(2) La sfera $x^2+y^2+z^2=2$ ed il paraboloide $z=x^2+y^2$ si intersecano nella circonferenza

$$\begin{cases} x^2+y^2+z^2=2 \\ z=\cancel{x}^2+y^2 \end{cases} \iff \begin{cases} \rho^4+\rho^2-2=0 \\ \rho^2=x^2+y^2 \end{cases}$$

cioè se $\rho^2=1$, ovvero $x^2+y^2=1$.

Posto $D=\{(x,y)\in\mathbb{R}^2 \,/\, x^2+y^2\le 1\}$

si ha pertanto

$$V=\{(x,y,z)\in\mathbb{R}^3 \,/\, (x,y)\in D,\ x^2+y^2\le z\le \le\sqrt{2-(x^2+y^2)}\}$$

(a) Il volume di $V$ si calcola dunque integrando „per fili"

$$\iiint_V dx\,dy\,dz = \iint_D dx\,dy \int_{x^2+y^2}^{\sqrt{2-(x^2+y^2)}} dz =$$

$$= \iint_D \left(\sqrt{2-(x^2+y^2)} - (x^2+y^2)\right) dx\,dy =$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(\rho\sqrt{2-\rho^2} - \rho^3\right) d\rho =$$

$$= 2\pi\left(\frac{1}{2}\int_0^1 \sqrt{2-t}\,dt - \frac{1}{4}\left[\rho^4\right]_0^1\right) =$$

$$= -\frac{2\pi}{3}\left[(2-t)^{3/2}\right]_0^1 - \frac{\pi}{2} = \left(\frac{4\sqrt{2}}{3} - \frac{7}{6}\right)\pi$$

(b) $\partial V = S_1 \cup S_2$, con $S_1$ sostegno della parametrizzazione cartesiana

$$\varphi_1 : D \to \mathbb{R}^3 \qquad \varphi_1(u,v) = (u, v, u^2+v^2)$$

e $S_2$ sostegno della parametrizzazione

$$\varphi_2 : D \to \mathbb{R}^3 \qquad \varphi_2(u,v) = \left(u, v, \sqrt{2-(u^2+v^2)}\right)$$

anch'essa cartesiana. Pertanto

$$\sqrt{1+\|\nabla\varphi_1\|^2} = \sqrt{1+4(u^2+v^2)}$$

$$\sqrt{1+\|\nabla\varphi_2\|^2} = \sqrt{1+\frac{u^2}{2-(u^2+v^2)}+\frac{v^2}{2-(u^2+v^2)}} =$$

$$= \sqrt{\frac{2}{2-(u^2+v^2)}}$$

da cui

$$\iint_{S_1} dS = \iint_D \sqrt{1+4(u^2+v^2)}\,du\,dv =$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho\sqrt{1+4\rho^2}\, d\rho = \pi \int_0^1 \sqrt{1+4t}\, dt =$$

$$= \pi/6 \left[(1+4t)^{3/2}\right]_0^1 = \frac{\sqrt{125}-1}{6}\pi$$

mentre

$$\iint_{S_2} dS = \int_0^{2\pi} d\theta \int_0^1 \rho \sqrt{\frac{2}{2-\rho^2}}\, d\rho =$$

$$= \sqrt{2}\pi \int_0^1 \frac{dt}{\sqrt{2-t}} = -2\sqrt{2}\pi\left[(2-t)^{1/2}\right]_0^1 =$$

$$= (4-2\sqrt{2})\pi.$$

Quindi $\displaystyle\iint_{\partial V} dS = \left(\frac{\sqrt{125}}{6} - 2\sqrt{2} + \frac{23}{6}\right)\pi.$