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Global analytic solutions to hyperbolic systems *

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Abstract

The aim of this paper is to extend to some classes of systems the global existence of analytic solutions to scalar equations of Kirchhoff type.

1 Introduction

The quasilinear integro-differential equations

$$u_{tt} - \varphi\left(\int_\Omega |\nabla u|^2\,dx\right)\triangle u = 0 \tag{1}$$

where $\varphi(r)$ is a continuous function ≥ 0 on $\{r \geq 0\}$ and Ω an open domain of $\mathbf{R}^n$, are currently called Kirchhoff equations; in the case when $\varphi(r) = 1 + r$ and $n = 1$, Equation (1) was proposed in [10], as a mathematical model for the small, transversal oscillations of an elastic string.

The first mathematical results for these equations were obtained by S. Bernstein [4], who considered the Cauchy problem for (1) with $n = 1$, $\Omega = [0, 2\pi]$ and looked for 2π-periodic solutions $u(t, \cdot)$: assuming that $\varphi(r)$ is a $\mathcal{C}^1$ function with $\varphi(r) \geq \nu > 0$, he proved the local well-posedness in suitable Sobolev spaces, as well as the global existence with real analytic data. After Bernstein, Kirchhoff type equations have been considered by several authors; we refer to [1] and [13] for a survey on the scalar case; for the vector case we mention [5] and [11] where a class

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of Kirchhoff type systems has been considered for which there is global existence for small, compact supported data.

In the non-coercive case, i.e. when the function $\varphi(r)$ is merely continuous and non negative, the global solvability for (1) with analytic initial data was firstly proved in [2] under the following additional assumption on $\varphi(r)$:

$$\textit{either} \qquad \varphi(r) \quad \textit{is bounded} \qquad \textit{or} \quad \int_0^\infty \varphi(r)\,dr \ = \infty.$$

This assumption was later removed in [6], where the same conclusion was obtained under the only condition that $\varphi(r) \geq 0$.

Such a result is based on the following facts:

- the global well-posedness in the analytic class (more exactly, the *a priori* estimate for the analytic solutions) of the linear equation
$$u_{tt} \ - \ a(t)\Delta u = 0, \quad \text{as soon as} \quad a(t) \geq 0, \quad a(t) \in L^1;$$
- the variational character of Eq.(1), which ensures that, if $\Phi'(r) = \varphi(r)$, $\Phi(0) = 0$, the positive functional
$$E(u,t) \ = \int_\Omega |u_t|^2\,dx + \Phi\left(\int_\Omega |\nabla u|^2\,dx\right),$$
keeps constant in time for any solution $u(t,x)$.

The purpose of this paper is to extend this global existence result to some Kirchhoff type systems. In particular we shall prove the global well-posedness in the class of analytic, 2π-periodic functions, for the Cauchy problem to the system

$$\begin{cases} v_t &= \psi\left(\int_0^{2\pi} v^2\,dx, \int_0^{2\pi} w^2\,dx\right) v_x + \alpha\left(\int_0^{2\pi} w^2\,dx\right) w_x \\ w_t &= \beta\left(\int_0^{2\pi} v^2\,dx\right) v_x + \psi\left(\int_0^{2\pi} v^2\,dx, \int_0^{2\pi} w^2\,dx\right) w_x \end{cases}$$

where $\psi(r,s), \alpha(s), \beta(r)$ are continuous functions and

$$\alpha(s) \geq 0, \quad \beta(r) \geq 0, \qquad \int_0^\infty \alpha(s)\,ds + \int_0^\infty \beta(r)\,dr = \infty.$$

Another system to which our global existence results apply, is

$$\begin{cases} v_t &= \left(C_1 + \int_0^{2\pi} v^2\,dx\right) w_x, \\ w_t &= \left(C_2 \ + \int_0^{2\pi} w^2\,dx\right) v_x \end{cases}$$

where C_i are constants ≥ 0.

2 Statements of the results

Let us consider the $N \times N$ systems of the general form

$$u_t - \sum_{j=1}^{n} A_j \left(\int_\Omega u_1^2 \, dx, \cdots, \int_\Omega u_N^2 \, dx \right) u_{x_j} = 0, \tag{2}$$

where $u = (u_1(t,x), \cdots, u_N(t,x)) \in \mathbf{R}^N$ and $A_j(r_1, \cdots, r_n)$ are real valued $N \times N$ matrices, continuous on $\mathbf{R}_+^N$. This system is *weakly hyperbolic* when the matrix

$$\sum_{j=1}^{n} \xi_j A_j(r_1, \cdots, r_n) \tag{3}$$

has real eigenvalues for all $\xi_j \in \mathbf{R}$ and all $r_i \geq 0$.

For simplicity, we shall consider here only the *periodic boundary condition* in x (however, see Remark 4 below), i.e., we take $\Omega = [0, 2\pi]^n$ and look for a solution $u(t,x)$, 2π-periodic in each space variable x_i. In this context, we denote by $[\mathcal{A}_{2\pi}(\mathbf{R}^n)]^N$ the class of $\mathbf{R}^N$ valued, 2π-periodic, analytic functions on $\mathbf{R}^n$.

We then prove:

Theorem 1 *The Cauchy-periodic problem for (2), with $\Omega = [0, 2\pi]^n$, is globally well posed in the analytic class $[\mathcal{A}_{2\pi}(\mathbf{R}^n)]^N$ whenever (2) is weakly hyperbolic and the coefficients $A_j(r_1, \cdots, r_n)$ are continuous and bounded on $\mathbf{R}_+^N$.*

Remark 1 *If (2) is a symmetric hyperbolic system, i.e. all the matrices A_j are symmetric, the global well posedness in $[\mathcal{A}_{2\pi}(\mathbf{R})]^2$ is obvious. Indeed in this case one has immediately:*

$$\frac{d}{dt} \left\| \frac{\partial^k u}{\partial x_j^k} \right\|_{L^2(\Omega)}^2 = 0 \qquad \forall j = 1, \ldots, n, \quad k \in \mathbf{N}.$$

In order to obtain some results without any boundedness assumption on the coefficients, we shall restrict ourselves to the 2×2 systems in one space dimension of the form

$$\begin{cases} v_t = \psi_1 \left(\|v(t)\|^2, \|w(t)\|^2 \right) v_x + \varphi_1 \left(\|v(t)\|^2, \|w(t)\|^2 \right) w_x \\ w_t = \varphi_2 \left(\|v(t)\|^2, \|w(t)\|^2 \right) v_x + \psi_2 \left(\|v(t)\|^2, \|w(t)\|^2 \right) w_x \end{cases} \tag{4}$$

where $\varphi_1(r,s), \varphi_2(r,s), \psi_1(r,s), \psi_2(r,s)$ are real, continuous functions on $\mathbf{R}_+^2$ and

$$\|v(t)\|^2 = \int_0^{2\pi} |v(t,x)|^2 \, dx, \quad \|w(t)\|^2 = \int_0^{2\pi} |w(t,x)|^2 \, dx.$$

