DISTRIBUTIVE GROUPOIDS IN KNOT THEORY

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ABSTRACT. A sequence of new knot invariants is constructed by using the relationship between the theory of distributive groupoids and knot theory.
   Bibliography: 3 titles.

The simplest and most common method of establishing the equivalence of two knots K_1, $K_2 \subset S^3$ which are given by their projections $\overline{K}_1$ and $\overline{K}_2$ consists in the step-by-step alteration of $\overline{K}_1$ to $\overline{K}_2$ (see Figure 1). A careful analysis of the process of alteration shows that $\overline{K}_2$ can always be obtained from $\overline{K}_1$ by a sequence of isotopic deformations and transformations of the form I–III (see Figure 2). From this it follows that any property of the projection of a knot which is preserved under transformations I–III is an invariant of the knot itself. A good example of such a property is found in the book of Crowell and Fox ([1], pp. 92–93, Exercises 6 and 7). Each passage of the knot projection is assigned one of three colors in such a way that at each double point the three passages either have all distinct colors or all the same color. The coloring will be called a *multicoloring* if more than one color is used. It is easy to verify that the property "the projection has a multicoloring" is invariant under transformations I–III and is thus an invariant of the knot. The standard circle does not have this property, since it has only one passage and therefore any coloring of it is singly-colored. Thus, any knot whose projection admits a multicoloring is nontrivial. As suggested in [1], this fact can also be proved by constructing a homomorphism of $\pi_1(S^3 \setminus K)$ onto the symmetric group of order 3. This property also helps to distinguish nontrivial knots. For example, the trefoil knot admits a multicoloring while the figure-eight knot does not.

   It is natural to try to generalize the above invariant by using a richer palette. Let Γ be an arbitrary set. We call the elements of Γ *colors*. Suppose that α: $\Gamma \times \Gamma \to \Gamma$ is an arbitrary binary operation (we denote $\alpha(a, b)$ by $a \circ b$). By a *colored oriented projection* $\overline{K}$ of a knot K we mean an assignment of a color to each passage of $\overline{K}$ in such a way that at each double point P the colors of the overpass (color b) and the passage on the left (color a) and the right (color c) are related by $a \circ b = c$ (see Figure 3). What sort of conditions must the operation α satisfy in order that the property of admitting a "multicoloring" be preserved under transformations I–III? It is not difficult to see that this property will be preserved under transformation I if α is left-invertible; that is, if for any a, $b \in \Gamma$ the

equation $x \circ a = b$ has a unique solution. To preserve the property under transformation II it is sufficient that every element be an idempotent: $a \circ a = a$. The property is preserved under III if α is right distributive; that is, if $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ for any $a, b, c \in \Gamma$. Thus, the property "the projection has a multicoloring" is a knot invariant if the palette is taken to be a left-invertible, idempotent, right-distributive groupoid Γ. For brevity, we will call such a groupoid simply a *distributive groupoid*. The set of three elements $\{a, b, c\}$ with the operation given by the table in Figure 4 is an example of a distributive groupoid.
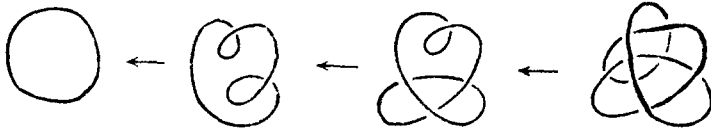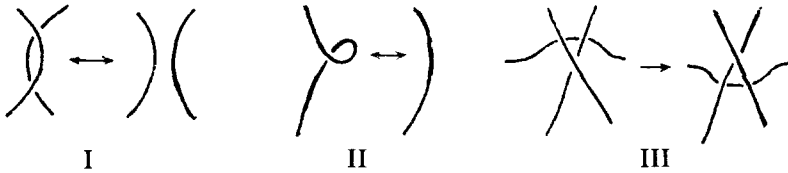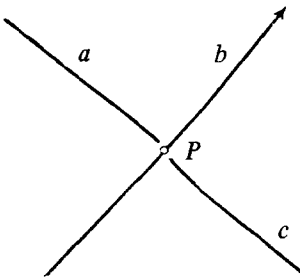

FIGURE 1

I   II   III

FIGURE 2

α	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

FIGURE 3   FIGURE 4

We have mentioned that a knot whose projection admits a multicoloring by the colors of some distributive groupoid is nontrivial. The converse is also true: the projection of any nontrivial knot admits a multicoloring by colors of some distributive groupoid Γ. The groupoid Γ may be taken, for example, to be the set $\pi_1(S^3 \setminus K)$ with the operation $a \circ b = bab^{-1}$. In this context, each passage of the projection $\overline{K}$ is colored by the associated generator in the corresponding Wirtinger presentation [1] of the group $\pi_1(S^3 \setminus K)$. Another method of constructing a groupoid which yields a multicoloring is to

define a distributive groupoid $\Gamma \langle A_{\bar{K}} | R_{\bar{K}} \rangle$ using generators and relations. The set of generators $A_{\bar{K}}$ consists of passages of the projection $\bar{K}$, and the set of relations $R_{\bar{K}}$ consists of relations of the form $a \circ b = c$—one for each double point of the projection $\bar{K}$. The groupoid $\Gamma \langle A_{\bar{K}} | R_{\bar{K}} \rangle$ does not change under transformations I–III and is therefore an invariant of the knot K itself. The invariant $\Gamma \langle A_{\bar{K}} | R_{\bar{K}} \rangle$ is universal in the sense that each coloring of the projection $\bar{K}$ by colors of an arbitrary distributive groupoid Γ is generated by some representation of the groupoid $\Gamma \langle A_{\bar{K}} | R_{\bar{K}} \rangle$ on Γ. The main result of this paper is the following theorem.

