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Conjugate points in length spaces

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Abstract

In this paper we extend the concept of a conjugate point in a Riemannian manifold to geodesic spaces. In particular, we introduce symmetric conjugate points and ultimate conjugate points and relate these notions to prior notions developed for more restricted classes of spaces. We generalize the long homotopy lemma of Klingenberg to this setting as well as the injectivity radius estimate also due to Klingenberg which was used to produce closed geodesics or conjugate points on Riemannian manifolds. We close with applications of these new kinds of conjugate points to $\mathrm{CBA}(\kappa)$ spaces: proving both known and new theorems. In particular we prove a Rauch comparison theorem, a Relative Rauch Comparison Theorem, the fact that there are no ultimate conjugate points less than π apart in a $\mathrm{CBA}(1)$ space and a few facts concerning closed geodesics. This paper is written to be accessible to students and includes open problems.

Keywords: Metric geometry; Conjugate points; Long homotopy; Klingenberg; Closed geodesic; Length space; Geodesic space; Rinow

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0. Introduction

Any student of classical Riemannian Geometry is introduced to the notions of geodesics, exponential maps, Jacobi fields and conjugate points. They also learn about the cut locus and how geodesics stop minimizing when they pass through cut or conjugate points. They learn how to construct continuous families about geodesics whose endpoints are not conjugate using the Inverse Function Theorem and how Klingenberg used this approach to prove theorems about long homotopies and the existence of geodesics. Naturally when they study spaces with sectional curvature bounds, they first learn the Rauch comparison theorem and its implications. Everything is proven using geometric analysis and the fact that the space is smooth (see for instance [10]).

On geodesic spaces things are not so simple. There is no exponential map in general and geodesics need not extend. Even if they do extend, there need not be a unique extension. Families of geodesics need not have Jacobi fields describing their infinitesimal behavior. There is not enough smoothness to apply the Inverse Function Theorem to prove the existence of families of geodesics. Rinow extended the notion of a conjugate point to geodesic spaces with extensible geodesics and unique local geodesics in his book [18]. His notion is equivalent to what we describe as a one-sided conjugate point: *a point q along a geodesic γ emanating from p is one-sided conjugate to p if there are points q_i converging to q and distinct geodesics σ_i, τ_i joining p to q_i converging to γ*. The same notion was applied in the setting of CBA(κ) spaces by Alexander and Bishop [1] and for exponential length spaces by the second author [20].

To enable our extension of these notions to geodesic spaces, we need to make a more precise definition of the convergence of the geodesics (see Definition 1.4). This in turn requires spaces that are locally uniformly minimizing, i.e., spaces with neighborhoods such that all geodesics of sufficiently small length are minimizing. We do not require local uniqueness of geodesics as in the above cited work. The various definitions are given in Section 1.

By a theorem of Warner [22] we see that the notion of a one-sided conjugate point is indeed equivalent to the usual notion in the Riemannian setting. However, the notion of one-sided conjugate point is not the only notion of conjugate point that is equivalent to the standard notion on a Riemannian manifold. While it was effective in proving the earlier results in [1] and [20], the fact that p does not vary restricts its applications significantly. It cannot, for example, be used to extend the work of Klingenberg. Jacobi fields on Riemannian manifolds are infinitesimal representations of families of geodesics without requiring a common basepoint and in Klingenberg's work he allows both ends of the geodesics to vary. This leads us naturally to the notion of a *symmetric conjugate point* (Definition 2.3). The definition is similar to one-sided conjugate except now we have sequences p_i, q_i converging to p, q respectively and pairs of distinct geodesics joining them converging to γ as before. Nevertheless, in order to prove the long homotopy lemma we need an even stronger notion of conjugate point to prove the existence of continuous families of geodesics (Definitions 3.1 and 3.3). This is developed in Sections 3 and 4 and leads to the notion of *unreachable conjugate point* (Definition 4.1). While the notion of unreachable conjugate point is needed to prove the existence of continuous families it is not equivalent to the Riemannian notion (see Example 4.7).

In Section 5 we define *ultimate conjugate points* as those that are either symmetric conjugate or unreachable conjugate (Definition 5.1). We prove this concept is an extension of the Riemannian notion (Theorem 5.3). In fact a geodesic whose endpoints are not ultimate conjugate has a unique family of geodesics about it (Proposition 5.4). It should be noted that Rinow was able to prove the existence of a fixed basepoint continuous family away from one-sided conjugate points in his book by essentially constructing a global exponential map [18]. Alexander and Bishop were able to prove the existence of continuous families under curvature bounds (see Theorem 11.1). Here we do not require the space to have curvature bounds or an exponential map.

In Section 6 we extend Klingenberg's Long Homotopy Lemma to geodesic spaces. That is we prove: *If a closed, contractible, nontrivial geodesic has length less than twice the ultimate conjugate radius, then any null homotopy $H(s,t)$ of the closed geodesic must pass through a curve $c_{t_0} = H(\cdot, t_0)$ of length at least twice the ultimate conjugate radius* (see Definition 6.1 and Theorem 6.2). To prove this we first construct *fans* of geodesics running along curves (see Definition 6.3, Lemma 6.4, Fig. 1). We carefully control the lengths of the geodesics in the fans (Lemma 6.5, Fig. 2, Corollary 6.6) and then apply these fans to fill in two-dimensional homotopies (Lemma 6.7).

In Section 7 we review various notions which extend the concept of the injectivity radius of a Riemannian manifold. Recall that on a Riemannian manifold geodesics stop minimizing when they are no longer the unique geodesic back to their starting point. On a geodesic space, like a graph, a geodesic may continue to be minimizing. To this end we offer various refined notions of injectivity radius (and corresponding cut loci) all of which extend the Riemannian notion: FirstInj, UniqInj, MinRad, SymInj and UltInj. We establish various inequalities between them. For example the unique injectivity radius (the notion in [7, p. 119] which is the supremum of the distance between points where geodesics are unique) is less than or equal to the minimal radius (which is defined in terms of geodesics being minimal). These radii and cut loci are related to the variety of notions defined by Burago, Burago and Ivanov, Miller and Pak, Otsu and Shioya, Plaut and Zamfirescu (cf. [8,14–16,23]).

In Section 8, we prove Klingenberg's Injectivity Radius Estimate (Theorem 8.3): *If a compact length space, X, with minimal injectivity radius,* $\mathrm{MinRad}(X) \in (0,\infty)$ *then either there is a pair of ultimate conjugate points, p, q with $d(p,q) \leqslant$* $\mathrm{MinRad}(X)$ *or there is a closed geodesic $\gamma : S^1 \to X$ with $L(\gamma) = 2\,\mathrm{MinRad}(X)$*. It should be noted that the minimal radius of a flat disk is infinity (Example 7.5) but that the minimal radius is finite in any compact space with at least one extensible geodesic (Lemma 8.2), so Theorem 8.3 can be used for a large class of spaces.

Section 9 gives a brief review of some essential points about $\mathrm{CBA}(\kappa)$ spaces including their definition. Recall that the $\mathrm{CBA}(\kappa)$ property is an extension to geodesic spaces of the Riemannian notion of having sectional curvature bounded above by κ (Theorem 9.3). We review work of Gromov, Charney and Davis, Ballmann and Brin and Otsu and Shioya.

In Section 10 we prove two Rauch Comparison Theorems for $\mathrm{CBA}(\kappa)$ spaces. The first is a known comparison theorem related to us by Stephanie Alexander for one-sided conjugate points and extended to symmetric conjugate points (Theorem 10.1). It states that *on a* $\mathrm{CBA}(\kappa)$ *space all symmetric conjugate points are further than D_κ apart*. To prove the theorem we describe the notion of a *bridge* (see Definition 10.3, Fig. 7) which may be regarded as a coarse analog of Jacobi field for length spaces. The second theorem in Section 10 extends the Jacobi field relative comparison theorem which says, for example, that on a Riemannian manifold with **sec** $\leqslant 1$, $J(t)/\sin(t)$ is nondecreasing. Since our Relative Rauch Comparison Theorem (Theorem 10.5) is stated with bridges and is not infinitesimal it has error terms depending on the height of the bridges. As the height decreases to 0 we get the same control on our bridges as one has on Jacobi fields in the Riemannian setting.

Section 11 concerns Theorem 11.1: *On a* $\mathrm{CBA}(\kappa)$ *space all ultimate conjugate points are further than D_κ apart*. To prove this one must show that any geodesic of length less than D_κ in a $\mathrm{CBA}(\kappa)$ space has a continuous family around it. This is a result of Alexander and Bishop stated within the proof of Theorem 3 in [1]. The proof is given in more detail in Ballmann's textbook on manifolds of nonpositive curvature (see Theorem 4.1 in [5]). A proof based on the

Rauch Comparison Theorem is described in this section. We have not included our proof in the published version of this paper but it is available on the arXiv preprint.

In Section 12 we apply the results in Section 11 to the work in the first half of the paper. First we prove Theorem 12.1: *On a locally compact* CBA(κ) *space, any null homotopy of a contractible closed geodesic of length* $< 2D_\kappa$ *passes through a curve of length* $\geqslant 2D_\kappa$. We then prove Corollary 12.3 of our Klingenberg Injectivity Radius Theorem: *A compact* CBA(κ) *length space, X, with $\kappa > 0$ such that* $\mathrm{MinRad}(X) \in (0, D_\kappa)$ *has a closed geodesic of length twice the minimal radius*. We relate this to the Charney–Davis proof of Gromov's systole theorem. We close with a discussion of possible applications of Theorem 6.2 (the generalized Long Homotopy Lemma) to CBA(κ) spaces with higher spherical rank. While Example 12.6 shows that one cannot hope to conclude rigidity as in [19], one may still be able to prove some interesting results for such spaces.

Open problems are suggested throughout the paper; see for instance Problems 2.5, 3.8, 4.5, 7.22, 12.2 and 12.5. Some of these are perhaps not too difficult and may be appropriate for graduate students while others like Problems 12.2 and 12.5 may require significant work.

1. Geodesic spaces

In this section we review the concept of geodesic spaces and geodesics in such spaces. We introduce a new kind of convergence of geodesics in such spaces which extends the concept of convergence of geodesics in Riemannian manifolds and is stronger than sup norm convergence (Definition 1.5 and Lemma 1.11). We then introduce the concept of a locally uniformly minimizing length space (Definition 1.9) which includes Riemannian manifolds and CBA(κ) spaces. A review of CBA(κ) spaces is given in Section 9. We then prove the Geodesic Arzela–Ascoli Theorem (Theorem 1.13) for compact length spaces which are locally uniformly minimizing.

Definition 1.1. A *geodesic space*, X, is a complete metric space such that any pair of points is joined by a rectifiable curve whose length is the distance between the points. This curve is called a minimizing geodesic and it is shorter than any other curve joining the two points.

These spaces are referred to as geodesic spaces in [7] and as strictly intrinsic geodesic spaces in [8]. A complete Riemannian manifold is a geodesic space by the Hopf–Rinow Theorem (cf. [10]). In fact Hopf and Rinow proved that any complete locally compact length space is a geodesic space (cf. [11]). Note however, that other aspects of the Riemannian Hopf–Rinow Theorem do not hold on these more general spaces.

Example 1.2. A metric space, X, can be created by taking a collection of line segments $[0, 1 + 1/j]$ and gluing all the left endpoints to a common point, p, and all the right endpoints to a common point, q. The metric on X can be defined as the infimum of the lengths of all rectifiable curves running between the given points. Note that p and q have infinitely many geodesics running between them and the distance between them is 1. Although X is a complete metric space, it is not a geodesic space because the distance between p and q is not achieved. If we add one more segment $[0, 1]$ running from p to q then we obtain a geodesic space, Y.

Definition 1.3. A *geodesic* in a geodesic space is a curve which is locally minimizing. A *closed geodesic* is a map $\gamma : S^1 \to X$ which is locally minimizing (generalizes smoothly closed). All geodesics are parametrized proportional to arclength.

Geodesics are sometimes called "local geodesics." Here we will always say minimizing geodesics when the geodesic achieves the distance between two points (following the convention in Riemannian geometry). Geodesics need not be unique (even locally), they may branch and they are not necessarily extensible.

Since there is no notion of a tangent space or a starting vector, the notion of convergence of geodesics needs to be clarified. In order to compare geodesic segments of different lengths, we will reparametrize them so that they are defined on unit intervals.

Definition 1.4. Let $\Gamma([0,1],X) \subseteq C_0([0,1],X)$ be the space of geodesic segments, $\gamma : [0,1] \to X$, parametrized proportional to arclength considered as a subset of the space of continuous functions on $[0,1]$ with the sup norm:

$$|\gamma_1 - \gamma_2|_0 = \sup_{t\in[0,1]} d_X\big(\gamma_1(t), \gamma_2(t)\big). \tag{1.1}$$

We say γ_i converge as geodesics to γ if they converge in this space once they are reparametrized to be defined on $[0,1]$.

Note that the sup norm is not in fact a norm but just a metric on $C_0([0,1],X)$ and $\Gamma([0,1],X)$.

Definition 1.5. Define a metric d_Γ on $\Gamma([0,1],X) \subseteq C_0([0,1],X)$ to be

$$d_\Gamma(\gamma_1,\gamma_2) = |\gamma_1 - \gamma_2|_0 + \big|L(\gamma_1) - L(\gamma_2)\big|. \tag{1.2}$$

See Example 1.12 as to why this definition gives a stronger definition of convergence than just the sup norm convergence. Later we also see that they give the same topology when the space is a Riemannian manifold.

Lemma 1.6. *In a compact length space, if γ_i are uniformly minimizing in the sense that they run minimally between points of some uniform interval of width 2δ, and if they have a uniform upper bound on length, then a subsequence converges in C_0 to γ and*

$$\lim_{i\to\infty} L(\gamma_i) = L(\gamma). \tag{1.3}$$

Proof. One first applies Arzela–Ascoli to show a limit γ exists. Since

$$\lim_{i\to\infty} d\big(\gamma_i(t_i-\delta), \gamma_i(t_i+\delta)\big) = d\big(\gamma(t-\delta), \gamma(t+\delta)\big) \leqslant L\big(\gamma\big([t-\delta, t+\delta]\big)\big) \tag{1.4}$$

$$\leqslant \lim_{i\to\infty} L\big(\gamma_i\big([t-\delta, t+\delta]\big)\big) = \lim_{i\to\infty} d\big(\gamma_i(t_i-\delta), \gamma_i(t_i+\delta)\big), \tag{1.5}$$

so all are equalities, we see that γ is a geodesic. To see that the lengths converge, we sum up the lengths of the minimizing segments and take the limit again. □

Without the uniform minimizing property this need not be true as can be seen in the well-known example of a cube.

Example 1.7. If X is a 2-dimensional cube, then squares which avoid the edges of the cube are geodesics, but when they reach a square lying on the edge, this last curve is not a geodesic. The edge square is not minimizing on any intervals about the points on the corners because there are short cuts in the face. Intuitively, wrap a rubber band around four faces of a cube and then slide it to one side. It snaps off when you reach the edge.