The hyperbolicity condition for (4) is

$$(\psi_1 - \psi_2)^2 + 4\varphi_1\varphi_2 \geq 0,$$

but in the following we shall always make the stronger assumption

$$\varphi_1 \cdot \varphi_2 \geq 0. \tag{5}$$

If we take

$$\psi_1 = \psi_2 \equiv 0, \quad \varphi_1 \equiv \varphi(s), \quad \varphi_2 \equiv 1, \quad \text{and} \quad v = u_x, \quad w = u_t,$$

we see that the class of systems of type {(4),(5)} includes the scalar equations of type (1). However, due to the lack of a conserved energy functional, there are systems of this type for which the Cauchy problem is not globally well-posed in $[\mathcal{A}_{2\pi}(\mathbf{R})]^2$. In the following example, the system is *strictly hyperbolic*, i.e. the eigenvalues of the matrix (3) are real and simple, and satisfies (5).

Example 1 *There exists a pair of initial data v_0, w_0 in $[\mathcal{A}_{2\pi}(\mathbf{R})]^2$ for which the problem*

$$\begin{cases} v_t = \left(1 + \int_0^{2\pi} v^2\, dx\right) w_x, \quad w_t = w_x \\ v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \end{cases} \tag{6}$$

has no global solution.

To obtain the global existence for a system of type (4), we are forced to make, besides (5), some additional assumption on the coefficients $\varphi_1(r,s), \varphi_2(r,s)$:

Theorem 2 *Let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be real and continuous functions on $\mathbf{R}^2_+$ and assume that $\varphi_1 \cdot \varphi_2 \geq 0$. Then, the Cauchy-periodic problem for (4) is globally well-posed in $[\mathcal{A}_{2\pi}(\mathbf{R})]^2$ in each of the following cases.*

- *If $\varphi_1(r,s)$ and $\varphi_2(r,s)$ are bounded on $\mathbf{R}^2_+$.*
- *If there is a $\mathcal{C}^1$ function $L(r,s)$ defined on $\mathbf{R}^2_+$, with*

$$\frac{\partial L}{\partial r} \cdot \varphi_1 = \frac{\partial L}{\partial s} \cdot \varphi_2, \tag{7}$$

such that, either

$$L(r,s) \to +\infty \quad as \quad r + s \to +\infty, \tag{8}$$

or

$$\begin{gathered} \inf_{s \geq 0} L(r,s) \to +\infty \quad as \quad r \to +\infty, \\ |\varphi_1(r,s)| + |\varphi_2(r,s)| \leq \Lambda(r) \end{gathered} \tag{9}$$

for some continuous function Λ.

Of course, (9) can be replaced by the symmetric conditions in (r,s).

The most common case to which Theorem 2 applies, is for

$$\varphi_1 = \alpha(r,s)\cdot\varphi(r,s), \qquad \varphi_2 = \beta(r,s)\cdot\varphi(r,s)$$

where α, β are functions ≥ 0 satisfying

$$\frac{\partial\alpha}{\partial r} = \frac{\partial\beta}{\partial s}. \tag{10}$$

In such a case (7) is fulfilled by the function

$$L(r,s) = \int_0^r \beta(\rho,s)\,d\rho + \int_0^s \alpha(0,\sigma)\,d\sigma \equiv \int_0^r \beta(\rho,0)\,d\rho + \int_0^s \alpha(r,\sigma)\,d\sigma.$$

Hence (8), is equivalent to

$$\int_0^\infty \alpha(0,s)\,ds = \int_0^\infty \beta(r,0)\,dr = +\infty,$$

and (9) to

$$\int_0^\infty \beta(r,0)\,dr = +\infty, \quad |\alpha(r,s)| + |\beta(r,s)| + |\varphi(r,s)| \leq \Lambda(r).$$

In particular, (10) is trivially fulfilled when $\alpha = C_1 + r, \quad \beta = C_2 + s$, or when $\alpha = \alpha(s),\ \beta = \beta(r)$. Thus we get:

Corollary 1

1. *The Cauchy-periodic problem for the system*

$$\begin{cases} v_t = \psi_1\left(\|v\|^2, \|w\|^2\right) v_x + \left(C_1 + \|v\|^2\right)\cdot\varphi\left(\|v\|^2, \|w\|^2\right) w_x \\ w_t = \left(C_2 + \|w\|^2\right)\cdot\varphi\left(\|v\|^2, \|w\|^2\right) v_x + \psi_2\left(\|v\|^2, \|w\|^2\right) w_x \end{cases} \tag{11}$$

where C_i are constants ≥ 0 and ψ_1, ψ_2, φ real continuous functions, is globally well-posed in $[\mathcal{A}_{2\pi}(\mathbf{R})]^2$.

2. *The same conclusion holds true for the system*

$$\begin{cases} v_t = \psi_1\left(\|v\|^2, \|w\|^2\right) v_x + \alpha\left(\|w\|^2\right)\cdot\varphi\left(\|v\|^2, \|w\|^2\right) w_x \\ w_t = \beta\left(\|v\|^2\right)\cdot\varphi\left(\|v\|^2, \|w\|^2\right) v_x + \psi_2\left(\|v\|^2, \|w\|^2\right) w_x \end{cases} \tag{12}$$

where $\alpha, \beta, \psi_1, \psi_2, \varphi$ are real continuous functions, with $\alpha \geq 0, \quad \beta \geq 0$ and, either

$$\int_0^\infty \alpha(s)\,ds = \int_0^\infty \beta(r)\,dr = \infty$$

or

$$\alpha(s) \quad \textit{is bounded}, \quad \int_0^{+\infty} \beta(r)\,dr = +\infty, \quad |\varphi(r,s)| \leq \Lambda(r).$$

Finally, we prove the following result which is an extension of the quoted result for the scalar Kirchhoff equations ([6]) and improves the second part of Corollary 1 for bounded φ and $\psi_1 = \psi_2$.