THEOREM. *The knots K_1 and K_2 are equivalent only if the groupoids $\Gamma \langle A_{\bar{K}_1} | R_{\bar{K}_1} \rangle$ and $\Gamma \langle A_{\bar{K}_2} | R_{\bar{K}_2} \rangle$ are isomorphic.*

Thus, the invariant $\Gamma \langle A_{\bar{K}} | R_{\bar{K}} \rangle$ is complete in the sense that it completely characterizes the knot K. All other invariants of the knot (the fundamental group, the Alexander module, and so on) must be expressible in terms of it. The existence of different knots with isomorphic fundamental groups shows that colorings of the projection of a knot by elements of groupoids do not reduce to coloring the projection by elements of groups; that is, to representations of the knot group.

This paper is arranged as follows. In §1, we introduce the notation we shall use. Definitions, examples, and properties of distributive groupoids are given in §2. A stock of examples of distributive groupoids can also be extracted from the theory of distributive semigroups, set forth in Belousiv's book [2]. However, as far as the author knows, there is no specific treatment of distributive groupoids in the literature. The definition of the associated group, introduced in §3, is somewhat different from the generally accepted definition of an associated group as a group generated by translations. The specification of a distributive groupoid by generators and relations, as well as the presentation of the group associated with it, is described in §§4 and 5. In §6, we construct a geometric groupoid Γ_K for each knot K. The elements of Γ_K are homotopy classes of paths beginning at a base point and ending on K. Following some necessary preliminaries, which are handled in §§7 and 8, a presentation of Γ_K (Theorem 1) is given in §9. The main result of the paper (Theorem 2) is proved in §10. In §11 we outline some applications of our results. In particular, a countable family of new knot invariants is constructed.

§1. Notation

A knot will be understood to be a tame simple closed curve K in S^3 together with a fixed orientation of its normal bundle. The knots K_1 and K_2 are equivalent if there exists a homeomorphism of the pair (S^3, K_1) to the pair (S^3, K_2) preserving the orientations of the normal bundles. Note that a choice of orientation of the sphere S^3 uniquely defines a direction of circuit of the knot K. For, it determines the third vector of an oriented frame in S^3; the first two define the orientation of the normal bundle. We will assume that each knot K is equipped with a tubular neighborhood N_K, a base point $x_K \in E_K$, a point $x'_K \in \partial N_K$ and a path $\dot{s}_K \subset E_K$ going from x_K to x'_K. Here, $E_K = S^3 \setminus \text{Int } N_K$. Let $\dot{s}_K$ define an inclusion of $\pi_1(\partial N_K, x'_K)$ into $G_K = \pi_1(E_K, x_K)$ by the formula $n \to [\dot{s}_K \dot{n} \dot{s}_K^{-1}]$, where $[\dot{n}] = n$. The image under this inclusion is the peripheral subgroup H_K. For a point $a \in \partial N_K$ let $\dot{m}_a$ denote the oriented rim of the disk cut out by the fiber of the normal bundle passing through a. The image of $\dot{m}_{x'_K}$ in H_K is denoted by m_K and is called a *meridian*. Any simple closed curve in ∂N_K which intersects $\dot{m}_{x'_K}$ only at the point x'_K, and

there transversally, is called a *parallel*. The image in H_K of a parallel will also be called a parallel. If s is an equivalence class, then $\dot{s}$ will denote a representative of the class. The class containing the element a is denoted by $[a]$. Thus $[\dot{s}] = s$.

§2. Definitions, examples, and properties of distributive groupoids

DEFINITION 1. A set Γ together with a bilinear operation is called a *distributive groupoid* if for any $a, b, c \in \Gamma$ the following axioms hold:

1) The equation $x \circ a = b$ has a unique solution.
2) $a \circ a = a$.
3) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$.

EXAMPLE 1. Let t be an invertible element of a ring A with unit and let M be a left A-module. Then M is a distributive groupoid with respect to the operation $a \circ b = ta + (1 - t)b$. This groupoid will be denoted by M_t.

EXAMPLE 2. Let G be an arbitrary group and n an integer. Then G is a distributive groupoid with respect to the operation $a \circ b = b^n a b^{-n}$. This groupoid will be denoted by $G^{(n)}$. Note that any homomorphism of a group G_1 to a group G_2 is simultaneously a homomorphism of the groupoid $G_1^{(n)}$ to the groupoid $G_2^{(n)}$.

EXAMPLE 3. Let H be a subgroup of a group G and let m be an element in the center of H. Define an operation $\circ$ on the set G/H of left residue classes by setting $aH \circ bH = bmb^{-1}aH$. This operation is well defined because m lies in the center of H. The resulting distributive groupoid will be denoted by $(G/H, m)$.

One verifies directly that each of the three examples above satisfies axioms 1)–3) of Definition 1.

The left inverse operation to $\circ$ in the distributive groupoid Γ will be denoted by $/$. In other words, b/a is the solution to the equation $x \circ a = b$. It follows from axioms 1)–3) of Definition 1 and the definition of $/$ that:

1) the equation $x/a = b$ has a unique solution;
2) $a/a = a$;
3) $(a/b)/c = (a/c)/(b/c)$;
4) $(a/b) \circ b = a$;
5) $(a \circ b)/b = a$;
6) $(a \circ b)/c = (a/c) \circ (b/c)$;
7) $(a/b) \circ c = (a \circ c)/(b \circ c)$.