A study of geodesics with uniform minimizing properties appears in [21]. Here however, we do not want to restrict our collection of geodesics and so we restrict our spaces instead. Recall the following definition; spaces with this property have been discussed in [21].

Definition 1.8. A geodesic space is uniformly minimizing if there exists a $\delta > 0$ such that all geodesic segments of length $\leqslant \delta$ are minimizing.

Definition 1.9. We say a geodesic space is locally uniformly minimizing if about every point p in the space there is a neighborhood U_p and a length ϵ_p, such that any geodesic in the neighborhood U_p of length $\leqslant \epsilon_p$ is minimizing.

This concept is distinct from that of a locally minimizing space where minimizing geodesics are required to be unique and lengths are not uniformly estimated. Note that Riemannian manifolds and $\mathrm{CBA}(\kappa)$ spaces are locally uniformly minimizing. See for instance Proposition 1.4 (1) and (2) on page 160 of [7] and recall that $\mathrm{CBA}(\kappa)$ spaces are locally $\mathrm{CAT}(\kappa)$.

Lemma 1.10. *Let X be a locally uniformly minimizing geodesic space and geodesics γ_j converge in the sup norm to a curve γ, then γ is a geodesic.*

Proof. Let U be a uniformly minimizing neighborhood about $\gamma(t)$. Take a quarter of the uniform minimizing radius of U and let $\delta < \epsilon_{\gamma(t)}/4$. So $\gamma([t-\delta, t+\delta])$ is a curve lying in U. Eventually $\gamma_i([t-\delta, t+\delta])$ lies in U so it is unique and minimizing. Then we apply (1.4) to prove $\gamma([t-\delta, t+\delta])$ is minimizing as well. □

The cube is not locally uniformly minimizing as can be seen by taking the point p to be one of its corners. The following lemma's proof requires the assumption of locally uniformly minimizing and uses Lemma 1.10. We leave the proof as an exercise.

Lemma 1.11. *In a locally uniformly minimizing length space X, if geodesics γ_i converge to γ in the sup norm, then their lengths converge. As a consequence Definitions* 1.4 *and* 1.5 *are equivalent on locally uniformly minimizing length spaces including* $\mathrm{CBA}(\kappa)$ *spaces and Riemannian manifolds.*

Example 1.12. Let X be the geodesic space formed by gluing an infinite collection of cylinders of different radii r_j together along a common line. Taking $r_j = 1/j$ we see this is the isometric product of the Hawaiian earring with a line. If we take γ to be a geodesic of length 1 lying on the line where all the cylinders were glued, then it is approached uniformly by geodesics γ_j of length 2 lying in the jth cylinder, wrapped extra times around.

Example 1.12 indicates the necessity of the locally uniformly minimizing condition. The necessity of the local compactness can be seen in the Y of Example 1.2 where the sequence of geodesics from p to q of decreasing lengths has no limit since they are uniformly bounded away from each other by a distance at least 1. We can now state the Geodesic Arzela–Ascoli Theorem which gives convergence in the sense of Definition 1.5.

Theorem 1.13 *(Geodesic Arzela–Ascoli Theorem). In a locally uniformly minimizing locally compact geodesic space, a sequence of geodesics with a common upper bound on their lengths whose endpoints converge to points p and q, will have a subsequence which converges in d_Γ to a geodesic with endpoints p and q.*

Proof. Since the endpoints converge and the length is bounded above, all the γ_i lie in a common closed ball. Since the space is locally compact, this closed ball is compact. Since the geodesics are parametrized proportional to arclength, they are equicontinuous and thus by Arzela–Ascoli, they have a subsequence which converges in the sup norm to a curve. By the local uniform compactness and Lemmas 1.10 and 1.11, the subsequence converges in d_Γ. □

We close with the statement of a well-known lemma that will be useful later.

Lemma 1.14. *In a Riemannian manifold, M, γ_i converge to γ in the sup norm if and only if $\gamma_i(0) \to \gamma(0)$ and $\gamma_i'(0) \to \gamma'(0)$ in TM.*

In general length spaces there is no concept of vectors or angles between geodesics emanating from a common point. Even in CAT(k) and CBA(k) spaces, where there is a concept of angle, the angle between two geodesics could be 0 and the geodesics could be distinct. In fact a tree is a length space, and there the geodesics start together and can agree on intervals only to diverge later.

2. Symmetric conjugate points

We can now rigorously state the definition of a one-sided conjugate point. This notion was first introduced by Rinow [18]. His definition of conjugate point used the notion of an exponential map that he had created using the fact that his spaces had extensible nonbranching geodesics which were locally unique. He then proved that his notion was equivalent to this one [18, pp. 414–415]. See Appendix A for a translation of these pages.

Definition 2.1. Given a pair of points p and q in a geodesic space, X, we say q is a *one-sided conjugate to* p along a geodesic γ running between them if there exists a sequence of q_i converging to q such that for each q_i there are two distinct geodesics running from p to q_i, σ_i and γ_i, both of which converge to γ as geodesics as in Definition 1.5. By distinct we mean that there exists t such that $\gamma_i(t) \neq \sigma_i(t)$.

Note that Alexander and Bishop [1] use the weaker definition Definition 1.4 for the convergence but they work in CBA(0) spaces and the definitions are equivalent there by Lemma 1.11. In [20] the spaces were exponential length spaces, where geodesics have initial vectors and the vectors were required to converge. Zamfirescu has a stronger notion of conjugate point which requires that the geodesics be minimizing [23]. In [22], F. Warner proves that Riemannian conjugate implies one-sided conjugate although he did not state it as such.

Theorem 2.2 *(Warner). If M is a complete Riemannian manifold then q is one-sided conjugate to p along γ in the sense of Definition* 2.1 *if and only if it is conjugate to p in the Riemannian sense, i.e., there is a nontrivial Jacobi field along γ which vanishes at p and at q.*

Warner's Theorem is easy to see in one direction just using the Inverse Function Theorem to show that the exponential map is locally one-to-one. However, the opposite direction uses special properties of the exponential map demonstrating that it cannot behave like $f(x) = x^3$ which is one-to-one globally but has a critical point.

In this paper we would like to make a more natural extension of the Riemannian definition of a conjugate point which allows for the variation of both endpoints.

Definition 2.3. Given a pair of points p and q in a geodesic space, X, we say q and p are *symmetrically conjugate* along a geodesic γ running between them if there exists a sequence of q_i converging to q and p_i to p such that for each pair p_i, q_i there are two distinct geodesics σ_i and γ_i running from p_i to q_i both of which converge to γ in $(\Gamma([0,1]), d_\Gamma)$.

Remark 2.4. One could also say they are C_0 symmetrically conjugate if one requires only that σ_i and γ_i converge to γ in the C_0 sense.

Clearly if q is one-sided conjugate to p along γ then p and q are symmetrically conjugate along γ. The converse is less clear.

Open Problem 2.5. Can one construct a length space with a symmetric conjugate point that is not a one-sided conjugate point?

The above question can however be answered for Riemannian manifolds; all notions of conjugate are equivalent in this setting. One of these equivalences in the theorem below is Warner's theorem.

Theorem 2.6. *Let M be a complete Riemannian manifold and suppose p and q are points on a geodesic γ. Then the following statements are equivalent:*

(i) *q is one-sided conjugate to p along γ.*
(ii) *q is symmetrically conjugate to p along γ.*
(iii) *q is conjugate to p in the Riemannian sense, i.e., there is a nontrivial Jacobi field along γ which vanishes at p and at q.*

Proof. If p and q are conjugate in the Riemannian sense then Warner's Theorem says that p is conjugate to q in the sense of Definition 2.1, which immediately implies they are symmetrically conjugate.

If p and $q = \exp_p(v)$ are not conjugate in the Riemannian sense then the map $F : TM \to M \times M$ defined as

$$F(x, w) = \big(x, \exp_x(w)\big) \tag{2.6}$$

has a nonsingular differential at (p, v). Thus by the Inverse Function Theorem this is locally invertible around (p, q) back to points near (p, v). Now suppose on the contrary that p and q

are symmetrically conjugate, so there exists $p_i \to p$ $q_i \to q$ and γ_i, σ_i distinct geodesics from p_i to q_i converging to γ as geodesics. Lemma 1.14 implies that $\gamma_i'(0)$ and $\sigma_i'(0)$ are converging to $\gamma'(0)$. Thus $F(p_i, \gamma_i'(0)) = F(p_i, \sigma_i'(0))$ and it is not locally one-to-one about (p, v). $\square$

3. Continuous families of geodesics

Recall from Definition 1.4 the space of geodesics, $\Gamma([0,1], X) \subset C_0([0,1], X)$ and d_Γ.

Definition 3.1. We say F is a continuous family of geodesics about γ if there are neighborhoods U of $\gamma(0)$ and V of $\gamma(1)$ such that

$$F : U \times V \to \Gamma\big([0,1], X\big), \quad F(x, y) = \gamma, \quad \text{where } x = \gamma(0) \text{ and } y = \gamma(1), \tag{3.7}$$

is continuous with respect to d_Γ. The map F is defined for any $u \in U$, $v \in V$ as $F(u, v) = \sigma$, where σ is a geodesic such that $\sigma(0) = u$ and $\sigma(1) = v$.

Remark 3.2. One could similarly define C_0 continuous families about geodesics.

The following weaker definition will be essential to understanding uniqueness of families. The crucial distinction is that here we do not assume continuity on the whole domain, but only at the endpoints of the given geodesic γ.

Definition 3.3. We say F is a family of geodesics which is continuous at γ if there are neighborhoods U of $\gamma(0)$ and V of $\gamma(1)$ such that

$$F : U \times V \to \Gamma\big([0,1], X\big), \quad F(x, y) = \gamma, \quad \text{where } x = \gamma(0) \text{ and } y = \gamma(1), \tag{3.8}$$

is continuous with respect to d_Γ at $(\gamma(0), \gamma(1))$. The map F is defined for any $u \in U$, $v \in V$ as $F(u, v) = \sigma$, where σ is a geodesic such that $\sigma(0) = u$ and $\sigma(1) = v$.

In the above definition if there are neighborhoods U, V such that the family is unique, i.e., for any other family of geodesics, there exist possibly smaller neighborhoods U', V' on which the two families are the same, then we say that there is a unique family continuous at γ.

Lemma 3.4. *Let X be a geodesic space. If all geodesics γ of length $L(\gamma) < R$ have unique families of geodesics about them which are continuous at $(\gamma(0), \gamma(1))$, then all geodesics γ of length $L(\gamma) < R$ have continuous families of geodesics about them.*

Proof. We begin with a family F about a geodesic γ which is continuous at $(\gamma(0), \gamma(1))$. Note that if we further restrict U and V we can guarantee that all geodesics in this family have lengths $< R$ by the definition of d_Γ.

We need only to show F is continuous on possibly smaller neighborhoods U and V. If not then F is not continuous at a sequence (p_i, q_i) converging to $(\gamma(0), \gamma(1))$. Let $\gamma_i = F(p_i, q_i)$. This means that there exists an $\epsilon_i > 0$ and $(p_{i,j}, q_{i,j})$ converging to (p_i, q_i) such that

$$\lim_{j \to \infty} F(p_{i,j}, q_{i,j}) \neq F(p_i, q_i) \tag{3.9}$$

where it is possible this limit does not exist.

Since $L(\gamma_i) < R$, each γ_i has a family F_i defined about it which is continuous at (p_i, q_i) so in particular

$$\lim_{j\to\infty} F_i(p_{i,j}, q_{i,j}) = \gamma_i = F(p_i, q_i). \tag{3.10}$$

Thus there exists a j_i sufficiently large that

$$F(p_{i,j}, q_{i,j}) \neq F_i(p_{i,j}, q_{i,j}) \quad \forall j \geqslant j_i. \tag{3.11}$$

Choosing j_i possibly larger, we can also guarantee that

$$d_\Gamma\big(F_i(p_{i,j_i}, q_{i,j_i}), \gamma_i\big) < 1/i \tag{3.12}$$

and that $(p_{i,j_i}, q_{i,j_i}) \to (\gamma(0), \gamma(1))$.

Thus we have two distinct geodesics $F(p_{i,j_i}, q_{i,j_i})$ and $F_i(p_{i,j_i}, q_{i,j_i})$ both converging to γ. This contradicts the fact that we have a unique family that is continuous at γ. □

Proposition 3.5. *On a Riemannian manifold two points on a geodesic γ, say $\gamma(0)$ and $\gamma(1)$ are not conjugate along γ if there is a continuous family about γ which is unique among all families of geodesics about γ which are continuous at γ.*

Proof. If $\gamma(0)$ and $\gamma(1)$ are not conjugate then the map $F : TM \to M \times M$ defined as

$$F(x, w) = \big(x, \exp_x(w)\big) \tag{3.13}$$

is locally invertible around $(\gamma(0), \gamma(1))$ back to points near $(\gamma(0), \gamma'(0))$. We then construct the continuous family using this inverse $F^{-1} : U \times V \to TM$ followed by the exponential geodesic map: $\mathrm{Exp} : TM \to \Gamma([0, 1])$ defined as

$$\mathrm{Exp}(p, v) = \exp_p(tv). \tag{3.14}$$

We then apply Lemmas 1.14 and 1.11 to see that $\mathrm{Exp} \circ F^{-1}$ provides a continuous family. If the family is not unique on possibly a smaller pair of neighborhoods, then there exists $(p_i, q_i) \to (\gamma(0), \gamma(1))$ with multiple geodesics γ_i and σ_i running between them (from different families). This implies that $\gamma(0)$ and $\gamma(1)$ are symmetric conjugate. By Theorem 2.6, they are then Riemannian conjugate which is a contradiction. □

The following proposition is classical so we do not include a proof.

Proposition 3.6. *On a Riemannian manifold two points on a geodesic γ say $\gamma(0)$ and $\gamma(1)$ are not conjugate along γ if and only if there is a continuous family about γ which is unique among all families of geodesics about γ which are continuous at γ.*

The following theorem also holds if one uses C_0 symmetric conjugate points and C_0 families rather than the stronger definition of convergence on any geodesic space X. There is no need to assume local compactness or local uniform minimality.

Theorem 3.7. *Assume X is a geodesic space. If $x, y \in X$ are not symmetrically conjugate along γ and there is a continuous family $F : U \times V \to \Gamma([0,1], X)$ about γ, then there exists possibly smaller neighborhoods $U' \subset U$ of x and $V' \subset V$ of y such that F restricted to $U' \times V'$ is the unique continuous family about γ defined on $U' \times V'$. In fact it is the only family continuous at $(\gamma(0), \gamma(1))$ on these restricted neighborhoods.*

Proof. Assume on the contrary that there exists $\delta_i \to 0$ and families

$$F_i : B_{\delta_i}(x) \times B_{\delta_i}(y) \to \Gamma\big([0,1], X\big) \tag{3.15}$$

about γ such that

$$\sigma_i = F_i(x_i, y_i) \neq F(x_i, y_i) = \gamma_i \tag{3.16}$$

for some $x_i \in B_{\delta_i}(x)$ and $y_i \in B_{\delta_i}(y)$. But then $\sigma_i(0) = x_i = \gamma_i(0)$ and $\sigma_i(1) = y_i = \gamma_i(1)$ and by the continuity of F and F_i at $(\gamma(0), \gamma(1))$ we know σ_i and γ_i both converge to γ causing a symmetric conjugate point. $\square$

We are not able to show in general that unique continuous families exist precisely when two points are not symmetrically conjugate.

