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LONGITUDES OF A LINK AND PRINCIPALITY OF AN ALEXANDER IDEAL

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ABSTRACT. In this note it is shown that the longitudes of a μ-component homology boundary link L are in the second commutator subgroup G'' of the link group G if and only if the μth Alexander ideal $\mathcal{E}_\mu(L)$ is principal, generalizing the result announced for $\mu = 2$ by R. H. Crowell and E. H. Brown. These two properties were separately hypothesized as characterizations of boundary links by R. H. Fox and N. F. Smythe.

For a μ-component homology boundary link L the first nonvanishing Alexander ideal is $\mathcal{E}_\mu(L)$. If L is actually a boundary link, then $\mathcal{E}_\mu(L)$ is principal and the longitudes of L lie in the second commutator subgroup of the link group [**2**], [**6**]. R. H. Crowell and E. H. Brown have announced that the latter two assertions are equivalent for a 2-component homology boundary link [**2**]. This note presents a proof of the following generalization.

THEOREM. *Let L: $\bigcup_{i=1}^{\mu} S_i^1 \to S^3$ be a (locally flat) μ-component homology boundary link, with group G. Then $\mathcal{E}_\mu(L) = (\Delta_\mu) \cdot A$ where A is contained in the annihilator ideal (in*

$$\Lambda = \mathbf{Z}[\mathbf{Z}^\mu] \approx \mathbf{Z}[t_1, t_1^{-1}, \dots, t_\mu, t_\mu^{-1}])$$

of the image of the longitudes in the Λ-module G'/G'', and A is contained in no proper principal ideal. Hence $\mathcal{E}_\mu(L)$ is principal if and only if the longitudes of L lie in G''.

PROOF. L extends to an imbedding N: $\bigcup_{i=1}^{\mu} S_i^1 \times D^2 \to S^3$, since it is locally flat. Let $X = S^3 - \operatorname{int}(\operatorname{Im}(N))$ have base point $x_0 \in X - \partial X$. Then $G \approx \pi_1(X, x_0)$. Let p: $X' \to X$ be the maximal abelian cover of X and choose $x_0' \in p^{-1}(x_0)$, so that $\pi_1(X', x_0') \approx G'$ and $H_1(X') = G'/G''$. By definition of homology boundary link there is a map

$$f: (X, x_0) \to \Big(\bigvee_{j=1}^{\mu} S_j^1, * \Big)$$

inducing an epimorphism of fundamental groups, and p is the pullback via f of the maximal abelian cover of $\bigvee_{j=1}^{\mu} S_j^1$. Thus X' may be constructed by splitting X along "Seifert surfaces", as was done in [**3**] for boundary links. For

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each j such that $1 \leqslant j \leqslant \mu$, choose $P_j \in S_j^1$ distinct from the wedge-point $*$, and let $V_j = f^{-1}(P_j)$. After homotoping f if necessary, each V_j may be assumed a connected, bicollared submanifold. Let $Y = X - \bigcup_{j=1}^{\mu} \operatorname{int} W_j$, where the W_j are disjoint regular neighborhoods of the V_j in X. There are two natural embeddings of each V_j in Y; call one ν_{j+} and the other ν_{j-}. (Making such a choice is equivalent to choosing a local orientation for each P_i in $\bigvee_{j=1}^{\mu} S_j^1$, or choosing orientations for the meridians of L.) Y is a deformation retract of $X - V$, where $V = \bigcup_{j=1}^{\mu} V_j$. Then one has

$$X' = Y \times \mathbf{Z}^{\mu}/\nu_{j+}(w) \times \langle n_1, \ldots, n_j + 1, \ldots, n_{\mu}\rangle$$
$$\sim \nu_{j-}(w) \times \langle \dot{n}_1, \ldots, n_j, \ldots, n_{\mu}\rangle, \quad \forall w \in V_j, \quad 1 \leqslant j \leqslant \mu.$$