EXAMPLE 4. If Γ is a distributive groupoid, then Γ is also a distributive groupoid with respect to the operation $/$. We denote this groupoid by Γ^{-1}.

EXAMPLE 5. Let Γ be a distributive groupoid. Define the operation $\circ^n$ inductively by the relations $a \circ^n b = (a \circ^{n-1} b) \circ b$ and $a \circ^1 b = a \circ b$. Then for each integer n the set Γ with the operation $\circ^n$ is a distributive groupoid which we will denote by Γ^n.

§3. The associated group of a groupoid

DEFINITION 2. A group G with a fixed homomorphism $f_0: \Gamma \to G^{(1)}$ is said to be *associated with* the distributive groupoid Γ if for any group G_1 and homomorphism $f: \Gamma \to G_1^{(1)}$ there exists a unique group homomorphism $h: G \to G_1$ such that $hf_0 = f$.

The uniqueness of the associated group follows from the usual category type arguments. The existence will be established in §5. We let $As(\Gamma)$ denote the group associated with the groupoid Γ.

Let $\operatorname{Aut}\Gamma$ denote the group of all isomorphisms of Γ to itself. For any $a \in \Gamma$ define the right translation $R_a: \Gamma \to \Gamma$ by setting $R_a(x) = x \circ a$. It follows from 3) of Definition 1

that right translation is a homomorphism. We can express the homomorphism R_a^{-1} by the formula $R_a^{-1}(x) = x/a$. Define a map $f: \Gamma \to (\text{Aut } \Gamma)^{(1)}$ by setting $f(a) = R_a$. Since

$$f(a \circ b)(x) = x \circ (a \circ b) = ((x/b) \circ a) \circ b = R_b R_a R_b^{-1}(x)$$
$$= (R_a \circ R_b)(x) = (f(a) \circ f(b))(x),$$

the map f is a homomorphism. By Definition 2, there exists a homomorphism h: $\text{As}(\Gamma) \to \text{Aut } \Gamma$ such that $hf_0 = f$. Thus, the group $\text{As}(\Gamma)$ acts on Γ by the formula $g(x) = h(g)(x)$, and $f_0(a)(x) = hf_0(a)(x) = f(a)(x) = x \circ a$. The homomorphism h need be neither a monomorphism nor an epimorphism.

LEMMA 1. *For any element a of a distributive groupoid Γ and any $g_1, g_2 \in \text{As}(\Gamma)$,*

$$g_1(a) \circ g_2(a) = g_2 f_0(a) g_2^{-1} g_1(a).$$

PROOF. We have

$$g_1(a) \circ g_2(a) = g_2 g_2^{-1} g_1(a) \circ g_2(a) = g_2(g_2^{-1} g_1(a) \circ a)$$
$$= g_2(f_0(a)(g_2^{-1} g_1(a))) = g_2 f_0(a) g_2^{-1} g_1(a).$$

§4. The specification of distributive groupoids by generators and relations

Let A be an arbitrary alphabet. By a *word* in the alphabet A we shall mean any arbitrary finite sequence consisting of elements of the set A and the symbols $(,),$ $\circ$ and $/$. Define inductively the set $\mathfrak{D}(A)$ of admissible words in the alphabet A as follows:

1) the word a, where $a \in A$ is admissible;

2) if w_1 and w_2 are admissible words, then the words $(w_1 \circ w_2)$ and (w_1/w_2) are also admissible.

Let R be a set of relations; that is, a set of sequences of the form $r = s$, where r, $s \in \mathfrak{D}(A)$. We introduce an equivalence relation $\backsim$ on the set $\mathfrak{D}(A)$. Write $w_1 \backsim w_2$ if w_1 can be transformed into w_2 by a sequence of operations of the form I), I'), II), III), IV):

I) $((x/y) \circ y) \leftrightarrow x$;

I') $((x \circ y)/y) \leftrightarrow x$;

II) $(x \circ x) \leftrightarrow x$;

III) $((x \circ y) \circ z) \leftrightarrow ((x \circ z) \circ (y \circ z))$;

IV) $r \leftrightarrow s$.

Here, x, y, $z \in \mathfrak{D}(A)$, $(r = s) \in R$, and $v_1 \leftrightarrow v_2$ denotes the operation of selecting a subword v_1 and replacing it by the subword v_2 or vice versa.

We introduce the operation $\circ$ on the set $\mathfrak{D}(A)/\backsim$ by setting $w_1 \circ w_2 = [(\dot{w}_1 \circ \dot{w}_2)]$. One then verifies that axioms 1)–3) of Definition 1 hold using transformations I, I', II, and III. For example, the equation $x \circ a = b$ has the solution $x = [(\dot{b}/\dot{a})] = b/a$ and if x_1 were another solution, then $x_1 = (x_1 \circ a)/a = b/a$. Thus, $\mathfrak{D}(A)/\backsim$ is a distributive groupoid with respect to the operation $\circ$. We will denote it by $\Gamma\langle A \mid R \rangle$.