Theorem 3 *The Cauchy-periodic problem for (4), where $\varphi_1, \psi_1, \varphi_2, \psi_2$, are real continuous functions, and $\varphi_i \geq 0$, is well- posed in $[\mathcal{A}_{2\pi}(\mathbf{R})]^2$ as soon as the following conditions are both fulfilled:*

(i) there is a $\mathcal{C}^1$ function $L(r,s)$, with

$$\frac{\partial L}{\partial r} \cdot \varphi_1 = \frac{\partial L}{\partial s} \cdot \varphi_2,$$

such that

$$\inf_{s \geq 0} L(r,s) \to +\infty \quad as \quad r \to +\infty, \tag{13}$$

and

$$|\varphi_2(r,s)| \leq \Lambda(r) < \infty. \tag{14}$$

(ii) there is a constant C such that

$$|\psi_2(r,s) - \psi_1(r,s)|^2 \leq C\varphi_1(r,s). \tag{15}$$

Of course, (13)–(15) can be replaced by the symmetric conditions in (r,s).

By this we obtain

Corollary 2 *The periodic-Cauchy problem for system (12), where $\alpha(s)$, $\beta(r)$, $\varphi(r,s)$, $\psi_1(r,s)$, $\psi_2(r,s)$ are real continuous functions, and $\alpha, \varphi, \beta \geq 0$, is well-posed in $[\mathcal{A}_{2\pi}(\mathbf{R})]^2$ as soon as:*

$$\begin{aligned} \int_0^\infty \beta(r)\, dr &= +\infty, \\ |\varphi(r,s)| &\leq \Lambda(r), \\ |\psi_2(r,s) - \psi_1(r,s)|^2 &\leq C\alpha(s)\varphi(r,s). \end{aligned}$$

Of course, the same conclusion holds under the symmetric assumptions

$$\int_0^\infty \alpha(s)\, ds = \infty, \quad |\varphi(r,s)| \leq \Lambda(s) < \infty$$

and

$$|\psi_2(r,s) - \psi_1(r,s)|^2 \leq C\beta(r)\varphi(r,s).$$

More generally, we have the following

Corollary 3 *The Cauchy-periodic problem for the system*

$$\left\{\begin{array}{lcl} v_t & = & \psi_1\left(\|v\|^2,\|w\|^2\right)v_x+\alpha_1(\|v\|^2)\cdot\alpha_2(\|w\|^2)\cdot\varphi(\|v\|^2,\|w\|^2)w_x \\ w_t & = & \beta_1(\|v\|^2)\cdot\beta_2(\|w\|^2)\cdot\varphi(\|v\|^2,\|w\|^2)v_x \ +\psi_2\left(\|v\|^2,\|w\|^2\right)w_x \end{array}\right. \tag{16}$$

is globally well-posed in $[\mathcal{A}_{2\pi}(\mathbf{R})]^2$ *as soon as:* $\alpha_1,\beta_2>0,\ \ \alpha_2,\beta_1,\varphi\geq 0,$

$$\int_0^\infty \frac{\beta_1(r)}{\alpha_1(r)}\,dr=+\infty,$$

and at least one of the following properties is verified:
either

- $\alpha_1(r)\beta_2(s)\varphi(r,s)\leq\Lambda(r)$, *and*

  $$|\psi_1(r,s)-\psi_2(r,s)|^2\leq C\alpha_1(r)\alpha_2(s)\varphi(r,s),$$

  or

- $$\int_0^\infty \frac{\alpha_2(s)}{\beta_2(s)}\,ds=+\infty.$$

Remark 2 *For* $\psi_i\equiv 0$ *and* $\varphi\equiv\beta\equiv 1$*, Corollary 2 give the result of [6] for Equation (1).*

Remark 3 *By effecting the Fourier transform we can obtain similar results to them of Theorems 2 and 3 for a pseudo- differential* 2×2 *system like*

$$U_t \ = \ A\left(r(t),s(t),D\right)|D|U.$$

This makes it possible, in particular, to deal with second order scalar equations in several space variables.

Remark 4 *The same results of Theorems 1, 2 and 3 hold true if we consider, instead of the Cauchy-periodic problem, the Cauchy problem on the whole* $\mathbf{R}^n$*. In this case the analytic class* $\mathcal{A}_{2\pi}(\mathbf{R}^n)$ *must be replaced by*

$$\mathcal{A}_{L_2}(\mathbf{R}^n) \ =\left\{w:\mathbf{R}^n\to\mathbf{R}^N:\|D^\alpha w\|_{L^2(\mathbf{R}^n)}\leq \ M\Lambda^{|\alpha|}\alpha!,\ \forall\alpha\in\mathbf{N}^n\right\}.$$

Remark 5 *Similar conclusions to those of Theorems 1, 2, 3 hold true for the more general systems*

$$\left\{\begin{array}{lcl} v_t & = & \psi_1 v_x+\varphi_1 w_x+\rho_1 v+\rho_2 w \\ w_t & = & \varphi_2 v_x\ +\ \psi_2 w_x+\mu_1 v+\mu_2 w \end{array}\right. \tag{17}$$

under suitable conditions on the lower order terms $\rho_1,\rho_2,\mu_1,\mu_2$ *(which, of course, are depending on* $\|v\|^2,\|w\|^2$*).*

3 Proofs

Let us firstly recall that, if

$$\varphi(x) = \sum_{h \in \mathbf{Z}^n} \hat{\varphi}_h e^{i(x,h)}$$

is the Fourier expansion of the 2π-periodic vector valued function $\varphi(x)$, then $\varphi \in [\mathcal{A}_{2\pi}(\mathbf{R}^n)]^N$ if and only if

$$\sum_{h \in \mathbf{Z}^n} e^{\delta|h|} |\hat{\varphi}_h|^2 < +\infty \tag{18}$$

for some $\delta > 0$.

Using this characterization we can easily prove (see [2], Section 2) the local well-posedness in $[\mathcal{A}_{2\pi}(\mathbf{R}^n)]^N$ for any system like (2), with $\Omega = [0, 2\pi]^n$.

As to the global existence results in Theorems 1, 2, 3, they rely on two Lemmata concerning the global solvability in $[\mathcal{A}_{2\pi}(\mathbf{R}^n)]^N$ for weakly hyperbolic *linear* systems.

Lemma 1 *Let $u \in C^1([0, T[, [\mathcal{A}_{2\pi}(\mathbf{R}^n)]^N)$ be a solution to the linear system*

$$u_t - \sum_{j=1}^{n} A_j(t) u_{x_j} = 0 \tag{19}$$

where $A_j(t)$ are $N \times N$ matrix valued, measurable functions on $[0, T[$ such that

$$A(t, \xi) = \sum_{j=1}^{n} \xi_j A_j(t) \tag{20}$$

has real eigenvalues for all $\xi \in \mathbf{R}^n$.

