on $[0,2\pi]$ of (4.59) and (4.60), and we find that $\overline{W}_2(t)$ is the solution of

$$\overline{W}_2''+m_0\overline{W}_2+m_0'\left\{-\tfrac{3}{2}\cos^3(\sqrt{m_0}\,t)+\tfrac{1}{2}\cos(\sqrt{m_0}\,t)\right\}=0$$

with initial data

$$\overline{W}_2(0)=\frac{m_0'}{4m_0}\qquad\overline{W}_2'(0)=0.$$

The solution of this Cauchy problem is written in (4.55).

4.4.4. Asymptotic behaviour of T_2. We prove that

$$\lim_{\varepsilon\to 0^+} T_2(\varepsilon,\theta) = \frac{m_0'}{(m_0)^{3/2}}\frac{\pi}{4} + \Phi(\theta) \tag{4.61}$$

where $\Phi : [0, 2\pi] \to \mathbb{R}$ is a periodic function whose average is zero.

Indeed, the denominator in (4.45) is

$$\varepsilon m(\varepsilon^2) = \varepsilon m_0 + \mathrm{o}(\varepsilon). \tag{4.62}$$

Now let us consider the numerator in (4.45). From (4.54) we have that

$$\begin{aligned} U_2'(\varepsilon,\theta,\tau_\varepsilon) &= \frac{\sqrt{m_0}}{2\varepsilon}\sin\left(\sqrt{m_0}\,\tau_\varepsilon\right) + \varepsilon W_2'(\theta,\tau_\varepsilon) + \mathrm{o}(\varepsilon) \\ &= \frac{\sqrt{m_0}}{2\varepsilon}\sin\left(2\pi - \frac{3\pi}{4}\frac{m_0'}{m_0}\varepsilon^2 + \mathrm{o}(\varepsilon^2)\right) + \varepsilon W_2'\left(\theta, \frac{2\pi}{\sqrt{m_0}}\right) + \mathrm{o}(\varepsilon) \\ &= \frac{\sqrt{m_0}}{2\varepsilon}\left(-\frac{3\pi}{4}\frac{m_0'}{m_0}\varepsilon^2 + \mathrm{o}(\varepsilon^2)\right) + \varepsilon W_2'\left(\theta, \frac{2\pi}{\sqrt{m_0}}\right) + \mathrm{o}(\varepsilon) \\ &= -\frac{3\pi}{8}\frac{m_0'}{\sqrt{m_0}}\varepsilon + \varepsilon W_2'\left(\theta, \frac{2\pi}{\sqrt{m_0}}\right) + \mathrm{o}(\varepsilon). \end{aligned} \tag{4.63}$$

Now from (4.55) we have that

$$W_2'\left(\theta, \frac{2\pi}{\sqrt{m_0}}\right) = \frac{5\pi}{8}\frac{m_0'}{\sqrt{m_0}} + \Psi'\left(\theta, \frac{2\pi}{\sqrt{m_0}}\right).$$

Setting

$$\Phi(\theta) := \frac{1}{m_0}\Psi'\left(\theta, \frac{2\pi}{\sqrt{m_0}}\right)$$

formula (4.61) follows from (4.62), (4.63) and (4.56).

4.4.5. Asymptotic behaviour of Θ_2. Passing to the limit as $\varepsilon \to 0^+$ in problem (4.43) it turns out that $\Theta_2(\varepsilon,\theta,t)$ converges to a function $\Theta_{2l}(\theta,t)$ which solves the following equation:

$$\Theta_{2l}' = -\frac{m_0'}{\sqrt{m_0}}\sqrt{\nu}\left\{\cos^4(\theta - \sqrt{\nu m_0}\,t) - \cos^2(\sqrt{m_0}\,t)\cos^2(\theta - \sqrt{\nu m_0}\,t)\right\} \tag{4.64}$$

with the initial condition $\Theta_{2l}(\theta, 0) = 0$. The convergence is uniform on $\theta \in [0, 2\pi]$ and on bounded time intervals.