Open Problem 3.8. Assume X is a locally compact length space which is locally minimizing. Is it true that two points $p, q \in X$ are not symmetrically conjugate along γ if and only if there are neighborhoods U of p and V of q and a unique continuous map $F : U \times V \to \Gamma([0,1], X)$ such that $F(p,q) = \gamma$ and for any $u \in U$, $v \in V$, $F(u,v) = \sigma$, where $\sigma(0) = u$ and $\sigma(0) = v$?

The main difficulty in attempting to answer this question is in the construction of the family of geodesics. This motivates the concept of an unreachable conjugate point and Proposition 4.2.

4. Unreachable conjugate points

Definition 4.1. We say that p and q are unreachable conjugate points along γ if there exist sequences $p_i, q_i \to p, q$ such that no choice of a sequence of geodesics γ_i running from p_i to q_i converges to γ. Otherwise we say that a pair of points is reachable.

Naturally as soon as pairs of points are reachable there are geodesics joining nearby points which can be used to build a family about γ which is continuous at γ. This is the content of the next proposition. Note that it does not require local compactness and holds for all pairs of reachable points including symmetric conjugate points. The proof follows almost immediately from the definitions.

Proposition 4.2. *Assume X is a geodesic space. Suppose $\gamma(0)$ and $\gamma(1)$ are reachable along γ. Then there exist neighborhoods U about $\gamma(0)$ and V about $\gamma(1)$ and a family F about γ which is continuous at $(\gamma(0), \gamma(1))$.*

Example 4.3. On a standard sphere the poles are unreachable conjugate points along any minimizing geodesic running between them. This can be seen by taking describing the geodesic γ

as the 0 degree longitude and taking p_i and q_i all lying on the 90 degree longitude where p_i approach the north pole p and q_i approach the south pole q. Since any geodesic running between p_i and q_i must lie on the great circle which includes the 90 degree longitude, none of them is close to our given geodesic γ.

Note that these poles are also symmetric conjugate points as can be seen by taking sequences $p_i = p$ and $q_i = q$ and γ_i the small positive degree longitudes and σ_i the small negative degree longitudes.

Lemma 4.4. *On a Riemannian manifold, unreachable conjugate points are Riemannian conjugate points and are therefore symmetric conjugate points as well.*

The proof follows from Proposition 3.6 and Theorem 2.6. On the other hand it is not clear what may happen in a general geodesic space. Rinow proved that unreachable conjugate points are one-sided conjugate points on geodesic spaces with extensible geodesics and local uniqueness by effectively constructing an exponential map such that whenever a point is not a one-sided conjugate point (an ordinary point), the exponential map provides a continuous family of geodesics based at the fixed point [18, pp. 414–415]. In other words, the problem below is settled for the spaces studied by Rinow which are generalizations of Riemannian manifolds. But it is still open for locally uniformly minimizing geodesic spaces.

Open Problem 4.5. Does there exist a geodesic space with a geodesic γ whose endpoints are unreachable conjugate points along γ but are not symmetric conjugate points along γ? See Remark 8.6.

The converse of Lemma 4.4 is not true. In fact already for Riemannian manifolds, symmetric conjugate does not imply unreachable conjugate. More precisely consider the following lemma.

Lemma 4.6. *On a Riemannian manifold if γ is the unique geodesic which is minimizing from $\gamma(0)$ to $\gamma(1)$ and these points are conjugate along γ, then the family of minimizing geodesics is continuous at $\gamma(0)$ and $\gamma(1)$ so the conjugate points are not unreachable conjugate points.*

We present a specific example of such a phenomenon pointed out to us by Viktor Schroeder.

Example 4.7. Let M^2 be an ellipsoid that is not a sphere, i.e., a surface of revolution of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $1 < a < b$. If we consider a generic geodesic γ, for example any geodesic that is not parallel to the coordinate planes, then the conjugate locus of such a geodesic looks like a diamond with inwardly curving sides (cf. [10, Chapter 13, Remark 2.6]). On the other hand the cut locus is a tree. If we consider a common vertex of the cut locus and the conjugate locus, then we obtain a point along γ that satisfies the hypotheses of Lemma 4.6.

Nevertheless the concept of unreachable conjugate points is crucial to the existence of unique continuous families and so we define a third kind of conjugate point which incorporates both unreachable and symmetric conjugate points.

5. Ultimate conjugate points

Definition 5.1. Two points p, q on a geodesic are said to be *ultimate conjugate points* along γ if they are either two-sided (symmetric) conjugate along γ or unreachable conjugate along γ.

Remark 5.2. Naturally one can define C_0 unreachable and C_0 ultimate conjugate points.

Theorem 5.3. *On a Riemannian manifold points are conjugate along a given geodesic if and only if they are ultimate conjugate along that geodesic.*

Proposition 5.4. *Suppose X is a locally uniformly minimizing and locally compact geodesic space. Suppose $\gamma(0)$ and $\gamma(1)$ are not ultimate conjugate points along γ. Then there exists a unique continuous family about γ as in Definition* 3.1. *In fact it is the only family continuous at γ.*

Proof. By Theorem 3.7 we need only to show there is a continuous family. To do this it suffices to show that the family F given in Proposition 4.2 is continuous on possibly smaller neighborhoods U and V. If not then F is not continuous at a sequence (p_i, q_i) converging to $(\gamma(0), \gamma(1))$. This means that there exists an $\epsilon_i > 0$ and $(p_{i,j}, q_{i,j})$ converging to (p_i, q_i) such that

$$\lim_{j\to\infty} F(p_{i,j}, q_{i,j}) \neq F(p_i, q_i) \tag{5.17}$$

where it is possible this limit does not exist. If there is a uniform upper bound on the lengths of a subsequence of these geodesics then by Lemma 1.10 a subsequence of these geodesics converges in C_0 to some geodesic $\sigma_i \neq F(p_i, q_i)$ running from p_i to q_i. This is where we use locally uniformly minimizing and locally compact. We now choose j_i such that

$$d_\Gamma\big(\sigma_i, F(p_{i,j_i}, q_{i,j_i})\big) < 1/i. \tag{5.18}$$

This implies that σ_i converges to

$$\lim_{i\to\infty} F(p_{i,j_i}, q_{i,j_i}) = \gamma. \tag{5.19}$$

So does $\gamma_i = F(p_i, q_i)$. Thus $\gamma(0)$ and $\gamma(1)$ are symmetric conjugate, and hence ultimate conjugate, along γ, which is a contradiction. □

The next proposition establishes the connection between unique continuous families and the non-existence of ultimate conjugate points (compare with Question 3.8). But note that its proof strongly requires the use of d_Γ in the definition of convergence, continuity and conjugate points. It may not hold if one uses C_0 versions of the definitions. We do not know of any examples.

Proposition 5.5. *Suppose X is a geodesic space. If there are no ultimate conjugate points along any geodesics of length less than R then there are unique continuous families about all geodesics γ of length $< R$. In fact on a sufficiently small domain the continuous family is unique among all families continuous at $(\gamma(0), \gamma(1))$.*

Proof. By Theorem 3.7 we need only to show there is a continuous family about any geodesic γ of length $< R$. By Proposition 4.2, we have a family F about any such γ which is continuous at $(\gamma(0), \gamma(1))$. The proposition then follows from Lemma 3.4. □

The following proposition is well known but we include its proof here for completeness of exposition. It gives us a local existence result. Note that here we use the locally minimizing set rather than a lack of conjugate points to obtain uniqueness. We not require locally uniformly minimizing.

Proposition 5.6. *Suppose X is a locally minimizing and locally compact geodesic space and suppose W is a minimizing neighborhood. For any pair of points in W there is a unique minimizing geodesic joining them. If $\gamma : [0,1] \to W$ is a geodesic, then there exists is a continuous family F extending γ on a sufficiently small pair of neighborhoods about $\gamma(0)$ and $\gamma(1)$ which in fact consists of minimizing geodesics.*

Proof. Let $U, V \subset W$ be neighborhoods of $\gamma(0), \gamma(1) \in W$ respectively. Define $F : U \times V \to \Gamma([0,1], X)$ so that $F(u, v)$ is the unique minimizing geodesic running from u to v. By construction $F(\gamma(0), \gamma(1)) = \gamma$. If $u_i \to u$ and $v_i \to v$ then $F(u_i, v_i) = \sigma_i$ are equicontinuous and a subsequence converges to some $\sigma_\infty \in C([0,1], X)$ running from u to v. Since

$$d(u, v) \leqslant L(\sigma_\infty) \leqslant \liminf_{i\to\infty} L(\sigma_i) \leqslant \lim_{i\to\infty} d(u_i, v_i) = d(u, v), \tag{5.20}$$

we know σ_∞ is a minimizing geodesic from u to v. Thus it is unique and so in fact σ_i converge to σ as geodesics without any need for a subsequence. So $\lim_{i\to\infty} F(u_i, v_i) = F(u, v)$. □

6. Long homotopy lemma à la Klingenberg

Now we proceed to generalize the long homotopy lemma of Klingenberg (cf. [13]) which states that if a c is a nontrivial, contractible closed geodesic in a compact Riemannian manifold M of length $\ell(c) < 2\,\mathrm{conj}(M)$, then any null homotopy c_s of c contains a curve c_{t_0} of length $\ell(c_{t_0}) \geqslant 2\,\mathrm{conj}\, M$. Given that we have a notion of conjugate point in locally uniformly minimizing length spaces, we may now define the conjugate radius of such a space.

Definition 6.1. The *ultimate conjugate radius* at a point $p \in M$, denoted $\mathrm{UltConj}(p)$, is the largest value $r \in (0, \infty]$ such that along all geodesics starting at p, there are no ultimate conjugate points up to length r. The ultimate conjugate radius of M is

$$\mathrm{UltConj}(M) = \inf_{p\in M} \mathrm{UltConj}(p). \tag{6.21}$$

Theorem 6.2 *(Long Homotopy Lemma). Let M be a locally minimizing, geodesic space and let $c : [0,1] \to M$ be a nontrivial, contractible closed geodesic of length $\ell(c) < 2\,\mathrm{UltConj}(M)$. Then any null homotopy $H(s,t)$ of $c(s)$ contains a curve $c_{t_0} = H(s, t_0)$ of length $\ell(c_{t_0}) \geqslant 2\,\mathrm{UltConj}(M)$.*

The following notion will be crucial in the proof of this theorem.

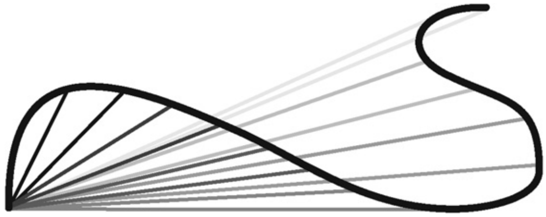

Fig. 1. A fan along an S shaped curve C.

Definition 6.3. Let M be a locally uniformly minimizing and locally compact geodesic space. Let $C : [0, L] \to M$ be any continuous curve. Then we define the *fan of* C as the family of geodesics,

$$F : [0, T) \times [0, 1] \to \Gamma\big([0, 1], M\big), \quad F(s, t) = \sigma_s(t), \quad \sigma_s(0) = C(0),$$

where $\sigma_s(t)$ is a geodesic joining the points $C(0)$ and $C(s)$. The map F is required to be continuous with respect to d_Γ.

The following lemma shows that for curves that are small enough, one is able to construct unique fans. We point out that the geodesics in the fan need not be minimizing, nor are they necessarily the unique geodesics joining $C(0)$ to $C(s)$. However, the fan is uniquely determined for C as in the lemma. See Fig. 1.

Lemma 6.4. *Let M be a locally uniformly minimizing and locally compact geodesic space. Let $C : [0, L] \to M$ be any continuous curve parametrized by s. Then there exists a fan $F : [0, T) \times [0, 1] \to \Gamma([0, 1], M)$ defined on C. Furthermore, if T is the maximal size of the interval on which the fan is defined, then either $T = L$ or $\limsup_{t\to T} L(\sigma_t) \geqslant \mathrm{UltConj}(M)$.*

Proof. We construct the fan by slowly unfolding it. Let S be the collection of $\bar{s}$ such that the fan of C is uniquely defined on $[0, \bar{s}]$ and $\ell(\sigma_s) < \mathrm{conj}_{C(0)}(M)$ for all $s \in [0, \bar{s}]$. Let $s_\infty = \sup S$; we need to show that either $s_\infty = L$ or $\ell(\sigma_{s_\infty}) = \mathrm{UltConj}(M)$. We know $0 \in S$, so S is nonempty. So we need only to show S is closed and open as a subset of $[0, L]$.

To show S is open about a given $\bar{s} \in S$ such that $\ell(\bar{s}) < \mathrm{UltConj}(M)$, we first extend C locally below 0 and above L so that it passes through $C(\bar{s})$. Since C is a continuous curve one may extend C beyond its endpoints, for example by backtracking on itself.

Then we just apply Proposition 5.4 to the geodesic $\sigma_{\bar{s}}$. This provides both the extension of the definition of the σ_s to all $s < \bar{s} + \delta$ and the uniqueness of such an extension. Then we restrict ourselves back to $s \in [0, L]$, so that we have shown S is an open subset of $[0, L]$.

To prove S is closed, take $s_i \in S$ converging to s_∞. Without loss of generality we may assume the s_i are increasing. So we have geodesics σ_s determined by the unique fan for all $s < s_\infty$. Since they have a uniform upper bound on length, they have a converging subsequence σ_i by Theorem 1.13, the Geodesic Arzela–Ascoli Theorem.

Let σ_∞ be such a limit and assume $L(\sigma_\infty) < \mathrm{UltConj}(M)$. By Proposition 5.4, we have a continuous family $\tilde{\sigma}_s$ about σ_∞ which must include the σ_i or i sufficiently large. But each σ_i also has a unique continuous family about it (in fact this family is σ_s), so $\sigma_s = \tilde{\sigma}_s$ near every s_i. Thus $\sigma_s = \tilde{\sigma}_s$ on $(s_\infty - \epsilon, s_\infty]$ unless there exists $t_i \to s_\infty$ where the two families agree on one

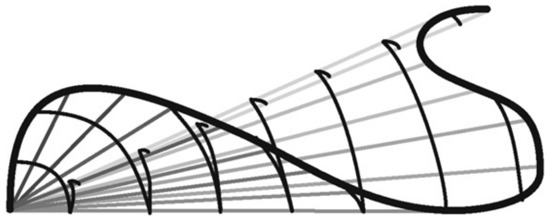

Fig. 2. A fan along an S shaped curve, C, with bands in black.

side of t_i and disagree on the other side. But this contradicts the existence of unique continuous families about $\sigma_{t_i} = \tilde{\sigma}(t_i)$.