§5. The presentation of the group associated with the groupoid $\Gamma\langle A \mid R \rangle$

Let A' be the alphabet obtained from the alphabet A by adding a prime to each letter. Let $\langle A' \mid \rangle$ be the free group generated by A'. Define the map $i: A \to A'$ by setting $i(a) = a'$ for all $a \in A$. Since the groupoid $\Gamma\langle A \mid \rangle$ is free, the map i can be extended to an isomorphism $i: \Gamma\langle A \mid \rangle \to \langle A' \mid \rangle^{(1)}$. If R is a set of relations, let R' denote the set of relations of the form $i(r) = i(s)$ where $(r = s) \in R$.

PROPOSITION 1. *The group* $\langle A' \mid R' \rangle$ *is associated with the groupoid* $\Gamma \langle A \mid R \rangle$.

PROOF. The homomorphism i: $\Gamma \langle A \mid \ \rangle \to \langle A' \mid \ \rangle^{(1)}$ carries relations to relations and thus induces a homomorphism f_0: $\Gamma \langle A \mid R \rangle \to \langle A' \mid R' \rangle^{(1)}$. We must show that f_0 is universal. Let f: $\Gamma \langle A \mid R \rangle \to G^{(1)}$ be another homomorphism. Then a unique "closing" homomorphism is defined on the generators by the formula $h(a') = f(a)$. It can be extended to a homomorphism of the entire group $\langle A' \mid R' \rangle$ to G, because if $i(r) = i(s)$ is a defining relation in $\langle A' \mid R' \rangle$, then $r = s$ is a relation in $\Gamma \langle A \mid R \rangle$ and, thus, $hi(r) = f(r) = f(s) = hi(s)$ in G.

COROLLARY. *If* Γ *is any distributive groupoid, then the group* $\mathrm{As}(\Gamma)$ *exists.*

The proof consists in observing that any distributive groupoid Γ can be written as $\Gamma \langle A \mid R \rangle$ upon taking A to be the set Γ and R the set of relations of the form $(a \circ b) = c$ whenever $a \circ b = c$ in Γ.
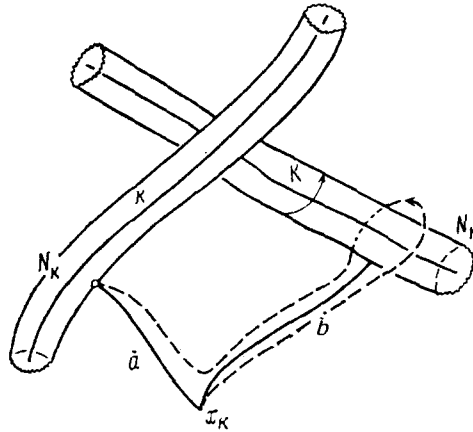

FIGURE 5

§6. The geometric groupoid of a knot

Let K be a knot in S^3. We let Γ_K denote the set of homotopy classes of loops in E_K which begin at x_K and end on ∂N_K (the condition on the initial points and the endpoints must be preserved in the process of homotopy). By way of definition, set $a \circ b = [\dot{b}\dot{m}_{\dot{b}(1)}\dot{b}^{-1}\dot{a}]$, where $\dot{b}(1)$ denotes the endpoint of the loop $\dot{b}$. In Figure 5, we have drawn a representative of the class $a \circ b$ using a dotted line. The arrow on ∂N_K indicates the orientation of the normal bundle. We verify that axioms 1)–3) of Definition 1 hold. The equation $x \circ a = b$ has the unique solution

$$x = \left[\dot{a}\dot{m}_{\dot{a}(1)}^{-1}\dot{a}^{-1}\dot{b} \right];$$

$$a \circ a = \left[\dot{a}\dot{m}_{\dot{a}(1)}\dot{a}^{-1}\dot{a} \right] = \left[\dot{a}\dot{m}_{\dot{a}(1)} \right] = a, \quad \text{since } \dot{m}_{\dot{a}(1)} \subset \partial N_K;$$

$$(a \circ c) \circ (b \circ c) = \left[(b \circ c)\dot{m}_{(b \circ c)(1)}(b \circ c)^{-1}(a \circ c) \right]$$

$$= \left[\dot{c}\dot{m}_{\dot{c}(1)}\dot{c}^{-1}\dot{b}\dot{m}_{(b \circ c)(1)}\dot{b}^{-1}\dot{c}\dot{m}_{\dot{c}(1)}^{-1}\dot{c}^{-1}\dot{c}\dot{m}_{\dot{c}(1)}\dot{c}^{-1}\dot{a} \right]$$

$$= \left[\dot{c}\dot{m}_{\dot{c}(1)}\dot{c}^{-1}\dot{b}\dot{m}_{\dot{b}(1)}\dot{b}^{-1}\dot{a} \right] = (a \circ b) \circ c.$$

In the second to last equality we have replaced $\dot{m}_{(b\,\circ\,c)(1)}$ by $\dot{m}_{\dot{b}(1)}$ because the endpoint of the path $(b \circ c)$ coincides with the endpoint of $\dot{b}$. Thus, 1)–3) are satisfied and Γ_K is a distributive groupoid.

We can define an action of the group G_K on Γ_K by setting $g(a) = [\dot{g}\dot{a}]$. Note that

$$g(a) \circ g(b) = \left[\dot{g}\dot{b}\dot{m}_{\dot{g}\dot{b}(1)}\dot{b}^{-1}\dot{g}^{-1}\dot{g}\dot{a}\right] = \left[\dot{g}\dot{b}\dot{m}_{\dot{b}(1)}\dot{b}^{-1}\dot{a}\right] = g(a \circ b)$$

and, thus, for each $g \in G_K$ the action of g on Γ_K is a homomorphism of Γ_K to Γ_K. It should be noted that, as a set, Γ_K coincides with the set $\pi_1(E_K, \partial N_K \cup x_K, x_K)$. Our definition of the action of the group $G_K = \pi_1(E_K, x_K)$ on Γ_K coincides with the canonical left action of G_K on $\pi_1(E_K, \partial N_K \cup x_K, x_K)$.

