A CHARACTERIZATION OF SURFACES WHOSE UNIVERSAL COVER IS THE BIDISK

FABRIZIO CATANESE, MARCO FRANCIOSI

ABSTRACT. We show that the universal cover of a compact complex surface X is the bidisk $\mathbb{H} \times \mathbb{H}$, or X is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, if and only if $K_X^2 > 0$ and there exists an invertible sheaf η such that $\eta^2 \cong \mathcal{O}_X$ and $H^0(X, S^2\Omega_X^1(-K_X) \otimes \eta) \neq 0$. The two cases are distinguished by the second plurigenus, $P_2(X) \geq 2$ in the former case, $P_2(X) = 0$ in the latter. We also discuss related questions.

1. INTRODUCTION

The beauty of the theory of algebraic curves is deeply related to the manifold implications of the:

Theorem 1.1 (Uniformization theorem of Koebe and Poincaré)**.** *A connected and simply connected complex curve $\tilde{C}$ is biholomorphic to:*

$$\tilde{C} \cong \begin{cases} \mathbb{P}^1 & if \ g = 0 \\ \mathbb{C} & if \ g = 1 \\ \mathbb{H} & if \ g \geq 2 \end{cases}$$

($\mathbb{H}$ denotes as usual the Poincaré upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$, but we shall often refer to it as the 'disk' since it is biholomorphic to $\{z \in \mathbb{C} : ||z|| < 1\}$).

Hence a smooth (connected) compact complex curve C of genus $g \geq 1$ admits a uniformization in the strong sense *(iii)* of the following definition:

Definition 1.2. *A connected complex space X of complex dimension n admits a* **uniformization** *if one of the following conditions hold:*

(i) *there is a connected open set $\Omega \subset \mathbb{C}^n$ and a surjective holomorphic map $f \colon \Omega \to X$ (**weak uniformization**);*
(ii) *there is a connected open set $\Omega \subset \mathbb{C}^n$ and a properly discontinuous group $\Gamma \subset Aut(\Omega)$ such that $\Omega/\Gamma \cong X$ (**Galois uniformization**).*

If X is a complex manifold, there are two stronger properties:

(iii) *there is a connected open set $\Omega \subset \mathbb{C}^n$ and a surjective holomorphic submersion $f \colon \Omega \to X$ (**étale uniformization**);*
(iv) *there is a connected open set $\Omega \subset \mathbb{C}^n$ biholomorphic to the universal cover of X (**strong uniformization**).*

Hence the universal cover of a compact complex curve is completely determined by its genus; in particular $\tilde{C} \cong \mathbb{H}$ if and only if $g \geq 2$, i.e., "C is of

general type", and we get then an isomorphism of $\pi_1(C)$ with a Fuchsian group $\Gamma \subset \mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2, \mathbb{R})$.

In higher dimension the condition that the universal cover be biholomorphic to a bounded domain Ω is quite exceptional; but still in the Galois étale case, where $\Omega/\Gamma \cong X$ and Γ acts freely with compact quotient, we have, if Ω is bounded, that the complex manifold X has ample canonical bundle K_X (see [Sieg73]), in particular it is a projective manifold of general type.

Even more exceptional is the case where the universal cover is biholomorphic to a bounded symmetric domain Ω, or where there is Galois uniformization (ii) of definition 1.2) with source a bounded symmetric domain, and there is a vast literature on a characterization of these properties (cf. [Yau77], [Yau88], [Yau93], [Bea00]).

The basic result in this direction is S.T. Yau's uniformization theorem (explained in [Yau88] and [Yau93]), and for which a very readable exposition is contained in the first section of [V-Z05], enphasyzing the role of polystability of the cotangent bundle for varieties of general type. One would wish nevertheless for more precise characterizations of the various possible cases.

For the sake of simplicity, we shall stick here to the case of smooth complex surfaces, where the former problem boils down to two very specific questions.

Question. When is the universal cover of a compact complex surface X biholomorphic to the two dimensional ball $\mathbb{B}_2 := \{z \in \mathbb{C}^2 : ||z|| < 1\}$, respectively to the bidisk $\mathbb{H} \times \mathbb{H}$?