Moreover suppose that

$$\int_0^T |A_j(t)|\, dt < +\infty \qquad (1 \le j \le n).$$

Then $u(t, \cdot)$ has a limit in $[\mathcal{A}_{2\pi}(\mathbf{R}^n)]^N$ for $t \to T^-$.

When $N = 2$, the sommability of the diagonal coefficients can be dropped in several important cases (cf. [14]):

Lemma 2 *Let $(v, w) \in C^1([0, T[, [\mathcal{A}_{2\pi}(\mathbf{R})]^2)$ be a solution to the linear system:*

$$\begin{cases} v_t &= \psi_1(t) v_x + \lambda(t) w_x, \\ w_t &= \mu(t) v_x + \psi_2(t) w_x, \end{cases} \tag{21}$$

where λ, μ, ψ_i are real valued measurable functions, on $[0,T[$ and

$$\lambda(t)\cdot\mu(t) \geq 0 \quad a.e.\ on \quad [0,T[. \tag{22}$$

Suppose that

$$\int_0^T |\lambda(t)|\,dt < +\infty, \quad \int_0^T |\mu(t)|\,dt < +\infty.$$

Then $v(t,\cdot)$ and $w(t,\cdot)$ have a limit in $\mathcal{A}_{2\pi}(\mathbf{R})$ for $t \to T^-$.

Proof of Lemma 1. This Lemma was proved by E. Jannelli in [9], under an integrability assumption on the eigenvalues of $A(t,\xi)$. We give here a proof for the sake of completeness.

The proof is based on the existence of a smooth *quasi-symmetrizer* for any weakly hyperbolic matrix $A(t,\xi)$ on $[0,T[\times\mathbf{R}^n$ such that A is homogeneous in ξ of order one and

$$|A(t,\xi)| \leq \Lambda(t)|\xi|$$

for some $\Lambda \in L^1(0,T)$. This quasi-symmetrizer is constructed in Appendix A.

For a quasi-symmetrizer we mean here a family $\{Q_\varepsilon(t,\xi)\}$, $\varepsilon > 0$, of $N\times N$ matrices such that one has on $[0,T[\times\mathbf{R}^n$:

$$\nu_\varepsilon I \leq Q_\varepsilon(t,\xi) = Q_\varepsilon^*(t,\xi) \leq I,$$

$$A(t,\xi)Q_\varepsilon(t,\xi) - Q_\varepsilon(t,\xi)A^*(t,\xi) \leq \varepsilon|\xi|\Lambda_\varepsilon(t)Q_\varepsilon(t,\xi)$$

with

$$\int_0^T \Lambda_\varepsilon(t)\,dt \leq C,$$

and

$$|Q'_\varepsilon(t,\xi)| \leq C_\varepsilon,$$

for some positive constants ν_ε, C_ε, C independent on (t,ξ). Here Q' denotes the time derivative of Q, and for two $N\times N$ matrices, $A \leq B$ means $(Av,v) \leq (Bv,v)$ for all $v \in \mathbf{C}^N$.

The conclusion of Lemma 1 then follows by a standard argument.

If $\{\hat{u}_h(t)\}$, are the Fourier coefficients of the solution $u(t,\cdot)$, we have:

$$\hat{u}'_h = iA(t,h)\hat{u}_h \qquad (h \in \mathbf{Z}^n).$$

Thus, defining the energy functions

$$E_{h,\varepsilon}(t) = (Q_\varepsilon(t,h)\hat{u}_h(t), \hat{u}_h(t))$$

we find

$$\begin{aligned} E'_{h,\varepsilon} &= (Q'_\varepsilon\hat{u}_h, \hat{u}_h) + 2Re[i(Q_\varepsilon A\hat{u}_h, \hat{u}_h)] \\ &\leq C_\varepsilon|\hat{u}_h|^2 + C\varepsilon|h|\Lambda_\varepsilon(t)E_{h,\varepsilon} \\ &\leq \left(\frac{C_\varepsilon}{\nu_\varepsilon} + C\varepsilon|h|\Lambda_\varepsilon(t)\right)E_{h,\varepsilon} \end{aligned}$$

and hence, for $0 \leq t < T$,

$$E_{h,\varepsilon}(t) \leq E_{h,\varepsilon}(0) \exp\left(\frac{C_\varepsilon}{\nu_\varepsilon} T + C\varepsilon|h| \int_0^T \Lambda_\varepsilon(t)\, dt\right).$$

In conclusion we have proved the inequality

$$|\hat{u}_h(t)|^2 \leq M(\varepsilon, T)|\hat{u}_h(0)|^2 e^{C\varepsilon|h|},$$

which, recalling the characterization (18) of the analytic functions, gives the conclusion of Lemma 1. □

Proof of Lemma 2. The proof is based on the following fact:
For any pair of functions $\lambda,\ \mu \in L^1(0,T)$ satisfying (22), and for all $0 < \varepsilon < 1$, it is possible to find two Lipschitz continuous functions $\lambda_\varepsilon,\ \mu_\varepsilon > 0$ on $[0,T]$, in such a way that

$$\int_0^T \frac{|\mu_\varepsilon \lambda - \lambda_\varepsilon \mu|}{\sqrt{\lambda_\varepsilon}\sqrt{\mu_\varepsilon}}\, dt \leq C\varepsilon \left(\|\lambda\|_{L^1(0,T)} + \|\mu\|_{L^1(0,T)}\right) \tag{23}$$

with C independent on ε, λ, μ, and

$$\varepsilon^2 \leq \frac{\lambda_\varepsilon(t)}{\mu_\varepsilon(t)} \leq \frac{1}{\varepsilon^2}. \tag{24}$$

Let us suppose for the moment to have constructed $\lambda_\varepsilon,\ \mu_\varepsilon$ as above.
Denoting by $\hat{v}_h, \hat{w}_h$ the Fourier coefficients of $v(t,\cdot), w(t,\cdot)$, we have by (21)

$$\begin{cases} \hat{v}_h' = ih\,\psi_1(t)\hat{v}_h + ih\,\lambda(t)\hat{w}_h, \\ \hat{w}_h' = ih\,\mu(t)\hat{v}_h + ih\,\psi_2(t)\hat{w}_h. \end{cases} \tag{25}$$