LEMMA 2. *The action of the group G_K on Γ_K is transitive. The isotopy subgroup of the element $s_K = [\dot{s}_K] \in \Gamma_K$ coincides with H_K.*

PROOF. Suppose that $a, b \in \Gamma_K$, and let $n \subset \partial N_K$ be a path starting from the endpoint of $\dot{b}$ and running to the endpoint of $\dot{a}$. Define $g \in G_K$ by setting $g = [\dot{b}\dot{n}\dot{a}^{-1}]$. We have $g(a) = [\dot{g}\dot{a}] = [\dot{b}\dot{n}\dot{a}^{-1}\dot{a}] = [\dot{b}\dot{n}] = b$. This establishes transitivity.

By the defintion of H_K, each $h \in H_K$ has the form $h = [\dot{s}_K\dot{n}\dot{s}_K^{-1}]$, where $\dot{n}$ is a loop in ∂N_K starting and ending at x_K'. Thus, $h(s_K) = [\dot{s}_K\dot{n}\dot{s}_K^{-1}\dot{s}_K] = [\dot{s}_K\dot{n}] = s_K$. This means that H_K acts trivially on s_K. On the other hand, if $g(s_K) = s_K$, then the paths $\dot{g}\dot{s}_K$ and $\dot{s}_K$ are homotopic. Let $\dot{n} \subset \partial N_K$ be the path traced by the endpoint of the path $\dot{g}\dot{s}_K$ under this homotopy. Then the path $\dot{g}\dot{s}_K\dot{n}\dot{s}_K^{-1}$ is homotopic to a constant; that is, $g = [\dot{s}_K\dot{n}^{-1}\dot{s}_K^{-1}]$. This means that g lies in H_K.

PROPOSITION 2. *The map $\gamma\colon (G_K/H_K, m_K) \to \Gamma_K$ defined by $\gamma(gH_K) = g(s_K)$ is an isomorphism of groupoids.*

PROOF. The map γ is well defined and injective, because H_K is the isotropy subgroup of the element s_K by Lemma 2. The surjectivity of γ follows from the transitivity of the action of G_K on Γ_K. We prove that γ is a homomorphism. We have

$$\gamma(g_1 H_K \circ g_2 H_K) = \gamma(g_2 m_K g_2^{-1} g_1 H_K)$$
$$= g_2 m_K g_2^{-1} g_1(s_K) = \left[\dot{g}_2\dot{s}_K\dot{m}_{x_K'}\dot{s}_K^{-1}\dot{g}_2^{-1}\dot{g}_1\dot{s}_K\right],$$

since $m_K = [\dot{s}_K\dot{m}_{x_K'}\dot{s}_K^{-1}]$. On the other hand,

$$\gamma(g_1 H_K) \circ \gamma(g_2 H_K) = g_1(s_K) \circ g_2(s_K)$$
$$= [\dot{g}_1\dot{s}_K] \circ [\dot{g}_2\dot{s}_K] = \left[\dot{g}_2\dot{s}_K m_{\dot{g}_2\dot{s}_K(1)}\dot{s}_K^{-1}\dot{g}_2^{-1}\dot{g}_1\dot{s}_K\right].$$

It remains to observe that the path $\dot{g}_2\dot{s}_K$ ends at the point x_K' and thus $\dot{m}_{x_K'} = \dot{m}_{\dot{g}_2\dot{s}_K(1)}$.

§7. The groupoid $\Gamma \langle A_{\bar{K}} | R_{\bar{K}} \rangle$ and the group associated with it

Let K be a knot in XYZ-space $\mathbf{R}^3 = S^3\backslash\{*\}$ which is in regular position with respect to the projection of $\mathbf{R}^3$ onto the place XOY. We adhere to the widely accepted convention of representing K by its projection $\bar{K}$. We break $\bar{K}$ into pieces by cutting at the lower double points. The words "lower" and "higher" are to be understood with reference to the values of the z coordinate. The set of connected components obtained by cutting the projection in this manner (the set of passages) will be denoted by $A_{\bar{K}}$. Note that the orientation of the normal bundle of K together with the XYZ-orientation of $\mathbf{R}^3$ defines a direction of circuit of K and of $\bar{K}$. Let P be the double point on the juncture of the passages $a, c \in A_{\bar{K}}$ and on
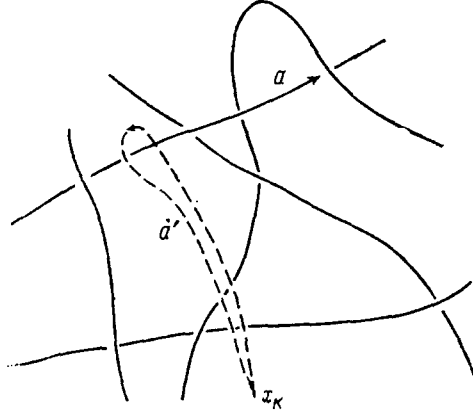
FIGURE 6

the overpass $b \in A_{\overline{K}}$ (see Figure 3 in the Introduction). We write the relation $(a \circ b) = c$, where a is the passage on the left of b and c is the passage on the right. The set of all these relations (one for each double point) is denoted by $R_{\overline{K}}$. Our next task is to identify the group $\mathrm{As}(\Gamma \langle A_{\overline{K}} | R_{\overline{K}} \rangle)$.