Once $\sigma_s = \tilde{\sigma}_s$ on $(s_\infty - \epsilon, s_\infty]$ then we see the limit σ_∞ must have been unique and the family is continuous on $[0, s_\infty]$ which is closed.

Since our collection is nonempty, open and closed, it must be all of $[0, L]$.

The fan is unique because if there were two distinct fans then there would be another family which is continuous at the point where they diverge from each other, and that would imply there is a symmetric conjugate point by Theorem 3.7. □

Lemma 6.5. *Let X be a locally uniformly minimizing locally compact geodesic space. Let C be a curve with a fan σ_s such that $L(\sigma_s) < \mathrm{UltConj}(M)$ for $s \in [0, S]$. Then*

$$L(\sigma_s) \leqslant L\big(C\big([0, s]\big)\big). \tag{6.22}$$

Proof. We begin by defining bands $H_r \subset \bigcup_{s\in[0,S]} \mathrm{Im}(\sigma_s)$ as follows; see Fig. 2:

$$H_r = \big\{\sigma_s\big(r/L(\sigma_s)\big)\colon\ s \in [0, S]\big\}. \tag{6.23}$$

Note that H_r is a connected set by the continuity of the family of geodesics and that

$$H_r \cap \mathrm{Im}(C) = C(s_r) \tag{6.24}$$

where $L(\sigma_{s_r}) = r$.

Note that $\bigcup_{s\in[0,S]} \mathrm{Im}(\sigma_s)$ is compact and our space is locally uniformly minimizing, so there exists some sufficiently small $\epsilon > 0$ such that all geodesics of length ϵ are minimizing. In particular all the σ_s run minimally between bands H_r and $H_{r+\epsilon/2}$.

Fixing $s \in [0, S]$ and taking a partition $0 = r_0 < r_1 < \cdots < r_k = L(\sigma_s)$ such that $r_{i+1} - r_i < \epsilon/2$ we have:

$$L(\sigma_s) = \sum_{i=1}^{k} (r_i - r_{i-1}) \leqslant \sum_{i=1}^{k} d\big(C(s_{r_i}), C(s_{r_{i-1}})\big) \tag{6.25}$$

$$\leqslant \sum_{i=1}^{k} L\big(C\big([s_{r_i}, s_{r_{i-1}}]\big)\big) \leqslant L\big(C\big([0, s]\big)\big). \tag{6.26}$$

Note that this last inequality follows both from the fact that C is not a geodesic and the fact that C may move back and forth in the bands. See Fig. 3. □

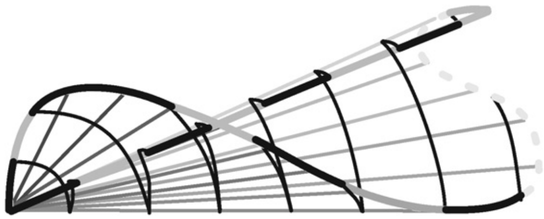

Fig. 3. The bands for each r_i are marked in black, the alternating minimizing segments along σ_s are in black and grey. The corresponding segments along C are marked in matching colors (and are longer). The total length of C in fact may include extra segments, like the dotted section, which do not contribute to the sum.

Lemmas 6.4 and 6.5 then immediately imply the next corollary.

Corollary 6.6. *Let M be a locally uniformly minimizing, locally compact, geodesic space. If $C : [0, L] \to M$ is a continuous curve of length $\leqslant \mathrm{UltConj}(M)$, then it has a fan defined on $[0, L]$.*

Lemma 6.7. *Let M be a locally uniformly minimizing locally compact geodesic space. Let $H : [0,1] \times [0,1] \to M$ be continuous such that for all t, $C_t(s) = H(s,t)$ implies $L(C_t) < \mathrm{UltConj}(M)$. Then there exists a unique continuous family of geodesics*

$$\sigma : [0,1] \times [0,1] \to \Gamma\big([0,1], M\big) \tag{6.27}$$

such that $\forall s, t \in [0,1]$ we have $\sigma_{s,t}(0) = H(0,0)$ and $\sigma_{s,t}(1) = H(s,t)$.

Proof. We know by the above that for each fixed t we have a unique continuous fan $\sigma_{t,s}$ running along C_t. So we have a family of geodesics $\sigma_{t,s}$ which is continuous with respect to the s variable.

Assume σ is continuous on $T \times S$ where S and T are intervals including 0. We know this is true for the trivial intervals $S = \{0\} = T$.

We claim S and T are open on the right: if $t_0 \in T$ and $s_0 \in S$ then there exists $\delta > 0$ such that $t_0 + \delta \in T$ and $s_0 + \delta \in S$. This follows because there are unique continuous family of geodesics about σ_{t,s_0} and about $\sigma_{t_0,s}$ for all $s \in S$ and $t \in T$ and so these continuous families must agree on the overlaps of all the neighborhoods. By choosing δ small enough that our fans land in this neighborhood, we see they must agree with this continuous family (by the uniqueness of the fans and the construction of the fans using the unique continuous families).

We claim S and T are closed on the right. Let s_i approach $\sup S = s_\infty$ and t_i approach $\sup T = t_\infty$. Then by local compactness and locally uniformly minimizing property, and the Geodesic Arzela–Ascoli Theorem, a subsequence of the σ_{t_i,s_i} must converge to some limit.

Let σ_∞ be such a limit and assume $L(\sigma_\infty) < \mathrm{UltConj}_{C(0)} M$. By Proposition 5.5, we have a continuous family $\tilde{\sigma}_{s,t}$ about σ_∞ which must include the σ_{t_i,s_i} for i sufficiently large. But each σ_{t_i,s_i} also has a unique continuous family about it (in fact this family is $\sigma_{s,t}$), so $\sigma_{s,t} = \tilde{\sigma}_{s,t}$ near every s_i. Thus $\sigma_{s,t} = \tilde{\sigma}_{s,t}$ on $(s_\infty - \epsilon, s_\infty] \times (t_\infty - \epsilon, t_\infty]$ unless there exists $s_i' \to s_\infty$ and $t_i' \to t_\infty$ where the two families agree on points near (t_i', s_i') and disagree nearby as well. But this contradicts the existence of unique continuous families about $\sigma_{t_i',s_i'} = \tilde{\sigma}(t_i', s_i')$.

Once $\sigma_{t,s} = \tilde{\sigma}_{t,s}$ on $(t_\infty - \epsilon, t_\infty] \times (s_\infty - \epsilon, s_\infty]$ then we see the limit σ_∞ must have been unique and the family is continuous on $[0, s_\infty] \times [0, t_\infty]$ which is closed. Since our collection is nonempty, open and closed, it must be all of $[0, L]$. The family is unique because if there

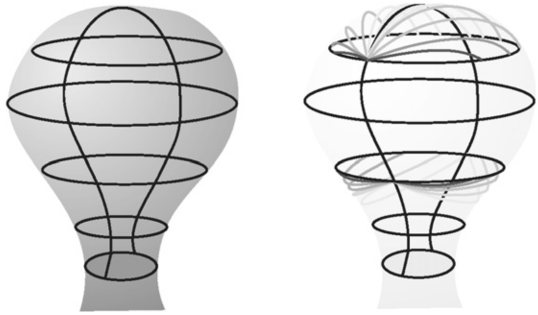

Fig. 4. The bands for each r_i are marked in black, the alternating minimizing segments along σ_s are in black and grey. The corresponding segments along C are marked in matching colors (and are longer). The total length of C in fact may include extra segments, like the dotted section, which do not contribute to the sum.

were two distinct families then there would be another family which is continuous at the point where they diverge from each other, and that would imply there is a symmetric conjugate point by Proposition 5.5. □

Proof of Theorem 6.2 (Long Homotopy Lemma). Each curve in the homotopy is denoted $c_t(s) = H(s,t)$, so $c_0(s) = p$ (the point curve) and $c_1(s) = c(s)$ (the given curve). Without loss of generality we may assume each curve c_t is parametrized proportional to arclength. We assume, by way of contradiction, that all closed curves in the homotopy have length $L(c_t) < 2\,\mathrm{UltConj}(M)$.

Let $\gamma(t) = H(0,t)$ (the starting point of each curve), and let $\sigma(t) = H(\frac{1}{2}, t)$ (the halfway point of each curve). Then $\gamma(0) = \sigma(0) = p$, $\gamma(1) = c(0)$ and $\sigma(1) = c(\frac{1}{2})$. In Fig. 4: on the left we see p at the top, γ running down the front of the surface and σ running down the back of the surface, with c at the neck and $c_{1/5}$, $c_{2/5}$, $c_{3/5}$ and $c_{4/5}$ drawn as ellipses for simplicity.

By our control on the lengths of c_t and by Corollary 6.6 we have two well defined fans: $h_{s,t}$ is the fan for $c_t(s)$ running along $s = [0, 1/2]$ and $\bar{h}_{s,t}$ which is the fan for $c_t(s)$ running backward from $s = 1$ down to $s = 1/2$. In Fig. 4 on the right we see two such pairs of fans, above one for $c_{1/5}$ and below one for $c_{4/5}$. By Lemma 6.7, the families $h_{s,t}$ and $\bar{h}_{s,t}$ are continuous in both s and t.

We will say $c_t(s)$ has a *closed pair of fans* when the two fans meet at a common geodesic, $h_{1/2,t}(r) \equiv \bar{h}_{1/2,t}(r)$. This happens trivially at $t = 0$, where

$$h_{s,0}(r) \equiv \bar{h}_{s,0}(r) = H(s,0) = p. \tag{6.28}$$

In Fig. 4 the pair of fans of $c_{1/5}$ is closed but not the pair for $c_{4/5}$. By the continuity of h and $\bar{h}$, $c_t(s)$ must have a closed fan for all t.

However $c_1(s)$ is a geodesic, so when one creates a fan about it one just gets $h_{s,1}(r) = c_1(r)$ and $\bar{h}_{s,1}(r) = c_1(L(c) - r)$ for all values of s. This is not closed. By continuity, nearby curves $c_t(s)$ for t close to 1 are also not closed. This is a contradiction. □

7. Cut loci and injectivity radii

In this section we define four kinds of cut loci and injectivity radii all of which extend the Riemannian definition for these concepts and yet take on different values on length spaces. Those interested in the CBA(κ) analog of the concepts we have already introduced may skip to the section reviewing that theory.

Definition 7.1. We say q is a cut point of p if there are at least two distinct minimizing geodesics from p to q.

The following definition then naturally extends the Riemannian definition of a cut locus; see for example [10].

Definition 7.2. A point q is in the *First Cut Locus* of p, $q \in \mathrm{1stCut}(p)$ if it is the first cut point along a geodesic emanating from p. Note that in the case where there is a sequence of cut points $\gamma(t_i)$ of a point $p = \gamma(0)$ with t_i decreasing to t_∞ then $\gamma(t_\infty)$ is considered to be a first cut point as long as there are no closer cut points along γ, even if $\gamma(t_\infty)$ is not a cut point itself.

We use the term "first" because we will define other cut loci which are not based on being the first cut point but also agree with the traditional cut locus on Riemannian manifolds.

Definition 7.3. Let the *First Injectivity Radius* at p be defined

$$\mathrm{1stInj}(p) = d\big(p, \mathrm{1stCut}(p)\big) \tag{7.29}$$

and $\mathrm{1stInj}(M) = \inf_{p \in M} \mathrm{1stInj}(p)$. If the cut locus of a point is empty the 1st injectivity radius of that point is infinity.

The next definition is the well-known notion that has appeared in various papers. In particular, this is the notion used in the book [7, p. 119].

Definition 7.4. The *Unique Injectivity Radius* of a geodesic space X denoted $\mathrm{UniqueInj}(X)$, is the supremum over all $r \geqslant 0$ such that any two points at distance at most r are joined by a unique geodesic.

Example 7.5. On the closed flat disk the Unique Injectivity Radius is equal to ∞ since for any $r > 0$, any two points at distance at most r can be joined by a unique geodesic (the statement is of course vacuous for $r > 1$). The first cut locus of the origin is empty for the same reason.

The following lemma is not too hard to show.

Lemma 7.6. *On a geodesic space*

$$\mathrm{1stInj}(p) = \mathrm{UniqueInj}(p). \tag{7.30}$$

The next definition of cut locus matches the definition on Riemannian spaces given in DoCarmo [10] and so it is another valid extension of the concept. It emphasizes the minimizing properties of geodesics rather than uniqueness.

Definition 7.7. We say q is in the *Minimal Cut Locus* of p, $q \in \mathrm{MinCut}(p)$, if there is a minimizing geodesic running from p through q which is not minimizing from p to any point past q. We are assuming this geodesic extends past q.

Fig. 5. The pinned sector and the pinned hemisphere.

Note that the $\mathrm{MinCut}(p)$ is the cut locus defined by Zamfirescu in his work on convex surfaces [23].

It is an essential point that this definition has no requirement that there be a unique geodesic. On a Riemannian manifold, geodesics stop minimizing past the First Cut Locus because as soon as geodesics are not unique, there are short cuts that can be taken which smooth out the corners. On many length spaces this is not the case:

Example 7.8. The pinned sector is a compact length space created by taking an "orange slice" or sector between poles on a sphere and adding line segments to each of the poles. See Fig. 5. We will denote the poles as p_i and the far ends of the segments as q_i. Then $p_2 \in \mathrm{UniqueCut}(p_1)$ and $p_2 \notin \mathrm{MinCut}(p_1)$. In contrast the pinned closed hemisphere has $p_2 \in \mathrm{MinCut}(p_1)$ because the geodesic running along the edge of the hemisphere stops minimizing at p_2 and continues along the edge. There is no such geodesic in the pinned sector.

Note that while compact Riemannian manifolds always have nonempty cut loci due to the extendability of geodesics, this is not the case on compact length spaces:

Example 7.9. On the flat disk, the center point, p, has an empty *Minimal Cut Locus* because every geodesic stops existing before it stops minimizing. It also has an empty *First Cut Locus* because there are unique geodesics joining every point to p.

Definition 7.10. Let the Minimal Radius at p be defined

$$\mathrm{MinRad}(p) = d\bigl(p, \mathrm{MinCut}(p)\bigr) \tag{7.31}$$

and $\mathrm{MinRad}(X) = \inf_{p \in X} \mathrm{MinRad}(p)$.

Note that, although the minimal radius agrees with the injectivity radius on a Riemannian manifold, we do not call it an injectivity radius because it does not imply any uniqueness of geodesics (like the injectivity of the exponential map).