Therefore, if we define

$$E_{\varepsilon,h}(t) = \lambda_\varepsilon|\hat{w}_h|^2 + \mu_\varepsilon|\hat{v}_h|^2,$$

we find a.e. on $[0,T]$:

$$\begin{aligned} E'_{\varepsilon,h} &= \frac{\lambda'_\varepsilon}{\lambda_\varepsilon}\,\lambda_\varepsilon|\hat{w}_h|^2 + \frac{\mu'_\varepsilon}{\mu_\varepsilon}\,\mu_\varepsilon|\hat{v}_h|^2 \\ &\quad +2(\lambda_\varepsilon Re(\hat{w}_h'\overline{\hat{w}_h}) + \mu_\varepsilon Re(\hat{v}_h'\overline{\hat{v}_h})) \\ &\leq \left(\frac{|\lambda'_\varepsilon|}{\lambda_\varepsilon} + \frac{|\mu'_\varepsilon|}{\mu_\varepsilon}\right) E_{\varepsilon,h} + 2|h|(\mu_\varepsilon\lambda - \lambda_\varepsilon\mu) Im(\hat{v}_h\overline{\hat{w}_h}) \\ &\leq \left(\frac{|\lambda'_\varepsilon|}{\lambda_\varepsilon} + \frac{|\mu'_\varepsilon|}{\mu_\varepsilon}\right) E_{\varepsilon,h} + |h|\frac{|\mu_\varepsilon\lambda - \lambda_\varepsilon\mu|}{\sqrt{\lambda_\varepsilon}\sqrt{\mu_\varepsilon}} E_{\varepsilon,h}. \end{aligned}$$

Hence, by (23), there exist some constants $C,\ C_\varepsilon$ such that

$$E_{\varepsilon,h}(t) \leq C_\varepsilon E_{\varepsilon,h}(0) e^{C|h|\varepsilon}. \tag{26}$$

Now if $r_0 > 0$ is such that

$$\sum_{-\infty}^{+\infty} e^{2r_0|h|}(|\hat{v}_{0,h}|^2 + |\hat{w}_{0,h}|^2) < +\infty,$$

we have

$$e^{r_0|h|}E_{\varepsilon,h}(t) \leq C_\varepsilon E_{\varepsilon,h}(0)e^{2r_0|h|}e^{-|h|(r_0-C\varepsilon)};$$

and hence for $\varepsilon \leq \dfrac{r_0}{C}$

$$\sum_{-\infty}^{+\infty} e^{r_0|h|}E_{\varepsilon,h}(t) \leq \overline{C}_\varepsilon, \qquad \text{on} \quad [0,T[.$$

Therefore v and w can be extended to the closed interval $[0,T]$ as analytic functions of x.

Now we prove the fact stated at the beginning.

Let us firstly assume that λ, μ are strictly positive, Lipschitz continuous functions on $[0,T]$.

Given $\varepsilon \in]0,1[$, we define the intervals

$$I_\varepsilon = \{t: \ \lambda(t) \geq \frac{1}{\varepsilon^2}\mu(t)\} \qquad J_\varepsilon = \{t: \ \mu(t) \geq \frac{1}{\varepsilon^2}\lambda(t)\},$$

and the positive, Lipschitz continuous functions

$$\lambda_\varepsilon(t) = \begin{cases} \dfrac{\mu(t)}{\varepsilon^2} & \text{on} \quad I_\varepsilon \\ \lambda(t) & \text{otherwise} \end{cases} \qquad \mu_\varepsilon(t) = \begin{cases} \dfrac{\lambda(t)}{\varepsilon^2} & \text{on} \quad J_\varepsilon \\ \mu(t) & \text{otherwise.} \end{cases}$$

Therefore, the function

$$\Lambda_\varepsilon(t) \equiv \Lambda_\varepsilon(\lambda,\mu,t) = \frac{|\mu_\varepsilon(t)\lambda(t) - \lambda_\varepsilon(t)\mu(t)|}{\sqrt{\lambda_\varepsilon(t)\mu_\varepsilon(t)}} \tag{27}$$

satisfies:

$$\begin{aligned} \Lambda_\varepsilon(t) &= \left|\frac{\mu(t)}{\varepsilon} - \varepsilon\lambda(t)\right| \leq 2\varepsilon\lambda(t) \qquad \text{on} \quad I_\varepsilon, \\ \Lambda_\varepsilon t) &= \left|\frac{\lambda(t)}{\varepsilon} - \varepsilon\mu(t)\right| \leq 2\varepsilon\mu(t) \qquad \text{on} \quad J_\varepsilon, \\ \Lambda_\varepsilon(t) &\equiv 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Hence, taking into account that I_ε and J_ε are disjoint, we get (23) with $C = 2$.

In the general case, when λ, μ are only integrable functions with $\lambda \cdot \mu \geq 0$, we approximate $|\lambda|$ and $|\mu|$ by Lipschitz continuous, strictly positive functions $\widetilde{\lambda}$, $\widetilde{\mu}$ such that

$$\|\,|\lambda| - \widetilde{\lambda}\,\|_{L^1(0,T)} \leq \delta \qquad \|\,|\mu| - \widetilde{\mu}\,\|_{L^1(0,T)} \leq \delta.$$

Therefore we can find $\widetilde{\lambda}_\varepsilon$, $\widetilde{\mu}_\varepsilon$ Lipschitz continuous and strictly positive, which satisfy (23) for $C = 2$, (with respect to $\widetilde{\lambda}$, $\widetilde{\mu}$) and (24).

But (22) implies

$$\left|\widetilde{\lambda}_\varepsilon|\mu| - \widetilde{\mu}_\varepsilon|\lambda|\right| = \left|\widetilde{\lambda}_\varepsilon\mu - \widetilde{\mu}_\varepsilon\lambda\right|,$$

hence, recalling (27), we get

$$\begin{aligned}
\|\Lambda_\varepsilon(\lambda,\mu,t)\|_{L^1} &= \|\Lambda_\varepsilon(|\lambda|,|\mu|,t)\|_{L^1} \\
&\le \|\Lambda_\varepsilon(\widetilde{\lambda},\widetilde{\mu},t)\|_{L^1} + \frac{2\delta}{\varepsilon} \\
&\le 2\varepsilon(\|\widetilde{\lambda}\|_{L^1} + \|\widetilde{\mu}\|_{L^1}) + \frac{2\delta}{\varepsilon} \\
&\le 2\varepsilon(\|\lambda\|_{L^1} + \|\mu\|_{L^1}) + 4\varepsilon\delta + \frac{2\delta}{\varepsilon}.
\end{aligned}$$

For $\delta = \varepsilon^2(\|\lambda\|_{L^1} + \|\mu\|_{L^1})$ we find (23) with $C = 8$.

This completes the proof of Lemma 2. □

Now we can prove our principal results.