$G'/G'' = H_1(X')$ then appears in the following segment of a Mayer-Vietoris sequence:

$$H_1(V) \otimes \Lambda \xrightarrow{d_1} H_1(Y) \otimes \Lambda \to H_1(X')$$
$$\to H_0(V) \otimes \Lambda \xrightarrow{d_0} H_0(Y) \otimes \Lambda \to \mathbf{Z} \to 0$$

where $d_*|H_*(V_j) \otimes \Lambda = (\nu_{j+})_* \otimes t_j - (\nu_{j-})_* \otimes 1$ and homology is taken with integral coefficients. The map f induces a map from this Mayer-Vietoris sequence to the corresponding one for the maximal abelian covering space of $\bigvee_{j=1}^{\mu} S_j^1$:

$$0 - F(\mu)'/F(\mu)'' \to \Lambda^{\mu} \to \Lambda \xrightarrow{\varepsilon} \mathbf{Z} \to 0.$$

(Here $F(\mu)$ is the free group of rank μ, and ε: $\Lambda \to \mathbf{Z}$ is the augmentation homomorphism.) Since each V_j is connected, the maps on the degree zero terms are all isomorphisms. Thus one concludes that

$$H_1(V) \otimes \Lambda \xrightarrow{d_1} H_1(Y) \otimes \Lambda \to K \to 0$$

is exact, where

$$K = \ker(: G'/G'' \to F(\mu)'/F(\mu)'') = \ker(: H_1(X') \to H_0(V) \otimes \Lambda).$$

Likewise f induces a map from the 4 term exact sequence of Crowell [**1**]

$$0 \to G'/G'' \to A(G) \to \Lambda \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

to the corresponding sequence for $F(\mu)$ (which is just the above Mayer-Vietoris sequence for $\bigvee_{j=1}^{\mu} S_j^1$) and so $0 - K \to A(G) \to A(F(\mu)) = \Lambda^{\mu} \to 0$ is exact. From this last short exact sequence one concludes that $\mathfrak{E}_k(L) = \mathfrak{E}_k(A(G))$ is equal to the ideal generated by $\bigcup_{l=0}^{k} \mathfrak{E}_l(K) \cdot \mathfrak{E}_{k-l}(\Lambda^{\mu})$; in particular $\mathfrak{E}_{\mu-1}(L) = 0$ and $\mathfrak{E}_{\mu}(L) = \mathfrak{E}_0(K)$.

Now the Λ-submodule of $H_1(X')$ generated by the longitudes is the image of $H_1(\partial X')$ via the inclusion map, and is contained in the image of $H_1(Y) \otimes \Lambda$, so is contained in K. Let B be this submodule, and let Q be the quotient Λ-module. Thus $0 - B - K \to Q \to 0$ is exact, and $\mathfrak{E}_0(K) = \mathfrak{E}_0(Q) \cdot \mathfrak{E}_0(B)$ (because Q has a square presentation matrix–see below). It is easy to see that $(\operatorname{Ann}(B))^{\mu} \subset \mathfrak{E}_0(B)$: if

$$\Lambda^{\lambda} \xrightarrow{M} \Lambda^{\mu} \xrightarrow{\varphi} B \to 0$$

is a presentation for B with $\varphi(e_i) = e$th longitude (where e_i is the ith standard basis element of Λ^{μ}), and if $\alpha_1, \ldots, \alpha_\mu \in \mathrm{Ann}(B)$ then

$$\Lambda^{\lambda} \oplus \Lambda^{\mu} \to \Lambda^{\mu} \xrightarrow{\tilde{M}} B \xrightarrow{\varphi} 0$$

is also a presentation for B, where $\tilde{M} = (M, \mathrm{diag}\{\alpha_1, \ldots, \alpha_\mu\})$, and so

$$\prod_{i=1}^{\mu} \alpha_i = \det(\mathrm{diag}\{\alpha_1, \ldots, \alpha_\mu\}) \in \mathfrak{E}_0(B).$$

It is scarcely more difficult to see that $\mathfrak{E}_0(B) \subset \mathrm{Ann}(B)$: let δ be the determinant of the $\mu \times \mu$ minor M'' of M. Then

$$\Lambda^{\mu} \xrightarrow{M''} \Lambda^{\mu} \to \mathrm{Coker}\, M'' \to 0$$

presents a module of which B is a quotient. Now if $\Sigma m_i e_i \in \Lambda^{\mu}$, then by Cramer's rule $\delta \cdot \Sigma m_i e_i = M''(\Sigma n_j e_j)$ where n_j is the determinant at the matrix obtained by replacing the ith column of M'' with the column of coefficients $\{m_i\}$. Hence δ annihilates Coker M'', and a fortiori, B. Therefore $\mathfrak{E}_0(B)$, which is generated by such determinants, is contained in $\mathrm{Ann}(B)$. Thus to prove the theorem it will suffice to show that $\mathfrak{E}_0(B)$ is not contained in any proper principal ideal, and that Q has a presentation of the form $\Lambda^q \xrightarrow{P} \Lambda^q \to Q \to 0$ so that $\mathfrak{E}_0(Q) = (\det P)$ is principal.