The first part of this question is fully answered by the well-known inequality by Miyaoka and Yau (cf. [Miy77], [Yau77] [Miy82]). Setting, as usual, $K_X =$ the canonical divisor, $\chi(X) := \chi(\mathcal{O}_X)$ the holomorphic Euler characteristic and $P_2(X) = h^0(X, 2K_X)$ the second plurigenus of X, we have the following characterization:

Theorem 1.3 (Miyaoka-Yau)**.** *Let X be a compact complex surface. Then $X \cong \mathbb{B}_2/\Gamma$ (with Γ a cocompact discrete subgroup of $\mathrm{Aut}(\mathbb{B}_2)$ acting freely on $\mathbb{B}_2$) if and only if*

(1) $K_X^2 = 9\chi(S)$;
(2) *the second plurigenus $P_2(X) > 0$.*

The above well known characterization is obtained combining Miyaoka's result ([Miy82]) that these two conditions imply the ampleness of K_X, with Yau's uniformization result ([Yau77]) which uses the existence of a Kähler Einstein metric; quite remarkably, it is given purely in terms of certain numbers which are either bimeromorphic or topological invariants.

In the case where $X = \mathbb{H} \times \mathbb{H}/\Gamma$, with Γ a discrete cocompact subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ acting freely, one has $K_X^2 = 8\chi(X)$.

But Moishezon and Teicher in [MT87] showed the existence of a simply connected surface of general type (whence with $P_2(X) > 0$) having $K_X^2 = 8\chi(X)$, so that the above conditions are necessary, but not sufficient. We observe however that (and our contribution here is a by-product of our attempt

to answer the latter question) it is still unknown if there exists a surface of general type with $\chi(X) = 1, K_X^2 = 8$ which is not uniformized by $\mathbb{H} \times \mathbb{H}$.

The purpose of this note is to point out a precise characterization of compact complex surfaces whose universal cover is the bidisk, and of the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, discussing whether some hypotheses can be dispensed with, and to pose an analogous question in higher dimension. Our characterization, which is of course based on Yau's results, relies on the following crucial

Definition 1.4. *Let X be a complex manifold of complex dimension n.*

*Then a **special tensor** is a non zero section $0 \neq \omega \in H^0(X, S^n\Omega_X^1(-K_X))$, while a **semi special tensor** is a non zero section $0 \neq \omega \in H^0(X, S^n\Omega_X^1(-K_X) \otimes \eta)$, where η is an invertible sheaf such that $\eta^2 \cong \mathcal{O}_X$.*

We shall say that X admits a unique semi special tensor if moreover $\dim(H^0(X, S^n\Omega_X^1(-K_X) \otimes \eta)) = 1$.

In fact, the existence of such tensors is a fundamental property of manifolds strongly uniformized by the polydisk as we are now going to see.

Recall that the group of automorphism of $\mathbb{H}^n$, $\mathrm{Aut}(\mathbb{H}^n)$, is the semidirect product of $(\mathrm{Aut}(\mathbb{H}))^n$ with the symmetric group $\mathfrak{S}_n$, hence for every subgroup Γ of $\mathrm{Aut}(\mathbb{H}^n)$ we have a diagram:

$$
\begin{array}{ccccccccc}
1 \to & (\mathrm{Aut}(\mathbb{H}))^n & \to & \mathrm{Aut}(\mathbb{H}^n) & \to \mathfrak{S}_n & \to 1 \\
& \cup & & \cup & \cup & \\
1 \to & \Gamma^0 & \hookrightarrow & \Gamma & \to H & \to 1.
\end{array}
$$

Proposition 1.5. *Let $X = \mathbb{H}^n/\Gamma$ be a compact complex manifold whose universal covering is the polydisk $\mathbb{H}^n$: then X admits a semi special tensor and K_X is ample, in particular $K_X^n > 0$.*

Proof. In $\mathbb{H}^n$ take coordinates $\{z_1, \ldots, z_n\}$ and define

$$
\tilde{\omega} := \frac{\mathrm{d}\, z_1 \otimes \cdots \otimes \mathrm{d}\, z_n}{\mathrm{d}\, z_1 \wedge \cdots \wedge \mathrm{d}\, z_n}.
$$

Observe that $\tilde{\omega}$ is clearly invariant for $(\mathrm{Aut}(\mathbb{H}))^n$ and for the alternating subgroup. Let η be the 2-torsion invertible sheaf associated to the signature character of $\mathfrak{S}_n$ restricted to H. Then clearly $\tilde{\omega}$ descends to a semi special tensor $\omega \in H^0(X, S^n\Omega_X^1(-K_X) \otimes \eta)$.