Proof of Theorem 1. Let $u(t,x)$ a (local) analytic solution of (2) defined on some strip $[0,T[\times\mathbf{R}^n$, and let $A_j(t) = A_j(\|u_1(t)\|_2^2,\dots,\|u_N(t)\|_2^2)$, $\quad j = 1,\dots,n$. Since the A_j's are bounded, we can apply Lemma 1 to extend u on the closed strip $[0,T]\times\mathbf{R}^n$ as an analytic periodic function. Thus we obtain the global existence of u. □

Proof of Theorem 2. Let T be such that Problem (4) has a local solution defined on $[0,T[\times\mathbf{R}^2$.
If φ_1,φ_2 are bounded functions we can conclude the proof as in Theorem 1, by using Lemma 2.

In the other case, there exists a conserved energy for our Problem (4). Indeed if we define:

$$E(t) = L(\|v(t)\|^2, \|w(t)\|^2)$$

we have:

$$\begin{aligned}
E'(t) &= 2\frac{\partial L}{\partial r}\cdot\int_0^{2\pi} vv_t\,dx + 2\frac{\partial L}{\partial s}\cdot\int_0^{2\pi} ww_t\,dx \\
&= 2\frac{\partial L}{\partial r}\cdot\varphi_1\int_0^{2\pi} vw_x\,dx + 2\frac{\partial L}{\partial s}\cdot\varphi_2\int_0^{2\pi} v_x w\,dx = 0.
\end{aligned}$$

Hence:

- if holds (8) there exists a constant $K = K(v_0,w_0)$ such that

$$\|v(t)\|^2 + \|w(t)\|^2 \le K \qquad \text{on} \quad [0,T[;$$

- if holds (9), there exists a constant $K = K(v_0, w_0)$ such that

$$\|v(t)\|^2 \leq K, \qquad \text{on} \quad [0, T[,$$

so that $\Lambda(\|v(t)\|^2)$ is bounded.

But therefore (9) implies that φ_1, φ_2 are bounded, and we can conclude the proof as above. □

Proof of Theorem 3. We shall follow an argument similar to [6].

We recall that $\|\phi\|^2, \langle \phi, \psi \rangle$ denote the L^2-norm and the L^2-inner product in $L^2(0, 2\pi)$.

Let (v, w) an analytic periodic solution of (4) on $[0, T[\times \mathbf{R}^2$. It is not restrictive to suppose that for all $0 \leq t < T$

$$\int_0^{2\pi} v(t,x)\, dx = \int_0^{2\pi} w(t,x)\, dx = 0.$$

Indeed the average

$$\mu(t) = \int_0^{2\pi} v(t,x)\, dx$$

satisfies

$$\begin{aligned} \mu'(t) &= \int_0^{2\pi} v_t(t,x)\, dx \\ &= \psi_1(\|v\|^2, \|w\|^2) \int_0^{2\pi} v_x(t,x)\, dx + \varphi_1(\|v\|^2, \|w\|^2) \int_0^{2\pi} w_x(t,x) \\ &= 0, \end{aligned}$$

and the same is true for

$$\nu(t) = \int_0^{2\pi} w(t,x)\, dx.$$

Hence the functions

$$\underline{v} = v - \int_0^{2\pi} v_0(t,x)\, dx, \quad \underline{w} = w - \int_0^{2\pi} w_0(t,x)\, dx$$

are solutions to system (4) with null average.

Now we have

$$E(t) \equiv L(\|v(t)\|^2, \|w(t)\|^2) = \text{constant},$$

thus by (13), (14), there exist two constants C_1, C_2 such that:

$$\|v(t)\|^2 \leq C_1, \quad \varphi_2(\|v(t)\|^2, \|w(t)\|^2) \leq C_2 \qquad \text{on} \quad [0, T[.$$

On the other hand, if $z(t,x)$ denotes the unique periodic function with null average in x, such that

$$z_x = w$$

(we recall that w has null average in x) we have

$$z_{tx} = \varphi_2(\|v\|^2, \|w\|^2)v_x + \psi_2(\|v\|^2, \|w\|^2)w_x,$$

hence also

$$z_t = \varphi_2(\|v\|^2, \|w\|^2)v + \psi_2(\|v\|^2, \|w\|^2)w.$$

Observing that

$$\langle w, z\rangle = \langle z_x, z\rangle = 0,$$

we then find

$$\begin{aligned}(\|z\|^2)' &= \varphi_2(\|v\|^2, \|w\|^2) \\ &\leq \sqrt{C_1}C_2\|z\|,\end{aligned}$$

and hence

$$\|z(t)\| \leq C_3 \quad \text{on } [0,\mathrm{T}[.$$

Moreover we have:

$$\begin{aligned}\langle v_t, z\rangle &= \psi_1(\|v\|^2, \|w\|^2)\langle v_x, z\rangle + \varphi_1(\|v\|^2, \|w\|^2)\langle w_x, z\rangle \\ &= -\psi_1(\|v\|^2, \|w\|^2)\langle v, w\rangle - \varphi_1(\|v\|^2, \|w\|^2)\|w\|^2,\end{aligned}$$

and

$$\langle v, z_t\rangle = \varphi_2(\|v\|^2, \|w\|^2)\|v\|^2 + \psi_2(\|v\|^2, \|w\|^2)\langle v, w\rangle.$$

From this, recalling (15), we obtain

$$\begin{aligned}\varphi_1(\|v\|^2, \|w\|^2)\|w\|^2 &= -\langle v, z\rangle' + \langle v, z_t\rangle - \psi_1(\|v\|^2, \|w\|^2)\langle v, w\rangle \\ &\leq -\langle v, z\rangle' + C_1C_2 + \\ &\quad + \left|\psi_1(\|v\|^2, \|w\|^2) - \psi_2(\|v\|^2, \|w\|^2)\right| \cdot |\langle v, w\rangle| \\ &\leq C_1C_2 - \langle v, z\rangle' + \sqrt{C}\sqrt{\varphi_1(\|v\|^2, \|w\|^2)}\sqrt{C_1}\|w\|\end{aligned}$$

and hence

$$\varphi_1(\|v\|^2, \|w\|^2)\|w\|^2 \leq C_4 - \langle v, z\rangle'.$$

for some constant C_4. Integrating on $[0,t]$ we find

$$\begin{aligned}\int_0^t \varphi_1(\|v(s)\|^2, \|w(s)\|^2)\|w(s)\|^2\, ds &\leq C_4T + \langle v_0, z_0\rangle - \langle v(t), z(t)\rangle \\ &\leq C_4T + |\langle v_0, z_0\rangle| + \sqrt{C_1}C_3,\end{aligned}$$