Choose base points in $V_i \cap \partial N(S_i^1 \times D^2)$ for each i, $1 \leqslant i \leqslant \mu$, and choose paths from these base points to α_0. (Equivalently, X' contains copies of V_i indexed by $\mathbf{Z}^{\mu}$. Choose one such lift, V_i', for each i.) If one now orients the link L, the longitudes are unambiguously defined, as elements of G. Let l_i be the image of the ith longitude in B. Since the ith longitude commutes with the ith meridian, one has $(t_i - 1)l_i = 0$. In contrast to the case of boundary links, ∂V_j will in general have several components; however $\partial V_j \cap \partial N(S_i^1 \times D^2)$ is always homologous in $\partial N(S_i^1 \times D^2)$ to the ith longitude, if $j = i$, and to 0 otherwise. $\partial V_i'$ is a union of translates of loops in the homology classes $l_1, \ldots, l_\mu$. Hence there are relations of the form

$$\sum_{i=1}^{\mu} p_{ij}(t_1, \ldots, t_\mu) l_j = 0$$

in B, and by the above remarks on ∂V_j, one has $p_{ij}(1, \ldots, 1) = 0$ for $i \neq j$ and $p_{ii}(1, \ldots, 1) = \pm 1$. Since $t_i \cdot l_i = 1 \cdot l_i$, one may assume that $p_i = p_{ii}(t_1, \ldots, t_\mu)$ does not involve t_i. Clearly $p_i \Pi_{j \neq i}(t_j - 1)$ is the determinant of a $\mu \times \mu$ matrix of relations for B, and so is in $\mathfrak{E}_0(B)$. (For what follows it would be sufficient to observe that it clearly annihilates B, and so the μth power is in $\mathfrak{E}_0(B)$.) Let (c) be a principal ideal containing $\mathfrak{E}_0(B)$. Since Λ is a factorial domain, c may be assumed irreducible. Therefore $p_1 \Pi_{j>1}(t_j - 1) \in (c)$ implies c divides p_1 or some $(t_j - 1)$ for $j > 1$. If $c = t_j - 1$, then c cannot divide $p_j \Pi_{k \neq j}(t_k - 1)$ which does not involve t_j. If c divides p_i for each i,

$1 \leqslant i \leqslant \mu$, then c involves none of the variables and hence is in $\mathbf{Z}$. Since $p_i(1, \ldots, 1) = \pm 1$, $c = \pm 1$ and so $(c) = \Lambda$.

Let $J = \ker(: H_1(X - V, \partial X - V) \to H_0(\partial X - V)) = H_1(X - V)/H_1(\partial X - V)$. From the following commutative diagram of Λ-modules

$$\begin{array}{ccccc}
H_1(\partial V) \otimes \Lambda & \longrightarrow & H_1(V) \otimes \Lambda & \longrightarrow & H_1(V, \partial V) \otimes \Lambda \\
\downarrow & & \downarrow & & \downarrow \\
H_1(\partial X - V) \otimes \Lambda & \longrightarrow & H_1(X - V) \otimes \Lambda & \longrightarrow & H_1(X - V, \ \partial X - V) \otimes \Lambda \\
\downarrow & & \downarrow & & \downarrow \\
H_1(\partial X') & \longrightarrow & H_1(X') & \longrightarrow & H_1(X', \partial X')
\end{array}$$