To each passage $a \in A_{\overline{K}}$ we associate a loop $\dot{a}' \in E_{\overline{K}}$, with endpoints at x_K, satisfying the following conditions:

1. The projection of $\dot{a}'$ is transverse to $\dot{K}$.

2. At every point of intersection of $\overline{K}$ and the projection of $\dot{a}'$, except one, $\dot{a}'$ passes above K.

3. The unique point of intersection at which $\dot{a}'$ passes below lies on a, and at this point $\dot{a}'$ intersects a from right to left (see Figure 6).

It is clear that the homotopy class a' of the loop $\dot{a}'$ is defined uniquely by conditions 1–3.

PROPOSITION 3. *The assignment $a \rightarrow a'$ induces a homomorphism f_0: $\Gamma \langle A_{\overline{K}} | R_{\overline{K}} \rangle \rightarrow G_K^{(1)}$. The group G_K is associated with $\Gamma \langle A_{\overline{K}} | R_{\overline{K}} \rangle$ with respect to this homomorphism.*

PROOF. Let $A'_{\overline{K}} = \{a' \mid a \in A_{\overline{K}}\}$. For each relation $(a \circ b) = c$ in $R_{\overline{K}}$, write the relation $b'a'(b')^{-1} = c'$. Denote the set of such relations by $R'_{\overline{K}}$. By Proposition 1 the presentation $\langle A'_{\overline{K}} | R'_{\overline{K}} \rangle$ is a presentation of the associated group. On the other hand, the presentation $\langle A'_{\overline{K}} | R'_{\overline{K}} \rangle$ coincides with the well-known Wirtinger presentation of G_K (see [1]).

§8. Two lemmas

We label the elements of the set $A_{\overline{K}}$ by $a_0, a_1, \ldots, a_{n-1}$ in order of their position on $\overline{K}$ with respect to the direction of circuit of $\overline{K}$. The passage a_0 is chosen to be the one on which the center of the normal disk with boundary $\dot{m}_{x_{K}} \subset \partial N_K$ is situated. The integer n denotes the number of passages in $\overline{K}$. Except for the case of the standard circle, the number of double points in $\overline{K}$ is also equal to n. The relations in $R_{\overline{K}}$ are of the form $(a_i \varepsilon_i a_{r_i}) = a_{i+1}$, where ε_i means $\circ$ if a_i lies to the left of a_{r_i} and $/$ if it lies on the right (in the latter case, the relation $(a_i / a_{r_i}) = a_{i+1}$ is equivalent to the relation $(a_{i+1} \circ a_{r_i}) = a_i$). The indices are taken modulo n. We can write the relations in $R'_{\overline{K}}$ in the form

$$\left(a'_{r_i}\right)^{\delta_i} a'_i \left(a'_{r_i}\right)^{-\delta_i} = a'_{i+1},$$

where $\delta_i = 1$ if ε_i signifies $\circ$ and $\delta_i = -1$ if ε_i is $/$. Let

$$\sigma_i'\left(a_{r_{i-1}}'\right)^{\delta_{i-1}}\left(a_{r_{i-2}}'\right)^{\delta_{i-2}}\cdots\left(a_{r_0}'\right)^{\delta_0} \in G_K, \qquad 0 \leqslant i \leqslant n.$$

By changing the path $\dot{s}_K$, if necessary, it may be assumed that the projection of $\dot{s}_K$ is transverse to $\overline{K}$ and that, at all points of intersection of the projection of $\dot{s}_K$ with $\overline{K}$, the path $\dot{s}_K$ passes above. Then we can take $\dot{a}_0'$ to be the path $\dot{s}_K \dot{m}_{x_K'} \dot{s}_K^{-1}$. This means that $a_0' = m_K$. It is not difficult to see that σ_n' is then a parallel in H_K.

LEMMA 3. *For each i, the relations $\sigma_i'(a_0) = a_i$ and $\sigma_i'a_0'(\sigma_i')^{-1} = a_i'$ hold.*

PROOF. Both relations hold for $i = 0$ because $\sigma_0' = 1$. Suppose that they hold for $i = m - 1$. Since $\sigma_m' = (a_{r_{m-1}}')^{\delta_{m-1}}\sigma_{m-1}'$, we have

$$\sigma_m'(a_0) = \left(a_{r_{m-1}}'\right)^{\delta_{m-1}}\sigma_{m-1}'(a_0) = \left(a_{r_{m-1}}'\right)^{\delta_{m-1}}(a_{m-1}) = (a_{m-1}\varepsilon_{m-1}a_{r_{m-1}}) = a_m.$$

Similarly,

$$\sigma_m'a_0'(\sigma_m')^{-1} = \left(a_{r_{m-1}}'\right)^{\delta_{m-1}}a_{m-1}'\left(a_{r_{m-1}}'\right)^{-\delta_{m-1}} = a_m'.$$

LEMMA 4. *The isotropy subgroup of the element a_0 contains H_K.*

PROOF. The group H_K is generated by the meridian a_0' and the parallel σ_n'. It remains to observe that $a_0'(a_0) = (a_0 \circ a_0) = a_0$, that $\sigma_n'(a_0) = a_n$ by Lemma 3, and that a_n coincides with a_0.