4.4.6. Asymptotic behaviour of β_2. From (4.47) and the results of sections 4.4.2–4.4.5, we have that

$$\lim_{\varepsilon\to 0^+}\beta_2(\varepsilon,\theta) = -\sqrt{\nu m_0}\left\{\frac{m_0'}{(m_0)^{3/2}}\frac{\pi}{4} + \Phi(\theta)\right\} + \Theta_{2l}\left(\theta, \frac{2\pi}{\sqrt{m_0}}\right).$$

Now we compute the average in $[0, 2\pi]$ of the two summands. Since the average of Φ is zero, then the average of the first summand is clearly

$$-\frac{m_0'}{m_0}\frac{\pi}{4}\sqrt{\nu}. \tag{4.65}$$

For the second summand, let us first note that using (4.64) and a simple change of variables

$$\frac{1}{2\pi}\int_0^{2\pi}\Theta'_{2l}(\theta,t)\,\mathrm{d}\theta = -\frac{1}{2\pi}\frac{m'_0}{\sqrt{m_0}}\sqrt{\nu}\int_0^{2\pi}\left\{\cos^4\theta - \cos^2(\sqrt{m_0}\,t)\cos^2\theta\right\}\mathrm{d}\theta$$
$$= -\frac{m'_0}{\sqrt{m_0}}\sqrt{\nu}\left\{\tfrac{3}{8} - \tfrac{1}{2}\cos^2(\sqrt{m_0}\,t)\right\}$$

so that, reversing the order of integration,

$$\frac{1}{2\pi}\int_0^{2\pi}\Theta_{2l}\left(\theta,\frac{2\pi}{\sqrt{m_0}}\right)\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi/\sqrt{m_0}}\Theta'_{2l}(\theta,t)\,\mathrm{d}t\,\mathrm{d}\theta$$
$$= \int_0^{2\pi/\sqrt{m_0}}\frac{1}{2\pi}\int_0^{2\pi}\Theta'_{2l}(\theta,t)\,\mathrm{d}\theta\,\mathrm{d}t$$
$$= -\frac{m'_0}{\sqrt{m_0}}\sqrt{\nu}\int_0^{2\pi/\sqrt{m_0}}\left\{\tfrac{3}{8} - \tfrac{1}{2}\cos^2(\sqrt{m_0}\,t)\right\}\mathrm{d}t$$
$$= -\frac{m'_0}{m_0}\frac{\pi}{4}\sqrt{\nu}. \tag{4.66}$$

From (4.65) and (4.66) we conclude that

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_0^{2\pi}\beta_2(\varepsilon,\theta)\,\mathrm{d}\theta = -\frac{m'_0}{m_0}\frac{\pi}{2}\sqrt{\nu}.$$

4.4.7. Asymptotic behaviour of γ_ε. In this final part of the proof, we show that the limit of $\beta_2(\varepsilon,\theta)$ coincides with the limit of $b(\varepsilon,\theta)$. We recall that b is defined in the following way: assume that the Poincaré map is written in Cartesian coordinates (X, Y) where L_ε takes its canonical form $R_{\omega(\varepsilon)}$; then, in the corresponding polar coordinates (I,σ), the Poincaré map becomes

$$P^*_\varepsilon\begin{pmatrix} I \\ \sigma\end{pmatrix} = \begin{pmatrix} I \\ \sigma - \omega(\varepsilon)\end{pmatrix} + \begin{pmatrix} a(\varepsilon,\sigma)I^3 \\ b(\varepsilon,\sigma)I^2\end{pmatrix} + \mathrm{o}(I^3).$$