In Example 7.8, we have

$$\mathrm{MinRad}(p) = \infty, \qquad 1\mathrm{stInj}(p) = \pi. \tag{7.32}$$

The following lemma follows immediately from Definition 1.9 of locally uniformly minimizing and standard compactness arguments.

Lemma 7.11. *A compact length space is locally uniformly minimizing if and only if its minimal radius is positive.*

Example 7.12. The Rationally Attached Line, is a countable collection of real lines all attached to a common real line in the following manner. The first real line is attached to the common line at corresponding integers, the second at corresponding half integers, the third at corresponding third integers and so on. Note that on this line there are infinitely many minimizing geodesics joining any two points on the common line and that the common line is minimizing between any pair of its values. The unique injectivity radius and the first injectivity radius are both zero. On the other hand, since any geodesic between any pair of points is minimizing, the minimal radius is ∞.

In order to compare the minimal radius with the unique injectivity radius, we recall that on a Riemannian manifold, any point in the cut locus is either a cut point or a conjugate point along a minimizing geodesic. This leads to an alternate extension of the notion of the Riemannian cut locus:

Definition 7.13. The Symmetric Cut Locus of p, denoted $\mathrm{SymCut}(p)$, is the collection of all cut points of p and all symmetric conjugate points to p which are symmetric conjugate along a minimizing geodesic.

This definition is equivalent to the others on Riemannian manifolds because the concept of a symmetric conjugate point is equivalent to the concept of a conjugate point on Riemannian manifolds (Theorem 2.6) and on a Riemannian manifold a symmetric conjugate point along a minimizing geodesic is a first conjugate point. We can then also extend the definition of injectivity radius using this notion:

Definition 7.14. The Symmetric Injectivity Radius,

$$\mathrm{SymInj}(p) = d\big(p, \mathrm{SymCut}(p)\big) \tag{7.33}$$

and $\mathrm{SymInj}(X) = \inf_{p \in X} \mathrm{SymInj}(p)$.

In Miller and Pak's work [14] on piecewise linear spaces, their cut locus is in fact $Cl(\mathrm{MinCut}(p))$.

Lemma 7.15. *On a geodesic space*

$$Cl\big(\mathrm{MinCut}(p)\big) \subset \mathrm{SymCut}(p) \tag{7.34}$$

and

$$\mathrm{MinRad}(p) \geqslant \mathrm{SymInj}(p). \tag{7.35}$$

Proof. Let $q_i \in \mathrm{MinCut}(p)$ and $q_i \to q$. Either a subsequence of q_i are conjugate to p along a minimizing geodesic γ_i or a subsequence of q_i are cut points of p. In either case there exists pairs of geodesics σ_i and $\bar{\sigma}_i$ running between common endpoints such that one of the two satisfies:

$$d_\Gamma(\sigma_i, \gamma_i) < 1/i \tag{7.36}$$

and the other also satisfies this equation or has the same length as γ_i.

Moreover, since M is a uniformly locally minimizing compact space, we can apply our Geodesic Arzela–Ascoli Theorem to find a subsequence such that γ_i converges to γ_∞ running minimally between p and q and σ_i converges to σ_∞. If $\sigma_\infty = \gamma_\infty$, then p and q are symmetric conjugate, otherwise they are cut points. □

This inequality is not an equality as can be seen by examining the pinned sector (Example 7.8). There the pole p_2 lying opposite p_1 in the sector is a symmetric conjugate point for p_1 so $p_2 \in \mathrm{SymCut}(p_1)$ and

$$\mathrm{SymInj}(p_1) = 1\mathrm{stInj}(p_1) = \pi. \tag{7.37}$$

On the rationally attached line (Example 7.12), every point is a symmetric conjugate for any other point and so

$$\mathrm{SymInj}(p_1) = 1\mathrm{stInj}(p_1) = 0. \tag{7.38}$$

In fact the symmetric injectivity radius always agrees with the first injectivity radius.

Lemma 7.16. *On a geodesic space X,*

$$1\mathrm{stInj}(X) = \mathrm{UniqueInj}(X) = \mathrm{SymInj}(X) \leqslant \mathrm{MinRad}(X). \tag{7.39}$$

Proof. If $\mathrm{UniqueInj} = r_0$, then all geodesics of length $\leqslant r_0$ are unique, so any sequence of pairs of nonunique geodesics approaching a pair of symmetric conjugate points has length at least r_0, thus $\mathrm{SymInj}(X) \geqslant r_0$.

On the other hand if $\mathrm{SymInj}(X) = r_0$, then for any $p \in X$ we have $d(p, \mathrm{SymCut}(p)) \geqslant r_0$ and so the nearest cut point to p is outside $\bar{B}_p(r_0)$, so all geodesics of length $\leqslant r_0$ are unique and $\mathrm{UniqueInj} \geqslant r_0$. The rest follows from Lemmas 7.6 and 7.15. □

Finally we introduce the ultimate cut locus and the ultimate injectivity radius.

Definition 7.17. The Ultimate Cut Locus of p, denoted $\mathrm{UltCut}(p)$, is the collection of all cut points of p and all ultimate conjugate points to p which are conjugate along a minimizing geodesic.

Definition 7.18. The Ultimate Injectivity Radius, denoted $\mathrm{UltInj}(p)$ is $d(p, \mathrm{UltCut}(p))$.

Note that $\mathrm{UltInj}(p) = \min\{\mathrm{UltConj}(p), \mathrm{UniqueInj}(p)\}$ where the ultimate conjugate radius was defined in Definition 6.1. By the definition of an ultimate conjugate point we have the following lemma.

Lemma 7.19. *On a geodesic space,*

$$\mathrm{SymCut}(p) \subset \mathrm{UltCut}(p) \tag{7.40}$$

and

$$\mathrm{UltInj}(p) \leqslant \mathrm{SymInj}(p) \leqslant \mathrm{MinRad}(p). \tag{7.41}$$

Lemma 7.20. *If X is locally uniformly minimizing and locally compact then the ultimate injectivity radius is the supremum over all $r > 0$ such that any minimizing geodesic, γ, of length $\leqslant r$ is unique between its endpoints and has a continuous family about it which is unique among all families continuous at γ and no ultimate conjugate points along γ.*

Proof. If $\mathrm{UltInj}(X) = r_0$, then there are no ultimate conjugate points along any minimizing geodesics of length less than r_0 and so by Proposition 5.4 there are continuous families about these geodesics. The uniqueness of the minimizing geodesics of length $\leqslant r_0$ follows from the definition of cut and the inclusion of cut points in the ultimate cut locus. The converse follows similarly by applying Proposition 5.5. □

Lemma 7.21. *If X is locally uniformly minimizing and locally compact then any geodesic, γ, of length less than the unique injectivity radius has a continuous family about it which is unique among all families continuous at γ.*

Proof. We will prove that the end points of γ are not ultimate conjugate points and apply Proposition 5.5 to complete the proof. By Definition 5.1 we need only to show they are neither unreachable conjugate nor symmetric conjugate points.

First note that $\gamma(0)$ and $\gamma(1)$ are not unreachable conjugate points. If they were, then there would be sequences of points $p_i \to \gamma(0)$ and $q_i \to \gamma(1)$ such that any choice of sequence of geodesics γ_i joining p_i to q_i will not converge to γ. On the other hand for i large enough, $d(p_i, q_i) < \mathrm{UniqueInj}(X)$ since $d(\gamma(0), \gamma(1)) < \mathrm{UniqueInj}(X)$, which implies that there is a unique minimal geodesic, γ_i. So γ_i must converge in the sup norm to γ otherwise $\gamma(0), \gamma(1)$ would have two distinct minimizing geodesics joining them, namely γ and $\lim_{i\to\infty} \gamma_i$. Note this limit geodesic exists by the Geodesic Arzela–Ascoli Theorem (Theorem 1.13) using the fact that X is locally compact and locally uniformly minimizing.

We now show $\gamma(0), \gamma(1)$ are not symmetrically conjugate. If they were then there exist sequences $p_i \to \gamma(0)$, $q_i \to \gamma(1)$ and geodesics σ_i, λ_i joining p_i and q_i that converge to γ in d_Γ. Since $L(\gamma_i)$ and $L(\sigma_i)$ both converge to $L(\gamma)$ eventually their lengths are less than $\mathrm{UniqueInj}(X)$, By Lemma 7.16, we eventually have

$$L(\gamma_i), L(\sigma_i) < \mathrm{UniqueInj}(X) \leqslant \mathrm{MinRad}(X)$$

so γ_i and σ_i are eventually minimizing geodesics. However, minimizing geodesics having length less than the Unique Injectivity Radius are unique, so eventually $\gamma_i = \sigma_i$ which is a contradiction. □

Open Problem 7.22. It would be interesting to find an example demonstrating the necessity of the locally uniformly minimizing condition in the last lemma.

Remark 7.23. We should point out that with any of the numerous definitions of cut loci and injectivity radii, the injectivity radius for an arbitrary compact length space can be zero and this is probably true generically.

8. Klingenberg's injectivity radius estimate

In the special case that q is the closest point in the minimal cut locus of p, Klingenberg's Lemma from Riemannian geometry extends to locally uniformly minimizing length spaces.

Lemma 8.1. *Let M be a locally uniformly minimizing, locally compact length space. Let $p \in M$ and suppose $q \in Cl(\mathrm{MinCut}(p))$ such that $d(q,p) = d(p, \mathrm{MinCut}(p)) > 0$ then either there is an ultimate conjugate point along a minimizing geodesic from p to q or there is a geodesic running from q to q and passing through p.*

As you will see the proof follows Klingenberg's original proof (cf. [10]) almost exactly except that we have Lemma 8.1, where one shows the nearest point in the minimal cut locus of a point either has a closer ultimate cut point or a geodesic loop running through it. The distinction here is that this ultimate conjugate point need not be the closest cut point because the geodesic running through it may still be minimizing past it.

Proof. Assuming there are no ultimate conjugate points, then by Lemma 7.15, we know there is a pair of distinct minimizing geodesics from p to q: γ and σ. If we can show there exists $\epsilon > 0$ such that

$$d\big(\gamma(1-\epsilon), \sigma(1-\epsilon)\big) = 2\epsilon d(p,q) \tag{8.42}$$

then we are done.

Assume, instead there exists $\epsilon_i \to 0$ such that

$$d\big(\gamma(1-\epsilon_i), \sigma(1-\epsilon_i)\big) < 2\epsilon_i d(p,q) \tag{8.43}$$

thus the midpoint x_i between these two points is closer to p than q is.

Since there is a continuous family about γ and a continuous family about σ we can find γ_i converging to γ running from p to x_i and we can find σ_i converging to σ running from p to x_i. Eventually γ_i and σ_i must be distinct.

By the triangle inequality

$$L(\gamma_i) < L\big(\gamma\big([0, 1-\epsilon_i]\big)\big) + \epsilon_i d(p,q) = L(\gamma) \tag{8.44}$$

and

$$L(\sigma_i) < L\big(\sigma\big([0, 1-\epsilon_i]\big)\big) + \epsilon_i d(p,q) = L(\sigma). \tag{8.45}$$

Thus if γ_i and σ_i are minimizing, then we have found a closer cut point, causing a contradiction.

By Proposition 5.5 our family of geodesics about γ is unique among all families continuous at γ. Choosing $\delta > 0$ sufficiently small and N sufficiently large that all the γ_i for $i \geqslant N$ have $(\gamma_i(0), \gamma_i(t))$ in the unique domain for all $t \geqslant t_i$ where

$$t_i = \min\big\{1, \big(d(p,q) - \delta\big)/L(\gamma_i)\big\}. \tag{8.46}$$

This implies that γ_i restricted to initial segments of length $> d(p,q) - \delta$ and γ similarly restricted must be part of this unique family.

Let us restrict γ and γ_i to segments of length $d(p,q) - \delta'$ for some $\delta' \in (0, \delta)$. The restricted γ is minimizing and is the unique minimizing geodesic between its end points by our definition of $\mathrm{Cut}(p)$. Thus if we replace the restricted γ_i by minimizing geodesics between their endpoints, we'd get another family which is continuous at γ. By the uniqueness in Proposition 5.5 this forces the restricted γ_i to be minimizing between their endpoints.

That is all the γ_i with $i \geqslant N$ are minimizing on initial intervals of length $< d(p,q)$. However, γ_i have length less than $d(p,q)$ so they are minimizing geodesics without any restriction. The same is true for σ_i with i taken sufficiently large. Thus x_i is a cut point which leads to a contradiction. $\square$

Lemma 8.2. *If X is a compact length space with one extensible geodesic $\gamma : [0,\infty) \to X$ parametrized by arclength then*

$$\mathrm{MinRad}(X) \leqslant \mathrm{MinRad}\big(\gamma(0)\big) < \infty. \tag{8.47}$$

Proof. The geodesic γ cannot be minimizing on all intervals $[0,T]$, so eventually it stops minimizing and we have an element in the minimal cut locus of $\gamma(0)$. $\square$

The next theorem is applied in the final section of this paper to recover a result of Charney and Davis on $\mathrm{CBA}(\kappa)$ spaces but is much more general (Theorem 12.3). Note that since the minimal injectivity radius is positive, X is locally uniformly minimizing so it satisfies the hypothesis of Lemma 8.1.

Theorem 8.3. *If X is a compact length space with a positive and finite minimal injectivity radius then either there exists p and q which are ultimate conjugate and $d(p,q) \leqslant \mathrm{MinRad}(X)$ or there is a closed geodesic $\gamma : S^1 \to X$ which has length twice the minimal injectivity radius.*

Proof. Let $r_0 = \mathrm{MinRad}(X)$. So there exists $p_i \in X$ possibly repeating such that $\mathrm{MinRad}(p_i)$ decreases to r_0. So $d(p_i, \mathrm{MinCut}(p_i))$ decreases to r_0. Since X is compact, there exists $q_i \in Cl(\mathrm{MinCut}(p_i))$ such that $\mathrm{MinRad}(p_i) = d(p_i, q_i)$. So by Lemma 8.1 either p_i and q_i are ultimate conjugate or there is a geodesic γ_i running from p_i to p_i through q_i of length $L(\gamma_i) = 2d(p_i, q_i)$.

Since X is compact p_i has a subsequence converging to some p and q_i has a subsequence converging to some q, and $d(p,q) = r_0$. Since γ_i is minimizing on $[0, 1/2]$ and on $[1/2, 1]$ it has a subsequence which converges to a piecewise geodesic γ_∞ which is minimizing on these intervals.

If γ_∞ runs back and forth on the same interval then p and q are symmetric conjugate and thus ultimate conjugate. Otherwise q is a cut point of p and q is a cut point of p. Applying Lemma 8.1 to this pair p and q we see that γ must be minimal about both p and q unless they have ultimate conjugate points on a minimizing geodesic running between them. Therefore $\gamma : S^1 \to X$ is a geodesic. $\square$

The following example is somewhat extreme.