(with rows from exact sequences of pairs and columns from Mayer-Vietoris sequences of $\mathbf{Z}^\mu$-covers), one deduces a commutative diagram

$$\begin{array}{ccccccc}
H_1(\partial V) \otimes \Lambda & \longrightarrow & H_1(V) \otimes \Lambda & \twoheadrightarrow & H_1(V)/H_1(\partial V) \otimes \Lambda & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_1(\partial X - V) \otimes \Lambda & \longrightarrow & H_1(X - V) \otimes \Lambda & \twoheadrightarrow & J \otimes \Lambda & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_1(\partial X') & \longrightarrow & K & \twoheadrightarrow & Q & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

in which all rows and the first two columns are exact. It follows that the third column is exact, and so

$$(H_1(V)/H_1(\partial V)) \otimes \Lambda \to J \otimes \Lambda \to Q \to 0$$

is a presentation for Q. Let $\rho = rk_{\mathbf{Z}} H_1(V)$, $\sigma = rk_{\mathbf{Z}} H_1(\partial V)$. Since $0 \to H_2(V, \partial V) \to H_1(\partial V) \to H_1(V)$ is exact, one has $rk_{\mathbf{Z}}(H_1(V)/H_1(\partial V)) = \rho - \sigma + \mu$. Similarly,

$$H_1(X - V, \partial X - V) \to H_0(\partial X - V) \to H_0(X - V) \to 0$$

is exact, and $rk_{\mathbf{Z}} H_0(\partial X - V) = \sigma$, $rk_{\mathbf{Z}} H_0(X - V) = 1$, so

$$\begin{aligned} rk_{\mathbf{Z}} J &= rk_{\mathbf{Z}} H_1(X - V, \partial X - V) - \sigma + 1 \\ &= rk_{\mathbf{Z}} H_1(S^3 - V, \operatorname{Im} N) - \sigma + 1. \end{aligned}$$

Now each component of the link is the homology boundary of a (singular) surface in $S^3 - V$, and so the natural map

$$H_1(\operatorname{Im} N) \to H_1(S^3 - V)$$

is null. Therefore

$$0 - H_1(S^3 - V) \to H_1(S^3 - V, \operatorname{Im} N) \to H_0(\operatorname{Im} N) \to H_0(S^3 - V) \to 0$$

is exact, and so $rk_{\mathbf{Z}} H_1(S^3 - V, \ \operatorname{Im} N) = rk_{\mathbf{Z}} H_1(S^3 - V) + \mu - 1 = rk_{\mathbf{Z}} H_1(V) + \mu - 1$ by Alexander duality $= \rho + \mu - 1$. Thus $rk_{\mathbf{Z}} J = \rho + \mu$

$-\sigma = rk_{\mathbf{Z}}(H_1(V)/H_1(\partial V))$, and so $\mathfrak{E}_0(Q)$ is principal. This completes the proof of the theorem.

REMARKS. 1. Brown and Crowell asserted the somewhat more precise result (for $\mu = 2$) that A could be generated by 3 elements, of the form $(t_1 - 1)p_1(t_1)$, $(t_2 - 1)p_2(t_2)$ and $p_1(t_1) + p_2(t_2) - 1$ where $p_i(1) = 1$, and that the ith longitude lay in G'' if and only if $p_{3-i}(t_{3-i})$ were a unit **[2]**. This follows readily from $A = A_1 \cap A_2$, where A_i is the annihilator of the ith longitude and equals $(t_i - 1, p_{3-i}(t_{3-i}))$ for some p_i, as above.

2. Fox and Smythe conjectured that if the longitudes were in G'', then the link would be a boundary link **[6]**. H. W. Lambert has constructed a 2-component homology boundary link which is not a boundary link, as a counterexample to this conjecture **[4]**. (Figure 1 of his paper is incorrectly drawn: the shorter longitude of this example does *not* map to 0 in the Alexander module (via Crowell's inclusion $0 - G'/G'' \to A(G)$ **[1]**) and hence this link is not such a counterexample.[1]) Notice also that boundary links have the stronger (but less tractable?) property that the longitudes are in $(G_\omega)'$ (where $G_\omega = \bigcap_{n=1}^{\infty} G_n$ is the intersection of the terms of the lower central series). This follows from the construction of the ω-covering by splitting the link complement along Seifert surfaces, as in **[3]**.

3. If L is trivial then $\mathfrak{E}_\mu(L) = \Lambda$, but the converse is false, even for knots ($\mu = 1$), for there exists nontrivial knots (for instance doubled knots with twist number 0) with Alexander polynomial 1 **[5]**.

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[1]Lambert has advised me that his argument is based on a slightly different figure.