We know that the coordinate change from (X, Y) to the original coordinates (x, y) is given by the diagonal matrix $D(\varepsilon)$ introduced in statement (4) of theorem 3.1. The expression of the corresponding change $D_*(\varepsilon)$ from (I,σ) to (ρ,θ) is not so simple: it is given by

$$\rho = \alpha_*(\varepsilon,\sigma)I \qquad \theta = \delta_*(\varepsilon,\sigma) \tag{4.67}$$

where $\alpha_*(\varepsilon,\sigma) = \left\{\cos^2\sigma + \delta^2(\varepsilon)\sin^2\sigma\right\}^{1/2}$, and $\delta_*(\varepsilon,\sigma) = \arctan(\delta(\varepsilon)\tan\sigma)$ for $\sigma \in (-\pi/2,\pi/2)$, and similarly for all other values of σ. In particular,

$$\alpha_*(\varepsilon,\sigma)\to 1 \qquad \delta_*(\varepsilon,\sigma)\to\sigma \qquad \frac{\partial\delta_*}{\partial\sigma}(\varepsilon,\sigma)\to 1 \tag{4.68}$$

as $\varepsilon\to 0^+$, uniformly in σ. The same holds true for the inverse change $D^*(\varepsilon) := [D_*(\varepsilon)]^{-1}$, whose components $\alpha^*(\varepsilon,\theta)\rho$ and $\delta^*(\varepsilon,\theta)$ are defined in analogy with α_*, δ_*, but with $\delta^{-1}(\varepsilon)$ instead of $\delta(\varepsilon)$. Considering the second component of $P^*_\varepsilon = D^*(\varepsilon)P_\varepsilon D_*(\varepsilon)$ we have that, up to $\mathrm{o}(I^3)$,

$$\sigma - \omega(\varepsilon) + b(\varepsilon,\sigma)I^2 = \delta^*\left[\varepsilon, \beta_0(\varepsilon,\delta_*(\varepsilon,\sigma)) + \beta_2(\varepsilon,\delta_*(\varepsilon,\sigma))\cdot\alpha_*^2(\varepsilon,\sigma)\,I^2\right]$$

so that, making the Taylor expansion of the right-hand side and looking at the coefficients of I^2, it turns out that

$$b(\varepsilon,\sigma) = \frac{\partial\delta^*}{\partial\sigma}[\varepsilon,\beta_0(\varepsilon,\delta_*(\varepsilon,\sigma))]\cdot\beta_2(\varepsilon,\delta_*(\varepsilon,\sigma))\cdot\alpha_*^2(\varepsilon,\sigma).$$

Therefore, the thesis follows from (4.68).

References

[1] Arosio A 1993 Averaged evolution equations. The Kirchhoff string and its treatment in scales of Banach spaces *Proc. 2nd Workshop on Functional-Analytic Methods in Complex Analysis (Trieste)* (Singapore: World Scientific)

[2] Cazenave T and Weissler F B 1980 Unstable simple modes of the nonlinear string *Q. Appl. Math.* **38** 287–305

[3] Dickey R W 1996 Stability of periodic solutions of the nonlinear string *Q. Appl. Math.* **54** 253–9

[4] Hale J and Koçak H 1991 *Dynamics and Bifurcations* (Berlin: Springer)

[5] Kirchhoff G 1876 *Vorlesungen über Mathematische Physik: Mechanik* (Leipzig: Teubner) section 29.7

[6] Lazutkin V F 1993 *KAM Theory and Semiclassical Approximation of Eigenfunctions* (Berlin: Springer)

[7] Meyer K R and Hall G R 1992 *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem* (Berlin: Springer)

[8] Moser J K 1973 *Stable and Random Motion in Dynamical Systems* (Princeton, NJ: Princeton University Press)

[9] Siegel C L and Moser J K 1971 *Lectures on Celestial Mechanics* (Berlin: Springer)

[10] Spagnolo S 1992 The Cauchy problem for the Kirchhoff equation *Rend. Sem. Fis. Matem. Milano* **62** 17–51