Example 8.4. Take a circle and let $d(p_1, p_2) = r_0 < \pi$. Join p_1 and p_2 by a second line segment of length d. This space has minimal injectivity radius and unique injectivity radius $= r_0$. So in this space with no ultimate conjugate points we have the obvious geodesic loop of length $2r_0$. See the upper left space in Fig. 6 where one identifies the left and right endpoints, the loop is marked in black.

Example 8.5. If we fill in the loop of length $2r_0$ with a sector of a sphere of intrinsic diameter r_0, then the line segments will still be geodesics, but the corners will provide short cuts and so the

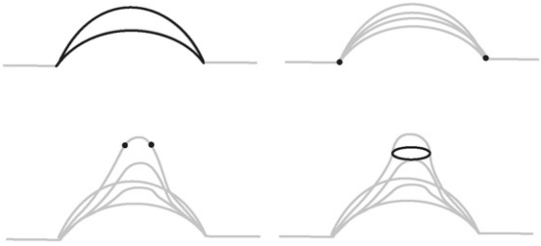

Fig. 6. The pinned sector and the pinned hemisphere.

loop will no longer be a geodesic. In fact while the first injectivity radius is still r_0 the minimal injectivity radius will increase to r_0 and there are now ultimate conjugate points on the sector a distance r_0 apart. See the upper right space in Fig. 6.

One wonders if one could change the metric on the sector so that it is still a minimizing neighborhood and somehow remove the ultimate conjugate points so that Theorem 8.3 forces us to see the loop which has length π. In other words can we introduce nonuniqueness to a circle without reducing its minimal injectivity radius or introducing ultimate conjugate points? In the bottom two spaces of Fig. 6 we stretch the sector and see that on one case we get closer ultimate conjugate points while in the other we get a short closed loop.

Remark 8.6. In light of the above example, one might consider using Theorem 8.3 to produce ultimate conjugate points in spaces where closed geodesics are well understood. This might help address Open Problem 4.5.

9. Reviewing CAT(κ) versus CBA(κ)

In the remainder of this paper we apply the theory we have developed to CBA(κ) spaces and so in this section we briefly recall the definitions of CAT(κ) and CBA(κ) spaces. We then give a very brief review of key results needed for this paper. For a more in depth approach see Bridson and Haefliger's text [7].

Let X be a metric space and κ a fixed real number. To define these spaces we need the notion of a comparison space. Let M_κ^2 denote the 2-dimensional, complete, simply connected Riemannian manifold of constant curvature κ. So M_1 is the round sphere of radius 1, M_0 is the Euclidean plane and M_{-1} is hyperbolic plane. Let D_κ denote the diameter of M_κ. Then

$$D_\kappa = \operatorname{diam}(M_\kappa) = \begin{cases} \frac{\pi}{\sqrt{\kappa}}, & \text{if } \kappa > 0, \\ +\infty, & \text{if } \kappa \leqslant 0. \end{cases} \tag{9.48}$$

Given a triangle constructed of three minimizing geodesics, Δ, in X with perimeter less than $2D_\kappa$, we can define a comparison triangle $\overline{\Delta}$ in M_κ whose sides have the same length. Given x and y in Δ we can construct unique comparison points $\bar{x}$ and $\bar{y}$ in $\bar{\Delta}$ which lie on corresponding sides with corresponding distances to the corresponding corners. Since the comparison triangle is unique up to global isometry the distance between these comparison points does not depend on the comparison triangle.

Definition 9.1. A geodesic space X is a $\mathrm{CAT}(\kappa)$ space if for all triangles Δ of perimeter less than $2D_\kappa$ and all points $x, y \in \Delta$ we have

$$d(x, y) \leqslant d(\bar{x}, \bar{y}) \tag{9.49}$$

where $\bar{x}$ and $\bar{y}$ are points on the comparison triangle $\overline{\Delta}$ in M^2_κ.

Note that this is a global definition; for example, a torus is not $\mathrm{CAT}(0)$. To extend the concept of a Riemannian manifold with sectional curvature bounded above one uses a localized version of this definition:

Definition 9.2. A geodesic space X is said to be $\mathrm{CBA}(\kappa)$ or to have *curvature bounded above by* κ if it is locally a $\mathrm{CAT}(\kappa)$ space. That is, for every $x \in X$, there exists $r_x > 0$ such that the metric ball $B_x(r_x)$ with the induced metric is a $\mathrm{CAT}(\kappa)$ space. We will abbreviate this to $\mathrm{CBA}(\kappa)$.

An example of a $\mathrm{CBA}(1)$ space which is not $\mathrm{CAT}(1)$ is a real projective space and an example of a $\mathrm{CBA}(0)$ space that is not $\mathrm{CAT}(0)$ is a flat torus.

Theorem 9.3 *(Alexandrov). A smooth Riemannian manifold M is* $\mathrm{CBA}(\kappa)$ *if and only if the sectional curvature of all* 2*-planes in M is bounded above by κ.*

The condition of $\mathrm{CAT}(\kappa)$ is a fairly strong condition on a space, much stronger than just having an upper curvature bound. For example it is easy to use the definition and degenerate comparison triangles to prove the following well-known proposition:

Proposition 9.4. *On a* $\mathrm{CAT}(\kappa)$ *space, X, all geodesics of length less than D_κ are minimizing and they are the unique minimizing geodesics running between their end points:*

$$\mathrm{MinRad}(X) \geqslant D_\kappa \quad \textit{and} \quad \mathrm{UniqueInj}(X) \geqslant D_\kappa. \tag{9.50}$$

The next proposition is then immediate implying in particular that the results from the previous sections apply to $\mathrm{CBA}(\kappa)$ spaces.

Proposition 9.5. *A compact* $\mathrm{CBA}(\kappa)$ *space is locally uniformly minimizing as in Definition* 1.9.

Naturally the minimal radius of a $\mathrm{CBA}(\kappa)$ space may be much smaller than D_κ just as the minimal radius of $\mathbf{RP}^2$ is only $\pi/2$ not π. We will later show its ultimate conjugate radius is $\geqslant \pi$; see Theorem 11.1. The following theorem distinguishes between $\mathrm{CBA}(\kappa)$ and $\mathrm{CAT}(\kappa)$ spaces. Note that the notion of injectivity radius of X used by Gromov in the theorem below corresponds to $\mathrm{UniqInj}(X)$ in our scheme; see Definition 7.4.

Theorem 9.6 *(Gromov). Let X be a compact length space that is* $\mathrm{CBA}(\kappa)$*. Then X fails to be* $\mathrm{CAT}(\kappa)$ *if and only if it contains a closed geodesic of length $\ell < 2D_\kappa$. Moreover, if it contains such a closed geodesic, then it contains a closed geodesic of length* $\mathrm{Sys}(X) = 2\,\mathrm{inj}(X)$*, where the systole* $\mathrm{Sys}(X)$ *is defined to be the infimum of the lengths of all closed geodesics.*

In the next section of our paper we will need an angle comparison theorem. We recall that on a $\mathrm{CBA}(\kappa)$ space the notion of angle is well defined (cf. [7]):

Definition 9.7. Let X be a $\mathrm{CAT}(\kappa)$ space and suppose $c, c' : [0,1] \to X$ are two geodesics issuing from the same point $p = c(0) = c'(0)$. Then the *Alexandrov angle* between c, c' at the point p is defined to be the limit of the κ-comparison angles $\lim_{t\to 0} \angle_p^{(\kappa)}(c(t), c'(t))$, i.e.,

$$\angle(c, c') = \lim_{t\to 0} 2 \arcsin\left(\frac{d(c(t), c'(t))}{2t}\right).$$

Let p, x, y be points in a length space X such that $p \neq x$, $p \neq y$. If there are unique geodesic segments $\overline{px}$, $\overline{py}$, then we write $\angle_p(x, y)$ to denote the Alexandrov angle between these segments. $\angle_p^{(\kappa)}(x, y)$ denotes the angle of the comparison triangle in M_κ^2. We then have the following angle comparison theorem for $\mathrm{CAT}(\kappa)$ spaces (see for instance [7, Proposition II,1.7]) which may be reformulated for a suitable $\mathrm{CAT}(\kappa)$ neighborhood in a $\mathrm{CBA}(\kappa)$ space.

Theorem 9.8 *(Alexandrov). Let X be a* $\mathrm{CAT}(\kappa)$ *metric space with* $\mathrm{MinRad}(X) \geqslant D_\kappa$ *(i.e., there is a unique geodesic between any pair of points less then distance D_κ apart). Furthermore if $\kappa > 0$, then the perimeter of each geodesic triangle considered is less than $2D_\kappa$. For every geodesic triangle $\Delta(p, q, r)$ in X and for every pair of points $x \in \overline{pq}$, $y \in \overline{pr}$, where $x \neq p$, $y \neq p$, the angles at the vertices corresponding to p in the comparison triangles $\overline{\Delta}(p, q, r)$, $\overline{\Delta}(p, x, y)$ in M_κ^2 satisfy*

$$\angle_p^{(\kappa)}(x, y) \leqslant \angle_p^{(\kappa)}(q, r).$$

This theorem was applied to small triangles in $\mathrm{CBA}(\kappa)$ spaces by Alexander and Bishop to prove the following Cartan–Hadamard Theorem originally stated by Gromov.

Theorem 9.9 *(Gromov). If X is a* $\mathrm{CBA}(\kappa)$ *length space for any $\kappa \leqslant 0$, then its universal cover $\widetilde{X}$ is a* $\mathrm{CAT}(\kappa)$ *space.*

The original result of Gromov appears in [12]. A detailed proof in the locally compact case was given by W. Ballmann (cf. [4]). Alexander and Bishop proved the theorem (cf. [1]) under the additional hypothesis that X is a geodesic metric space.

Another key property of Alexandrov spaces is that they have the nonbranching property. That is, if two geodesics agree on an open set then they cannot diverge later. In particular, if there is more than one minimal geodesic running from p to q, then neither geodesic can extend minimally past q. As a consequence we have:

Lemma 9.10. *All cut points q of p are in* $1\mathrm{stCut}(p)$ *and if a minimizing geodesic from p to q extends as a geodesic past q, then $q \in$* $\mathrm{MinCut}(p)$ *as well.*

Otsu and Shioya have defined a larger cut locus, C_p, for Alexandrov spaces in [15] to be the collection of points p which do not lie within minimal geodesics. So $\mathrm{MinCut}(p) \subset C_p$ and $1\mathrm{stCut}(p) \subset C_p$. Note that on the flat disk with p at the center, C_p is the boundary rather than the empty set. Otsu and Shioya prove that the Hausdorff measure of C_p is zero [15, Proposition 3.1], thus:

Proposition 9.11 *(Otsu–Shioya). In a* $\mathrm{CBA}(\kappa)$ *space,* $\mathrm{MinCut}(p)$ *and* $1\mathrm{stCut}(p)$ *have Hausdorff measure zero.*

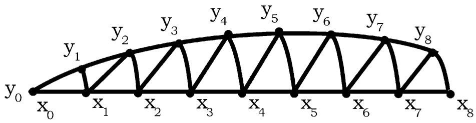

Fig. 7. A bridge with $x_j = \gamma(t_j)$ and $y_j = \sigma(t_j)$.

Shioya believes he can extend this to nonbranching spaces with weaker curvature conditions like the BG scaling condition he has been investigating recently.

10. The Rauch Comparison Theorem

In this section we prove Rauch Comparison Theorems for CBA(κ) spaces (Theorems 10.1 and 10.5). They will be applied in the next section to prove Theorem 11.1 that ultimate conjugate points are never less than D_κ apart and so there are continuous families about geodesics of length $< D_\kappa$ regardless of whether they are minimizing or not.

The following theorem and proof were outlined to us by Stephanie Alexander for a one-sided conjugate points on CBA(1). It is essentially a Rauch Comparison Theorem. We include its proof below for completeness since it is necessary to motivate our extension of the theorem, the Relative Rauch Comparison Theorem (Theorem 10.5). Similar techniques were used by Alexander and Bishop in their proof of the Cartan–Hadamard Theorem for CBA(0) spaces [1].

Theorem 10.1. *Let X be a* CBA(κ)*, geodesic space. If $\gamma(0)$ and $\gamma(1)$ are symmetric conjugate along γ, then $L(\gamma) \geqslant D_\kappa$.*

To prove this we will build a *bridge*. In order to build the bridge we need *struts*. We first provide precise definitions.

Definition 10.2. A strut is a quadrilateral with a diagonal. That is, it is a pair of minimizing geodesic segments σ and γ with four sides: $S = L(\sigma)$, $T = L(\gamma)$ $A = d(\gamma(0), \sigma(0))$ $B = d(\gamma(1), \sigma(1))$ and a diagonal $D = d(\gamma(0), \sigma(1))$. The entire strut must lie within a CAT(κ) neighborhood.

A comparison strut can then be set up in the simply connected two-dimensional comparison space M^2_κ. It is built by joining two geodesic triangles with sides SAD and TBD. So the comparison strut is unique when $S + A + D$ and $T + B + D$ are less that D_κ. The side lengths of the comparison strut are the same but the angles all increase by the angle version of the comparison theorem: Definition 9.7 and Theorem 9.8.

Definition 10.3. A *bridge* on γ is a nearby geodesic σ and a collection of struts between them. That is we selected $0 = s_0 < s_1 < \cdots < s_N = 1$ and $0 = t_0 < t_1 < \cdots < t_N = 1$ and partition γ and σ into segments such that $\gamma(t_j), \gamma(t_{j+1}), \sigma(s_j), \sigma(s_{j+1})$ form a strut for $j = 0 \dots N$.

A bridge has *length* $\leqslant L$ if both γ and σ have length $\leqslant L$ and it has *height* $\leqslant h$ if all struts have all their sides of length $\leqslant h$.

See Fig. 7.

One can build a comparison bridge in M^2_κ by piecing together the comparison struts. Unlike the bridge in X, this comparison bridge does not have geodesic beams. The struts in fact glue

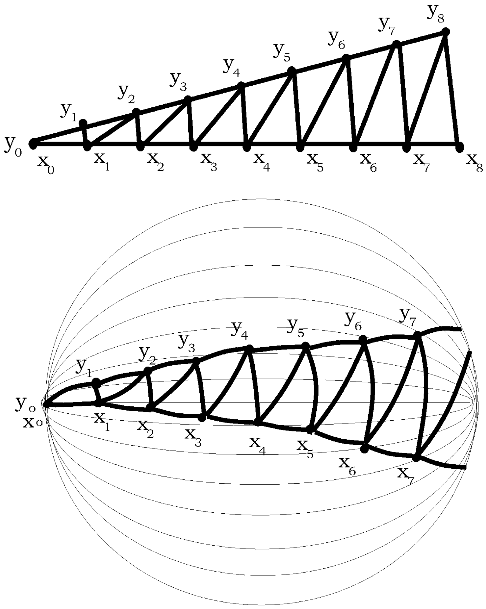

Fig. 8. Here the comparison space is a sphere.

together to form a bridge whose upper and lower decks are piecewise geodesics: $\bar{\gamma}$ and $\bar{\sigma}$. The lengths of all the triangles match and the interior angles in the comparison bridge do not decrease. See Fig. 8.