in particular

$$\varphi_1(\|v\|^2, \|w\|^2)\|w\|^2 \in L^1(0,T).$$

Hence:

$$\begin{aligned}\int_0^T \varphi_1(\|v(s)\|^2, \|w(s)\|^2)\,ds &= \int_{0\le t<T,\,\|w\|\le 1} \varphi_1(\|v\|^2, \|w\|^2)\,ds + \\ &\quad + \int_{0\le t<T,\,\|w\|>1} \varphi_1(\|v\|^2, \|w\|^2)\,ds \\ &\le MT + \int_0^T \varphi_1(\|v\|^2, \|w\|^2)\|w\|^2\,ds < +\infty,\end{aligned}$$

where we have put

$$M := \sup\{\varphi_1(r,s) : \quad r \le C_1, \quad s \le 1\}.$$

In conclusion we have proved that

$$\lambda(t) \equiv \varphi_1(\|v(t)\|^2, \|w(t)\|^2) \in L^1(0,T)$$

and

$$\mu(t) \equiv \varphi_2(\|v(t)\|^2, \|w(t)\|^2) \in L^\infty(0,T),$$

and therefore we can apply Lemma 2 to conclude that the solution $v(t,x),\ w(t,x)$ can be continued behind $t = T$. □

Proof of Corollary 2. We have only to remark that all the hypotheses of Theorem 3 are satisfied with:

$$L(r,s) = \int_0^r \beta(\rho)d\rho + \int_0^s \alpha(\rho)d\rho.$$ □

Proof of Corollary 3. We rewrite system (16) in the form

$$\begin{cases} v_t = \psi_1(\|v\|_2^2, \|w\|_2^2)v_x + \alpha(\|w\|_2^2)\theta(\|v\|_2^2, \|w\|_2^2)w_x \\ w_t = \beta(\|v\|_2^2)\theta(\|v\|_2^2, \|w\|_2^2)v_x + \psi_1(\|v\|_2^2, \|w\|_2^2)w_x \end{cases}$$

where

$$\theta(r,s) = \alpha_1(r)\beta_2(s)\varphi(r,s),$$

and

$$\alpha(s) = \frac{\alpha_2(s)}{\beta_2(s)}, \qquad \beta(r) = \frac{\beta_1(r)}{\alpha_1(r)}.$$

Then we are reduced to the cases of Theorems 2, 3. □

Proof of Remark 4. The proofs are similar to those of Theorems 1, 2, 3. We only remark two facts.

- One can easy prove the existence of a local solution by using a version of the abstract Cauchy-Kowalewsky Theorem (see [8]).
- For all $g \in \mathcal{A}_{L_2}(\mathbf{R}^n)$ there exists some $r_0 > 0$ such that

$$\int_{\mathbf{R}^n} e^{r_0|\xi|} |\hat{g}(\xi)|^2 d\xi < +\infty. \qquad \square$$

Proof of Example 1. We have:

$$\begin{aligned} (\|w\|^2)' &= 0, \\ (\|v\|^2)' &= -2(\|v\|^2+1)\sum_{-\infty}^{+\infty} h\, Im(\overline{\hat{v}_h}\,\hat{w}_h), \\ (\overline{\hat{v}_h}\,\hat{w}_h)' &= ih\,\overline{\hat{v}_h}\,\hat{w}_h - ih(\|v\|^2+1)|\hat{w}_h|^2, \end{aligned}$$

and hence

$$\overline{\hat{v}_h}\,\hat{w}_h = e^{iht}\,\overline{\hat{v}_{0,h}}\,\hat{w}_{0,h} - ih\int_0^t e^{ih(t-s)}(\|v\|^2+1)|\hat{w}_h|^2\, ds.$$

By this, if the initial data satisfy $\overline{\hat{v}_{0,h}}\hat{w}_{0,h} = 0$ for every h, we get:

$$(\|v\|^2)' = 2(\|v\|^2+1)\sum_{-\infty}^{+\infty} h^2 \int_0^t \cos((t-s)h)|\hat{w}_{0,h}|^2(\|v(s)\|^2+1)\, ds.$$

Let us now suppose that $\hat{w}_{0,h} \neq 0$ only for a finite number of h. For τ sufficiently small with respect to w_0, say $\tau \leq \tau_0$, we have:

$$\sum_{-\infty}^{+\infty} h^2 \cos(\tau h)|\hat{w}_{0,h}|^2 \geq \frac{1}{2}\sum_{-\infty}^{+\infty} h^2|\hat{w}_{0,h}|^2$$

so that for $t \leq \tau_0$:

$$(\|v\|^2)' \geq (\|v\|^2+1)\|(w_0)_x\|^2 \int_0^t (\|v(s)\|^2+1)\, ds.$$

In conclusion, if $\|(w_0)_x\|^2 \neq 0$, we can find some v_0 in such a way that $\|v(t)\|^2$ blows-up in a time $T \leq \tau_0$. $\square$

A Appendix

Denoting by $\mathbf{M_N}$ the linear space of $N \times N$ matrices, we have the following

Proposition 1 *Let $A : [0,T] \times \mathbf{R}^n \to \mathbf{M_N}$ such that:*

- *$A(t,\xi)$ is integrable in t and continuous in ξ,*
- *$A(t,\xi)$ has real eigenvalues for all (t,ξ),*
- *A is homogeneous in ξ of order one and*

$$|A(t,\xi)| \leq \Lambda(t)|\xi|;$$

for some $\Lambda \in L^1(0,T)$.

Then there exists a family $\{Q_\varepsilon(t,\xi)\}$, $\varepsilon > 0$ of $N \times N$ matrix valued smooth functions such that one has on $[0,T[\times \mathbf{R}^n$:

$$\nu_\varepsilon I \leq Q_\varepsilon(t,\xi) = Q^*_\varepsilon(t,\xi) \leq I, \tag{28}$$

$$A(t,\xi)Q_\varepsilon(t,\xi) - Q_\varepsilon(t,\xi)A^*(t,\xi) \leq \varepsilon|\xi|\Lambda_\varepsilon(t)Q_\varepsilon(t,\xi) \tag{29}$$

with

$$\int_0^T \Lambda_\varepsilon(t)\,dt \leq C,$$

and

$$|Q'_\varepsilon(t,\xi)| \leq C_\varepsilon, \tag{30}$$

for some positive constants ν_ε, C_ε, C independent on (t,ξ).

Proof. We shall use the following lemma of real Analysis (cf. [12])

Lemma *Let S be a compact subset of $\mathbf{R}^n$ and $f(t,\xi) : [0,T[\times S \to \mathbf{R}$ a Carathéodory function, i.e. integrable in t and continuous in ξ, such that:*

$$|f(t,\xi)| \leq \Lambda(t)$$

with $\Lambda \in L^1(0,T)$.