§9. The isomorphism between $\Gamma\left\langle A_{\overline{K}} \mid R_{\overline{K}} \right\rangle$ and Γ_K

THEOREM 1. *For any projection $\overline{K}$ of the knot K the groupoids $\Gamma\langle A_{\overline{K}} \mid R_{\overline{K}}\rangle$ and Γ_K are isomorphic.*

PROOF. By Proposition 2, the groupoids Γ_K and $(G_K/H_K, m_K)$ are isomorphic. Thus, it suffices to prove that the groupoids $\Gamma\langle A_{\overline{K}} \mid R_{\overline{K}}\rangle$ and $(G_K/H_K, m_K)$ are isomorphic. We define $\Phi_1 \colon A_{\overline{K}} \to (G_K/H_K, m_K)$ by setting $\Phi_1(a_i) = \sigma_i'H_K$. Recall that we have identified a_0' and m_K. Since

$$\sigma_i'H_K\varepsilon_i\sigma_{r_i}'H_K = \left(\sigma_{r_i}'a_0'(\sigma_{r_i}')^{-1}\right)^{\delta_i}\sigma_i'H_K = \left(a_{r_i}'\right)^{\delta_i}\sigma_i'H_K$$

by Lemma 3, and $(a_{r_i}')^{\delta_i}\sigma_i'H_K = \sigma_{i+1}'H_K$, the map Φ_1 carries relations in $R_{\overline{K}}$ to relations which are valid and thus extends to a homomorphism $\Phi \colon \Gamma\langle A_{\overline{K}} \mid R_{\overline{K}}\rangle \to (G_K/H_K, m_K)$. We remark that Φ is equivariant with respect to the action G_K. This is the case because if a_i' is any generator of G_K and a_j any generator of $\Gamma\langle A_{\overline{K}} \mid R_{\overline{K}}\rangle$, then

$$\Phi\left(a_i'(a_j)\right) = \Phi(a_j \circ a_i) = \sigma_j'H_K \circ \sigma_i'H_K = \sigma_i'a_0'(\sigma_i')^{-1}\sigma_j'H_K = a_i'\sigma_j'H_K$$

by Lemma 3.

Define the map $\Psi \colon (G_K/H_K, m_K) \to \Gamma\langle A_{\overline{K}} \mid R_{\overline{K}}\rangle$ by setting $\Psi(gH_K) = g(a_0)$. This is well defined because the group H_K acts trivially on a_0 by Lemma 4. We prove that Ψ is a homomorphism. In fact,

$$\Psi(g_1H_K \circ g_2H_K) = \Psi(g_2a_0'g_2^{-1}g_1H) = g_2a_0'g_2^{-1}g_1(a_0) = g_1(a_0) \circ g_2(a_0)$$

by Lemma 1. It remains to check that $\Phi\Psi = 1$ and $\Psi\Phi = 1$. We have $\Psi\Phi(a_i) = \Psi(\sigma_i'H_K) = \sigma_i'(a_0) = a_i$ by Lemma 3. Since Φ is equivariant,

$$\Phi\Psi(gH_K) = \Phi(g(a_0)) = g\Phi(a_0) = gH_K.$$

Theorem 1 is proved.

REMARK. An analysis of the isomorphisms $\gamma\colon (G_K/H_K, m_K) \to \Gamma_K$ and $\Phi\colon \Gamma\langle A_K^- \mid R_K^- \rangle \to (G_K/H_K, m_K)$ shows that as a path representative of a class $\gamma\Phi(a_j) \in \Gamma_K$ one can take any path $\dot{a}_j$ joining x_K to the passage a_j such that the projection of $\dot{a}_j$ is transverse to $\overline{K}$ and such that at all points where it intersects $\overline{K}$ the path $\dot{a}_j$ passes above $\overline{K}$. In particular, $\gamma\Phi(a_0) = s_K$. Thus, there is one and only one element a_0' in G_K corresponding to the elements a_0 and s_K and, as noted in §8, this path coincides with the meridian m_K.

COROLLARY. *For any knot K the group G_K is associated with the groupoid Γ_K. The isotropy subgroup of the element s_K is H_K, and the meridian $m_K \in H_K$ is the image of s_K under the homomorphism $f_0\colon \Gamma_K \to G_K^{(1)}$.*

The corollary follows from Theorem 1, Lemma 2 and the preceding remark.

§10. Knots with isomorphic groupoids are equivalent

We need a theorem proved by Waldhausen in [3].

THEOREM OF WALDHAUSEN. *Suppose that M and N are irreducible and boundary irreducible. Let M be sufficiently large and let $\psi\colon \pi_1(N) \to \pi_1(M)$ be an isomorphism preserving the peripheral structure. Then there exists a homeomorphism $f\colon N \to M$ inducing ψ.*

THEOREM 2. *For the knots K_1 and K_2 to be equivalent, it is necessary and sufficient that the groupoids Γ_{K_1} and Γ_{K_2} be isomorphic.*