Note it is crucial that the comparison space is two-dimensional. In the setting where one proves a Rauch Comparison Theorem for spaces with curvature bounded below, this leads to some complications. See the work of Alexander, Ghomi and Wang [3].

Proof of Theorem 10.1. Assume on the contrary that $L(\gamma) < D_\kappa$ and there are γ_i and σ_i converging to γ which share endpoints. They are eventually length $< T < D_\kappa$ as well by Lemma 1.11 since CBA(κ) spaces are locally uniformly minimizing.

Now we can cover γ with a collection of balls such that each ball is CAT(κ). For i sufficiently large both γ_i and σ_i are both within the union of these neighborhoods. We now use γ_i and σ_i to build a bridge. We do this by partitioning γ_i and σ_i into N subsegments of equal length with N large enough that each strut formed by the corresponding subsegments of γ_i and σ_i fit within a CAT(κ) neighborhood.

We now build a comparison bridge in M^2_κ using the comparison struts, obtaining the piecewise geodesics: $\bar{\gamma}_i$ and $\bar{\sigma}_i$. Note that here we build each triangle one by one with matching lengths starting from $t = s = 0$. See Fig. 8.

We claim that $\bar{\gamma}_i$ and $\bar{\sigma}_i$ are bending apart as depicted in Fig. 8 because the interior angles are greater than π. This can be seen because they are built with comparison triangles. Comparison triangles have larger angles than the corresponding angles in the space X. Since the geodesics γ and σ are straight where the triangles meet them, the sum of the three interior angles meeting at each partition point is $\geqslant \pi$, and so the sum of the comparison angles are only larger. Since the comparison space is two-dimensional this forces the piecewise geodesics apart.

Thus if we were to straighten out $\bar{\gamma}_i$ and $\bar{\sigma}_i$, creating smooth geodesics of the same length with the same initial opening angle by bending each geodesic segment inward, then they would meet before 1. That is they would meet before they are as long as T. But two distinct geodesics cannot meet before D_κ in M_κ^2 which gives us a contradiction. □

Notice that in Theorem 10.1 the bridge is replacing the role of Jacobi fields in the standard differentiable Rauch comparison theorem. Recall that Rauch proved in [17] that a Jacobi field $J(t)$ which has $J(0) = 0$ in a space with sectional curvature less than $\kappa = 1$ then has nondecreasing $|J(t)|/\sin(t)$. Thus $|J(t)| \geqslant \sin(t)$ and it cannot hit 0 before π. Imitating Gromov's proof of the Relative Volume Comparison Theorem [11], one can say that for all $r < R < \pi$

$$\frac{|J(r)|}{\sin(r)} < \frac{|J(R)|}{\sin(R)}, \tag{10.51}$$

which we call the smooth version of the Relative Rauch Comparison Theorem. The inequality is opposite that of Gromov's because the curvature is bounded above here.

We now prove the Relative Rauch Comparison Theorem for CBA(κ) spaces with bridges replacing Jacobi fields. It will be applied in the next section to prove that there are no ultimate conjugate points before D_κ in such spaces.

Recall Definition 10.3. First we need to insure that the bridge is short enough that its comparison bridge lies in a single hemisphere of the comparison sphere. See Fig. 8.

Lemma 10.4. *If a bridge has length $L < \pi$ and height $h \leqslant (\pi - L)/4$ then its comparison bridge lies in a hemisphere.*

Proof. By the triangle inequality, all points on the bridge are within a distance $L/2 + 2h < \pi/2$ from the halfway point of one of the geodesics. □

Theorem 10.5 *(Relative Rauch Comparison). Let X be a* CBA(κ) *space. Suppose γ and σ form a bridge in X with length $L < D_\kappa$ and height $h \leqslant (D_\kappa - L)/4$ with $\gamma(0) = \sigma(0)$. Then for all $r \in (4h, L)$ and $R \in (r + 4h, L - 4h)$, if $r_\gamma, r_\sigma \in (r - h, r + h)$ and $R_\gamma, R_\sigma \in (R - h, R + h)$ lie near partition points $t_i = r/L$ and $t_j = R/L$ of the bridge, we have:*

$$\frac{d(\gamma(r_\gamma/L), \sigma(r_\sigma/L))}{d(\gamma(R_\gamma/L), \sigma(R_\sigma/L))} \leqslant \left(1 + \frac{4h}{D_\kappa}\right) \sup_{\bar{r} \in [r-h, r]} \frac{f_\kappa(\bar{r})}{f_\kappa(R - r + \bar{r})} + \frac{4h + \alpha_\kappa(R - r + \bar{r}, h)}{D_\kappa} \tag{10.52}$$

where f_κ is the warping function of the space of constant curvature κ and $\alpha_\kappa(s, h)$ is an error term satisfying

$$\lim_{h \to 0} \frac{\alpha_\kappa(h, s)}{h} = 0. \tag{10.53}$$

In particular $f_1(r) = \sin(r)$, $f_0(r) = r$, $\alpha_1(s, h) = |s - \arccos(\cos(s)/\cos(h))|$ and $\alpha_0(s, h) = 2(s - \sqrt{s^2 - h^2})$.

One can immediately see that as the bridge decreases in height, $h \to 0$, our $\bar{r}$ approaches r and the right-hand side of (10.52) converges to (10.51). Note that for $\kappa \leqslant 0$ there is no restriction on the length or height of the bridge for this theorem to hold other than requirement that the struts lie in CAT(κ) neighborhoods.

Remark 10.6. Note also that to apply this theorem we need only to show that γ and σ form a bridge with heights $\leqslant h/2$ in a space where all triangles of circumference $\leqslant 3h$ fit in CAT(κ) neighborhoods, and then we can always add extra partition points to that bridge to ensure that we pass through r and R.

Proof. As in the proof above we build a comparison bridge in M^2_κ by piecing together the comparison struts where the struts glue together to form a comparison bridge whose upper and lower decks are only piecewise geodesics, $\bar{\gamma}$ and $\bar{\sigma}$, such that

$$d_M\big(\gamma(t_j),\sigma(s_j)\big) = d_{M_\kappa}\big(\bar{\gamma}(t_j),\bar{\sigma}(s_j)\big) \tag{10.54}$$

for all j including $t_j = r/L$ and $t_j = R/L$. These piecewise geodesics bend apart because the sum of the interior angles meeting at a given point on a deck is greater than π (each angle has increased in size) exactly as depicted in Fig. 8.

This is the step where we use the CAT(κ) property.

By the triangle inequality

$$d_M\big(\gamma(r_\gamma/L),\sigma(r_\sigma/L)\big) \leqslant d_{M_\kappa}\big(\bar{\gamma}(t_j),\bar{\sigma}(s_j)\big) + 2h \tag{10.55}$$

and

$$d_M\big(\gamma(R_\gamma/L),\sigma(R_\sigma/L)\big) \geqslant d_{M_\kappa}\big(\bar{\gamma}(R/L),\bar{\sigma}(R/L)\big) - 2h. \tag{10.56}$$

Let $\bar{r}_\gamma = d_{M_\kappa}(\bar{\gamma}(r/L),\bar{\gamma}(0))$ and $\bar{r}_\sigma = d_{M_\kappa}(\bar{\sigma}(r/L),\bar{\gamma}(0))$. Note that

$$\bar{r}_\gamma < L\big(\bar{\gamma}\big([0,r/L]\big)\big) = r \quad \text{and} \quad \bar{r}_\sigma L\big(\bar{\sigma}\big([0,r/L]\big)\big) = r \tag{10.57}$$

because piecewise geodesics are always longer than the distance between their endpoints.

Since the distance is measured in M_κ and the geodesics γ and σ are bending apart towards a maximum distance apart of h, we know that their length r is less than the worst path between their endpoints which runs straight and then make a right turn and runs a distance h:

$$r < \bar{r}_\gamma + h \quad \text{and} \quad r < \bar{r}_\sigma + h. \tag{10.58}$$

So

$$\bar{r}_\gamma, \bar{r}_\sigma \in [r-h, r]. \tag{10.59}$$

Now we draw minimizing geodesics of length L, $\tilde{\gamma}$ and $\tilde{\sigma}$ from $\bar{\gamma}(0) = \bar{\sigma}(0)$ through $\bar{\gamma}(r/L)$ and $\bar{\sigma}(r/L)$ respectively so that $\tilde{\gamma}(\bar{r}_\gamma/L) = \bar{\gamma}(r/L)$ and $\tilde{\sigma}(\bar{r}_\sigma/L) = \bar{\sigma}(r/L)$. Notice these geodesics lie outside the wedge formed by $\bar{\gamma}$ and $\bar{\sigma}$ before they hit these points but then extend into the wedge afterwards because the piecewise geodesics are bending apart. See Fig. 9 where $\tilde{\gamma}$ and $\tilde{\sigma}$ are in grey and $i = 6 < j = 7$.

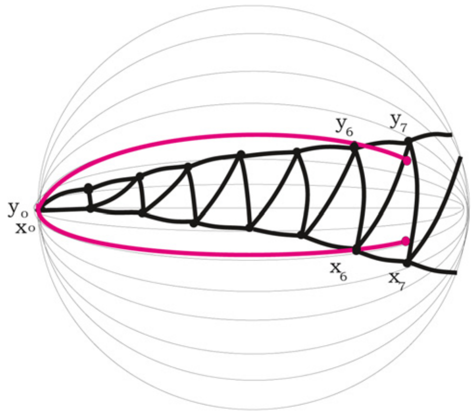

Fig. 9. Here the shaded geodesics are $\tilde\sigma$ and $\tilde\gamma$ which pass through $y_6 = \bar\sigma(r/L)$ and $x_6 = \bar\gamma(r/L)$. One sees the endpoints of $\tilde\gamma$ and $\tilde\sigma$ are closer together than $y_7 = \bar\sigma(R/L)$ and $x_7 = \bar\gamma(R/L)$ since they extend the same length from x_6 and y_6 but are bending together relative to the comparison bridge's geodesics.

In particular, the points $\bar\gamma(R/L)$ and $\bar\sigma(R/L)$ must be further apart than their counterparts on $\tilde\gamma$ and $\tilde\sigma$ also lying a distance $R-r$ out from their common points $\tilde\gamma(\bar r_\gamma/L) = \bar\gamma(r/L)$ and $\tilde\sigma(\bar r_\sigma/L) = \bar\sigma(r/L)$. More precisely:

$$d_{M_\kappa}\big(\tilde\gamma\big((R-r+\bar r_\gamma)/L\big), \tilde\gamma\big((R-r+\bar r_\sigma)/L\big)\big) \geqslant d_{M_\kappa}\big(\bar\gamma(R/L), \bar\sigma(R/L)\big). \tag{10.60}$$

Here we have used the restriction on the height and length of the bridge when $\kappa > 0$ so that we avoid passing through a pole or having a short cut running between $\bar\gamma(R/L)$ $\bar\gamma(R/L)$ on the opposite side of the sphere.

Applying (10.59) and selecting any

$$\bar r \in [r-h, r] \tag{10.61}$$

the triangle inequality combined with (10.60) gives us

$$d_{M_\kappa}\big(\bar\gamma(R/L), \bar\sigma(R/L)\big) + 2h \geqslant d_{M_\kappa}\big(\tilde\gamma\big((R-r+\bar r)/L\big), \tilde\gamma\big((R-r+\bar r)/L\big)\big) \tag{10.62}$$

and the triangle inequality combined with the choice of $\tilde\gamma$ and $\tilde\sigma$ gives us

$$d_{M_\kappa}(\bar\gamma\big((r/L), \bar\gamma(r/L)\big) - 2h \leqslant d_{M_\kappa}\big(\tilde\gamma(\bar r_\gamma/L), \tilde\sigma(\bar r_\sigma/L)\big). \tag{10.63}$$

Thus

$$\frac{d_{M_\kappa}(\bar\gamma(R/L), \bar\sigma(R/L)) + 2h}{d_{M_\kappa}(\bar\gamma(r/L), \bar\gamma(r/L)) - 2h} \geqslant \frac{d_{M_\kappa}(\tilde\gamma((R-r+\bar r)/L), \tilde\gamma((R-r+\bar r)/L))}{d_{M_\kappa}(\tilde\gamma(\bar r/L), \tilde\sigma(\bar r/L))}. \tag{10.64}$$

Combining this with (10.55) and (10.56) we have

$$\frac{d_M(\gamma(R_\gamma/L), \sigma(R_\sigma/L)) + 4h}{d_M(\gamma(r_\gamma/L), \sigma(r_\sigma/L)) - 4h} \geqslant \frac{d_{M_\kappa}(\tilde\gamma((R-r+\bar r)/L), \tilde\sigma((R-r+\bar r)/L))}{d_{M_\kappa}(\tilde\gamma(\bar r/L), \tilde\sigma(\bar r/L))}. \tag{10.65}$$

Note that both $\tilde{\gamma}$ and $\tilde{\sigma}$ are minimizing geodesics since $L < D_\kappa$ and they lie on M_κ. Thus there distances can be computed using the warping function f_κ.

If we take d'_{M_κ} to be the distance in M_κ measured by taking arc paths in spheres about $\tilde{\gamma}(0) = \tilde{\sigma}(0)$ then:

$$d'_{M_\kappa}\big(\tilde{\gamma}\big((R-r+\bar{r})/L\big), \tilde{\sigma}\big((R-r+\bar{r})/L\big)\big) = d'_{M_\kappa}\big(\tilde{\gamma}(\bar{r}/L), \tilde{\sigma}(\bar{r}/L)\big)H \tag{10.66}$$

where

$$H = \frac{f_\kappa(R-r+\bar{r})}{f_\kappa(\bar{r})} \leqslant \sup_{\bar{r}\in[r-h,r]} \frac{f_\kappa(\bar{r})}{f_\kappa(R-r+\bar{r})}. \tag{10.67}$$

Actual distances d_{M_κ} are somewhat shorter but can be estimated by this d'_{M_κ}:

$$d_{M_\kappa}\big(\tilde{\gamma}(s), \tilde{\sigma}(s)\big) \leqslant d'_{M_\kappa}\big(\tilde{\gamma}(s), \tilde{\sigma}(s)\big) \leqslant d_{M_\kappa}\big(\tilde{\gamma}(s), \tilde{\sigma}(s)\big) + \alpha_\kappa(h,s) \tag{10.68}$$

where $\alpha_\kappa(h,s) = 2(s-s')$ where $s-s'$ is the maximal distance between the arc joining $\tilde{\gamma}(s)$ and $\tilde{\sigma}(s)$ and the minimizing geodesic between these curves. On Euclidean space we have $s' = \sqrt{s^2-h^2}$ and on S^2 we have $\cos(s) = \cos(h)\cos(s')$. In general we can say that

$$\lim_{h\to 0} \frac{\alpha_\kappa(s,h)}{h} = 0. \tag{10.69}$$

Thus (10.66) implies

$$d_{M_\kappa}\big(\tilde{\gamma}\big((R-r+\bar{r})/L\big), \tilde{\sigma}\big((R-r+\bar{r})/L\big)\big) \geqslant d_{M_\kappa}\big(\tilde{\gamma}(\bar{r}/L), \tilde{\sigma}(\bar{r}/L)\big)H - \alpha_\kappa(R-r+\bar{r},h). \tag{10.70}$$

Putting this together with (10.65) we obtain

$$\frac{d(\gamma(r_\gamma/L), \sigma(r_\sigma/L)) - 4h}{d(\gamma(R_\gamma/L), \sigma(R_\sigma/L)) + 4h + \alpha_\kappa(R-r+\bar{r},h)} \leqslant \sup_{\bar{r}\in[r-h,r]} \frac{f_\kappa(\bar{r})}{f_\kappa(R-r+\bar{r})}. \tag{10.71}$$

Since $d(\gamma(R_\gamma/L), \sigma(R_\sigma/L)) \leqslant D_\kappa$ and

$$(x-4h)/(y+4h+\alpha) \leqslant H \quad \text{implies} \quad x \leqslant Hy + 4hH + \alpha H + 4h \tag{10.72}$$

implies

$$x/y \leqslant H + (4hH + \alpha h + 4h)/y \leqslant (1 + 4h/D_\kappa)H + (4h+\alpha)/D_\kappa \tag{10.73}$$

we have (10.52). $\square$

11. Ultimate conjugate points after π

In this section we describe a proof of the following theorem as an application of our Relative Rauch Comparison Theorem (Theorem 10.5). We will apply it combined with our Long Homotopy Lemma (Theorem 6.2) to prove Lemma 12.1. We then combine it with Klingenberg's Injectivity Radius Estimate (Theorem 8.3) to prove a result of Charney and Davis used in their proof of Gromov's Systole Theorem (Corollary 12.3).