Then for all $\delta > 0$ there exist $I_\delta \subseteq [0,T]$, $\Lambda_\delta \in L^1(0,T)$, and $f_\delta(t,\xi)$ continuous on $[0,T] \times S$ in such a way that:

- *$f(t,\xi) = f_\delta(t,\xi)$ for $t \notin I_\delta$,*
- *$|f_\delta(t,\xi)| \leq \Lambda_\delta(t)$ for $t \in I_\delta$,*
- *$\displaystyle\int_{I_\delta} (\Lambda(t) + \Lambda_\delta(t))\,dt \leq \delta$.*

Now, let $\sigma = \sigma(\delta) > 0$ be such that

$$|f_\delta(y) - f_\delta(y')| \leq \delta \qquad \text{for} \quad |y - y'| \leq \sigma$$

and let us consider a finite covering $\{B_1, \ldots, B_m\}$ of $[0,T] \times S$ by open sets with diameter $\leq \sigma$. We can assume that for some $m' \leq m$ one has

$$B_k \subseteq I_\delta \times S \qquad \Longleftrightarrow \qquad k = m'+1, \ldots, m.$$

Thus, taking a partition of the unity $\{\chi_k\}$ with $supp(\chi_k) \subseteq B_k$, we obtain the following

Corollary *There exist some nonnegative, smooth functions $\chi_1(t,\xi), \ldots, \chi_m(t,\xi)$ on $D = [0,T] \times S$ such that, for some $m' \leq m$ and some $(t_k, \xi_k) \in supp(\chi_k)$, one has*

- $\sum_1^m \chi_k(t,\xi) \equiv 1$ *on D;*
- $\sum_1^{m'} \chi_k(t,\xi)|f(t,\xi) - f(t_k,\xi_k)| \leq \varphi_\delta(t)$
- $\sum_{m'+1}^m \chi_k(t,\xi)|f(t,\xi)| \leq \varphi_\delta(t)$

where $\int_0^T \varphi_\delta(t)\, dt \ \leq \delta.$

Now we can prove Proposition 1.

For any constant matrix A with real eigenvalues, it is easy to construct (see [9] and [7]) a family of matrices $Q_\varepsilon = Q_\varepsilon(A)$, $\varepsilon > 0$, with the following properties:

$$\nu_\varepsilon I \leq Q_\varepsilon = Q_\varepsilon^* \leq I,$$

$$AQ_\varepsilon - Q_\varepsilon A^* \leq C_0|A|\varepsilon Q_\varepsilon.$$

Now let us set $S^1 = \{\xi \in \mathbf{R}^n : |\xi| = 1\}$, and for $\delta > 0$ (δ will be chosen suitably small with respect to ε) let us consider a smooth partition of the unity $\{\chi_k(t,\xi)\}_{1 \leq k \leq m}$ of $D = [0,T[\times S^1$, as in the previous Corollary. Then we define

$$A_k = A(t_k, \xi_k), \qquad Q_{k,\varepsilon} = Q_\varepsilon(A_k) \qquad \text{for } k = 1, \ldots, m'$$

and

$$Q_{\delta,\varepsilon}(t,\xi) = \sum_{k=1}^{m'} \chi_k(t,\xi) Q_{k,\varepsilon} + \sum_{m'+1}^{m} \chi_k(t,\xi) I.$$

Clearly, the family $Q_{\delta,\varepsilon}$ satisfies conditions (28) and (30) on D, as soon as $\nu_\varepsilon \leq 1$.

As to (29), we have the equality:

$$\begin{aligned} A(t,\xi)Q_{\delta,\varepsilon}(t,\xi) &= A(t,\xi)\sum_{k=1}^{m'}\chi_k(t,\xi)Q_{k,\varepsilon} + A(t,\xi)\sum_{k=m'+1}^{m}\chi_k(t,\xi)I \\ &= \sum_{k=1}^{m'}\chi_k(t,\xi)(A(t,\xi)-A_k)Q_{k,\varepsilon} + \\ &\quad + \sum_{k=1}^{m'}\chi_k(t,\xi)A_kQ_{k,\varepsilon} + A(t,\xi)\sum_{k=m'+1}^{m}\chi_k(t,\xi)I \end{aligned}$$

and a similar equality holds for $Q_{\delta,\varepsilon}A^*$.

On the other hand, by the Corollary we have:

$$\begin{aligned} \sum_{k=1}^{m'}\chi_k(t,\xi)(A_kQ_{k,\varepsilon} - Q_{k,\varepsilon}A_k^*) &\leq C_0\varepsilon\sum_{k=1}^{m'}\chi_k(t,\xi)Q_{k,\varepsilon}|A_k| \\ &\leq C_0\varepsilon\sum_{k=1}^{m'}\chi_k(t,\xi)|A_k - A(t,\xi)|I + \\ &\quad + C_0\varepsilon\sum_{k=1}^{m'}\chi_k(t,\xi)Q_{k,\varepsilon}|A(t,\xi)| \\ &\leq C_0\varepsilon\left(\frac{\varphi_\delta(t)}{\nu_\varepsilon} + \Lambda(t)\right)Q_{\delta,\varepsilon}(t,\xi). \end{aligned}$$

Hence using again the Corollary, we get

$$\begin{aligned} A(t,\xi)Q_{\delta,\varepsilon}(t,\xi) - Q_{\delta,\varepsilon}(t,\xi)A^*(t,\xi) \leq \\ \leq C_0\varepsilon\left(\frac{\varphi_\delta(t)}{\nu_\varepsilon} + \Lambda(t)\right)Q_{\delta,\varepsilon}(t,\xi) + 2\varphi_\delta(t)I, \end{aligned}$$

which gives (29) for

$$\Lambda_\varepsilon(t) = C_0\left(\frac{\varphi_\delta(t)}{\nu_\varepsilon} + \Lambda_\delta(t)\right) + 2\frac{\varphi_\delta(t)}{\nu_\varepsilon}.$$

But $\int_0^T \varphi_\delta(t)\ dt \leq \delta$; thus if we take δ small enough with respect to ε we see that

$$\int_0^T \Lambda_\varepsilon(t)\ dt \leq C < +\infty.$$

Finally we extend $Q_\varepsilon(t,\xi) \equiv Q_{\delta(\varepsilon),\varepsilon}(t,\xi)$ on $[0,T]\times\mathbf{R}^n$ as a homogeneous function in ξ of degree zero. □

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