PROOF. The necessity is obvious. We prove sufficiency. Let $\varphi\colon \Gamma_{K_1} \to \Gamma_{K_2}$ be an isomorphism. We remark that the manifolds E_{K_1} and E_{K_2} are irreducible and sufficiently large. If the knots K_1 and K_2 are nontrivial, then they are also boundary irreducible. Suppose, first, that one of the knots (say K_1) is trivial. Then $G_{K_1} \simeq \mathbf{Z}$. Since $G_{K_2} \simeq \mathrm{As}(\Gamma_{K_2}) \simeq \mathrm{As}(\Gamma_{K_1}) \simeq G_{K_1}$, the knot K_2 is also trivial and the theorem holds in this case. Consider the case that both K_1 and K_2 are nontrivial. Since the action of G_{K_2} on Γ_{K_2} is transitive by Lemma 2, there exists $g \in G_{K_2}$ such that $g(\varphi(s_{K_1})) = s_{K_2}$. By composing the isomorphism φ with the action of g we may assume at the outset that φ carries s_{K_1} to s_{K_2}. By the corollary to Theorem 1, the groups G_{K_i} are associated to Γ_{K_i} ($i = 1, 2$). Thus, the isomorphism $\varphi\colon \Gamma_{K_1} \to \Gamma_{K_2}$ induces an isomorphism $\psi\colon G_{K_1} \to G_{K_2}$. Since H_{K_i} coincides with the isotropy subgroup of the element s_{K_i} and since $\varphi(s_{K_1}) = s_{K_2}$, the map ψ is an isomorphism of H_{K_1} to H_{K_2} and it carries the meridian m_{K_1} to m_{K_2}. According to Waldhausen's theorem, there exists a homeomorphism $f\colon E_{K_1} \to E_{K_2}$ inducing ψ. Since $\psi(m_{K_1}) = m_{K_2}$, the map f extends to the desired homeomorphism $f'\colon S^3 \to S^3$.

§11. Applications

Let $\mathfrak{I}$ denote the category of distributive groupoids. The objects of $\mathfrak{I}$ are distributive groupoids and the morphisms are homomorphisms of groupoids. Let $F\colon \mathcal{C} \to \mathfrak{I}$ be a functor from some category $\mathcal{C}$ to $\mathfrak{I}$. Let $\Gamma \in \mathrm{Ob}\,\mathfrak{I}$.

DEFINITION. An object $A \in \mathrm{Ob}\,\mathcal{C}$ is said to be *associated to* the groupoid Γ by the homomorphism $f_0\colon \Gamma \to F(A)$ if for any other object $B \in \mathrm{Ob}\,\mathcal{C}$ and any other homomorphism $f\colon \Gamma \to F(B)$ there is a unique homomorphism $h\colon A \to B$ such that $f = F(h)f_0$.

The uniqueness of the associated object follows from general category theoretic considerations. We will denote it by $\mathrm{As}_F(\Gamma)$. It is clear that, for any knot K, the object $\mathrm{As}_F(\Gamma_K)$ is an invariant of the knot K.

EXAMPLE 1. Let $\mathcal{G}$ be the category of groups and homomorphisms. For every integer n let F_n be the functor which associates to each $G \in \mathrm{Ob}\,\mathcal{G}$ the groupoid $G^{(n)}$. Then the groups $\mathrm{As}_{F_n}(\Gamma_K)$ are invariants of the knot K. The invariants $\mathrm{As}_{,1}(\Gamma_K)$ coincide with the fundamental group of the knot K. The invariant $\mathrm{As}_{F_0}(\Gamma_K)$ is trivial. It is unclear as to whether the invariants $\mathrm{As}_{F_n}(\Gamma_K)$, $n \neq 0$, ± 1, can be expressed in terms of known invariants.

EXAMPLE 2. The association to each group G of its core (that is, the set G with the operation $a \circ b = ba^{-1}b$) defines a functor $F: \mathcal{G} \to \mathcal{T}$. The object $\mathrm{As}_F(\Gamma_K)$ is called the *core* of the knot K (the terminology is borrowed from the theory of semigroups). The core of a knot is a new invariant.

EXAMPLE 3. Let $\mathfrak{M}$ be the category of left modules over the ring $\mathbf{Z}[t, t^{-1}]$ of Laurent polynomials, and let F be the functor which associates to each module $M \in \mathrm{Ob}\,\mathfrak{M}$ the groupoid M_t (see Example 1 in §2). Then the module $\mathrm{As}_F(\Gamma_K)$ coincides with the Alexander module of the knot K.

EXAMPLE 4. Let $\mathcal{K}$ denote the category of commutative distributive semigroups and let $F: \mathcal{K} \to \mathcal{T}$ be the natural inclusion. Then the commutative semigroup $\mathrm{As}_F(\Gamma_K)$ is a new invariant of the knot K. In fact it coincides with the abelianization of Γ_K. Unlike the abelianized knot group, the invariant $\mathrm{As}_F(\Gamma_K)$ is nontrivial. It differentiates the trefoil from the figure-eight knot.

REMARK 1. The existence of the associated objects in Examples 1–4 is established by construction. If Γ_K is given by generators $A_{\bar{K}}$ and relations $R_{\bar{K}}$, then the associated object is given with the help of the generators $A_{\bar{K}}$ and relations $R'_{\bar{K}}$. The set of relations $R'_{\bar{K}}$ consists of relations of the form $b^n a b^{-n} = c$ in Example 1, of the form $ba^{-1}b = c$ in Example 2, and of the form $ta + (1 - t)b = c$ in Example 3. We get one such relation for each relation $a \circ b = c$ in $R_{\bar{K}}$. In Example 4, the sets of relations coincide.

REMARK 2. A groupoid Γ_K can be defined for links K in an arbitrary oriented 3-manifold M in a manner similar to what has been done for knots in S^3.

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