Theorem 11.1 *(Alexander–Bishop). If M is a* CBA(κ) *space and $\gamma : [0,1] \to M$ is a geodesic of length $< D_\kappa$, then γ has a unique continuous family of geodesics about it.*

One may wish to keep the following example in mind when considering this theorem in contrast to the usual smooth projective spaces:

Example 11.2 *(Tetrahedral Bi-sphere).* Suppose two copies of the round sphere $\mathbf{S}^2$ are glued to each other at 4 points that form the vertices of a regular tetrahedron on each sphere. The resulting space is CBA(1), but it does not have unique minimizing geodesics between all pairs of points of distance less than π.

Note that by Theorem 10.1, we already know there are no symmetric conjugate points along γ of length less than D_κ. So if we show every geodesic of length less than π has a continuous family about it (and thus no unreachable conjugate points) then there are no ultimate conjugate points and, by Proposition 5.5, we know that there is a unique continuous family about γ. In fact one can see that Alexander–Bishop proved exactly what we require in [1] and it is applied in [2]. A precise statement of their result and proof can be also be found in Ballmann's book on manifolds with nonpositive curvature [5, Chapter 1, Theorem 4.1]), as pointed out to us by Lytchak.

For completeness of exposition we prove a short description of a proof based on the Relative Rauch Comparison Theorem (Theorem 10.5). A completely rigorous write up of our proof can be found in the arXiv posting of this paper.

In our proof, we assume there is a continuous family about γ restricted to some interval $[0, T]$ and extend that family out to $[0, T + T_1]$ for $T_1 > 0$ but close enough that $\gamma([T - T_1, T + T_1])$ is in a minimizing neighborhood so that we can apply Lemma 7.21 and Proposition 9.5. To do this we glue geodesics in the continuous family to geodesics in the minimizing neighborhood. However it is not clear how one can glue geodesics to form geodesics rather than just piecewise geodesics. In fact, this is impossible when $T = D_\kappa$.

Instead we start with a pair of points u near $\gamma(0)$ and w near $\gamma(T + T_1)$ and define a sequence of piecewise geodesics using the continuous family about $\gamma([0, T])$ and the short minimal geodesics. We then prove this carefully controlled sequence is Cauchy and in fact converges to a geodesic from u to w. See Fig. 10.

We start with σ_0 running minimally from w to $y_0 = \gamma(T - T_1)$ and γ_0 in the given family from u to the midpoint, x_1, of σ_0. Then we take σ_1 running minimally from w to $y_1 = \gamma_0(T - T_1)$ and γ_1 in the given family from u to the midpoint, x_2, of σ_1, and so on. We show both sequences, γ_i and σ_i, are Cauchy using the Relative Rauch Comparison Theorem (Theorem 10.5) and the fact that their combined lengths are kept less than D_κ. We then prove they converge to a pair of geodesics γ_∞ and σ_∞ that glue together to form a geodesic $\gamma_{u,v}$ running from u to w.

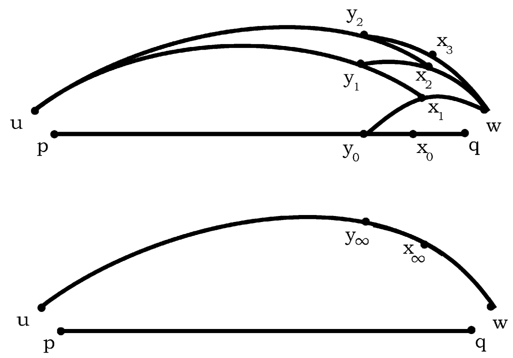

Fig. 10. Building the sequence of piecewise geodesics, then taking the limit.

The proof is very technical because we control the total height of the bridges as they build upon one another by choosing very small neighborhoods for our u and w. We ensure that the accumulated Rauch estimates and their errors due to the heights remains bounded by using a geometric series. As this proof was many pages long in the style of an analysis paper rather than a geometry paper, we have not included it here in the published version.

12. Applications to CBA(κ) spaces

In this section we briefly survey some applications of our results on geodesic spaces to CBA(κ) spaces and potential further directions of research. We begin with the long homotopy lemma and then the injectivity radius theorem and finally discuss some further directions.

Theorem 12.1. *If M is a locally compact* CBA(κ) *space and $c : [0, 1] \to M$ is a nontrivial contractible closed geodesic of length $l(c) < 2D_\kappa$, then any null homotopy for c contains a curve of length $\geqslant 2D_\kappa$.*

Proof. We begin by noting that M is locally uniformly minimizing (Lemma 9.5). By Theorem 11.1 we know $\mathrm{conj}(M) \geqslant D_\kappa$. Thus this follows immediately from Theorem 6.2, our generalization of the long homotopy lemma. □

The standard application of the theorem to Riemannian manifolds is depicted in Fig. 4. An application to a CBA(κ) space can be seen by taking two copies of that example and joining them together at finitely many corresponding points similar to the Tetra bi-sphere (Example 11.2). Note how it is crucial in this application that the fans constructed do not require uniqueness of geodesics between the points.

Open Problem 12.2. The Riemannian long homotopy lemma has also often been used in combination with Morse Theory to prove the existence of smooth closed geodesics. Possible extensions of Morse Theory to length spaces appear in [21]. Thus one might try to extend some of this existence theory to length spaces and CBA(κ) spaces.

Our next application is closely related to a step in Charney–Davis' proof of Gromov's Systole Theorem [9] (cf. [7, II.4.16]).

Corollary 12.3 *(Charney–Davis). A compact* CBA(κ) *space, X, with $\kappa > 0$ such that* $\mathrm{MinRad}(X) \in (0, D_\kappa)$ *has a closed geodesic with length twice the minimal injectivity radius.*

Proof. We just combine Theorem 8.3 with Theorem 11.1. □

In the Charney and Davis proof of Gromov's systole theorem they essentially prove the following statement: *If the Unique Injectivity radius is less than D_κ then there is a closed geodesic of length equal to twice the unique injectivity radius which is in fact a digon, so its length is twice the minimal injectivity radius as well.* They do not explicitly state this but the essence of the idea is there so we consider Corollary 12.3 to already be known. The argument in their situation is easier than ours because it is easy to construct continuous families about unique geodesics. So Corollary 12.3 is an over simplification of our Klingenberg Injectivity Radius Theorem (Theorem 8.3) but our theorem holds in a much wider setting.

We close with a discussion of the implications of equality in Theorem 11.1.

Definition 12.4. A CBA(1) space is said to have positive spherical rank if any geodesic segment of length π has conjugate endpoints.

On Riemannian manifolds, one has the immediate consequence that every geodesic has a Jacobi field running along it of the form $J(t) = \sin(t)E(t)$ where E is parallel. In [19], the authors called closed manifolds with this property to have *positive spherical rank*. In that paper it is shown: *If a Riemannian manifold of sectional curvature at most* 1 *has positive spherical rank, then its universal cover is isometric to a compact, rank one, symmetric space*.

Open Problem 12.5. Can one classify CBA(κ) spaces with positive spherical rank?

Note that following example demonstrates that the classification cannot be restricted to symmetric spaces at least without the further assumption of a lower curvature bound.

Example 12.6. The triple hemisphere is a compact length space created by gluing three hemispheres together along a common equator. It is a CBA(1) space with infinite negative curvature along the equator. See for example [8] for theorems about constructing CBA(κ) spaces using gluing. One can find the distance between any pair of points on a common hemisphere using standard spherical geometry and the distance between any pair of points on different hemispheres by considering the two hemispheres as a single sphere.

Any geodesic of length $< \pi$ will pass through at most two hemispheres and so can be seen to be running along a great circle on the sphere formed by those two hemispheres. Thus it has no ultimate conjugate points before π (otherwise the sphere would have a conjugate point by Proposition 5.5). So the triple hemisphere is a space with positive spherical rank.

While the above example is not a symmetric space, it is an example of a *spherical building*. One might hope that a classification of spaces with positive spherical rank might be limited to spherical buildings (or their generalizations). In the 2-dimensional case, W. Ballmann and M. Brin showed that if one further assumes that the space is obtained by gluing spherical simplices together then one has rigidity. Specifically they showed that all such examples are spherical buildings, spherical joins or one class of examples that is neither; see [6]. One can see that the examples in [6] that are neither buildings nor joins do not have positive spherical rank. So in this case one may be able to prove that such spaces are rigid.

Acknowledgments

It is a pleasure to thank Ben Schmidt for his assistance early in the project, particularly for pointing us to Warner's Theorem (Theorem 2.2). Thanks are also due to Viktor Schroeder for pointing out Example 4.7 and allowing us to include it. We are also very grateful to Stephanie Alexander for sharing with us the proof of Theorem 10.1. In addition we would like to thank both Stephanie Alexander and Dick Bishop for their graciousness in dealing with our reproving of Theorem 11.1 which we had not discovered in the literature and accidentally claimed as our own in the first version of this paper. We also want to thank Ezra Miller and Sasha Lytchak for offering several valuable comments and corrections after the first posting of our preprint. Finally, we also want to thank Werner Ballmann, Karsten Grove and Wolfgang Ziller for useful discussions.

Appendix A. Translated excerpt from Rinow [18]

As Rinow's text is in German, we have translated relevant sections here and added comments in italics.

A.1. Absolute conjugate points; p. 172

Rinow is first interested in minimizing geodesics:

Let $f(s), 0 \leqslant s < \beta$ denote the normal representation of a geodesic ray S_a with starting point $a = f(0)$. Let $k(S_a)$ denote the supremum over all $s' \in [0, \beta)$ for those s' for which $f|_{[0,s']}$ is a shortest curve (i.e., geodesic segment) between a and $f(s')$. Obviously, $0 < k(S_a) \leqslant \beta$.

Rinow then defines a notion of conjugate point which is unrelated to our notion and not equivalent to the Riemannian notion:

The **absolute conjugate point of the ray** S_a is defined to be the point $f(k(S_a))$ if $k(S_a) < \beta$, or in the case $k(S_a) = \beta < \infty$ the limit point of S_a if it exists. We also denote it as $f(k(S_a))$. In case $k(S_a) < \beta$, $f(s)$ always represents a shortest curve on $[0, k(S_a)]$, however, not on any interval $[0, \alpha]$, where $k(S_a) < \alpha \leqslant \beta$. It follows that all inner points for $f(k(S_a))$ are shortest curves for S_a. In an inner metric space, obviously $k(S_a) = d(a, f(k(S_a)))$, if $f(k(S_a))$ always exists.

Rinow later defines a notion equivalent to a one-sided conjugate point.

A.2. Conjugate points; pp. 414–415

X is an inner metric space with the following properties (p. 414):

(a) X is locally compact.
(b) X does not have any branching points (*i.e., X does not have bifurcation of geodesics; see p.* 162).
(c) For each point $a \in X$, there exists $\epsilon_a > 0$ so that

$$\inf\{k(y):\ y \in U(a, \epsilon_x)\} > 0.$$

Note that due to the nonbranching, there is local uniqueness of geodesics of length less than $k(y)$. One can also piece together geodesics to extend them:

Let $a \in X$ be any point. Let α be a real number so that $0 < \alpha < k(a)$. Let $\sum_\alpha$ denote the peripheral sphere of a with radius α (points in X at distance α to a). Every point $\xi \in \sum_\alpha$ can be connected to a by exactly one shortest curve and there is exactly one geodesic ray S_ξ which is infinitely extendable.

Rinow then defines an exponential map, f, using this uniqueness:

Let $f(\xi, s), 0 \leqslant s < \infty$ be the normal parameter representation of S_ξ. We set $P = \sum_\alpha \times [0, \infty)$ with the product metric

$$\rho\big((\xi, s), (\xi', s')\big) = \sqrt{d(\xi, \xi')^2 + |s - s'|^2}.$$

Then $f : P \to X$ is a well-defined map. Every point $x \neq a$ can be connected to a by a shortest curve. Consequently, we get a geodesic ray through x. Moreover, $f(\xi, 0) = a$ for every $\xi \in \sum$ and f is continuous on P.

Rinow uses this notion to define conjugate points which he next proves is equivalent to what we call a one-sided conjugate point:

Given a point $(\xi_0, s_0) \in P$ with $s_0 > 0$, we say s_0 is an *ordinary point* for the ray $f(\xi_0, s_0)$ if for some $\delta > 0$, f restricted to $V_\delta(\xi_0, s_0) = \{(\xi, s)\colon\ d(\xi, \xi') < \delta, |s - s_0| < \delta\}$, is a topological map (i.e., homeomorphism onto its image; see p. 11) and in addition $f(V_\delta(\xi_0, s_0))$ is open in X. If no such $\delta > 0$ exists, then s_0 is called a *conjugate point*. Obviously, the set of ordinary points is open while the set of conjugate points is closed in $[0, \infty)$.

Rinow then proves the following theorem which we only state here:

Theorem 1. *Let X be an n-dimensional manifold. $s_0 > 0$ is an ordinary point of $f(\xi_0, s)$ if and only if there exists $\delta > 0$ such that f restricted to $V_\delta(\xi_0, s_0)$ is injective.*

One then sees that this is equivalent to the notion of a one-sided conjugate point because f is an exponential map.

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