The other assertions are well known (cf. [Sieg73] and [K-M71]). $\square$

Remark 1.6. *We observe that also $(\mathbb{P}^1)^n$ admits the following special tensor ω, given on $\mathbb{C}^n \subset (\mathbb{P}^1)^n$ by $\omega := \frac{\mathrm{d}\, z_1 \otimes \cdots \otimes \mathrm{d}\, z_n}{\mathrm{d}\, z_1 \wedge \cdots \wedge \mathrm{d}\, z_n}$.*

In dimension two we have then the following

Theorem 1.7. *Let X be a compact complex surface.*

Then the following two conditions:

(1) *X admits a semi special tensor;*
(2) *$K_X^2 > 0$*

hold if and only if either

(i) $X \cong \mathbb{P}^1 \times \mathbb{P}^1$; or
(ii) $X \cong \mathbb{H} \times \mathbb{H}/\Gamma$ (where Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ acting freely).

In particular one has the following reformulation of a theorem of S.T. Yau (theorem 2.5 of [Yau93], giving sufficient conditions for (ii) to hold).

Theorem 1.8. (Yau) X *is strongly uniformized by the bidisk if and only if*

(1) X *admits a semi special tensor;*
(2) $K_X^2 > 0$;
(3) *the second plurigenus $P_2(X) \geq 1$.*

One can indeed be even more precise:

Theorem 1.9. X *is strongly uniformized by the bidisk if and only if*

(1*) X *admits a unique semi special tensor;*
(2) $K_X^2 > 0$;
(3*) *the second plurigenus $P_2(X) \geq 2$.*

X *is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if (1*), (2) hold and $P_2(X) = 0$.*

It is interesting to see that none of the above hypotheses can be dispensed with.

Remark 1.10. *The following examples show the existence of surfaces which satisfy two of the three conditions stated in Thm. 1.8, respectively in Thm. 1.9, but are not uniformized by the bidisk*

(i) $\mathbb{P}^1 \times \mathbb{P}^1$ *satisfies (1*) and (2);*
(ii) *A complex torus $X = \mathbb{C}^2/\Lambda$ satisfies (1) and (3), but neither (1*) nor (3*) (obviously, it does not satisfy (2));*
(iii) $X = C_1 \times C_2$ *with $g(C_1) = 1$, $g(C_2) = 2$ satisfies (1*) and (3*), but its universal cover is $\tilde{X} \cong \mathbb{C} \times \mathbb{H}$.*

The most intriguing examples are provided by

Proposition 1.11. *There do exist properly elliptic surfaces X satisfying*

- (1) X *admits a special tensor;*
- (3*) *the second plurigenus $P_2(X) \geq 2$;*
- $q(X) := dim(H^1(\mathcal{O}_X)) > 0$;
- $K_X^2 = 0$;
- X *is not birational to a product.*

We would like to pose then the following

Question. *Let X be a surface with $q(X) = 0$ and satisfying (1*) and (3*): is then X strongly uniformized by the bidisk?*

Concerning the above question, recall the following

Definition 1.12. $\Gamma \subset \mathrm{Aut}(\mathbb{H}^n)$ *is said to be reducible if there exists Γ^0 as above (i.e., such that $\gamma(z_1, ..., z_n) = (\gamma_1(z_1), ..., \gamma_n(z_n))$ for every $\gamma \in \Gamma^0$) and a decomposition $\mathbb{H}^n = \mathbb{H}^k \times \mathbb{H}^h$ (with $h > 0$) such that the action of Γ^0 on $\mathbb{H}^k$ is discrete.*

For $n = 2$ there are then only two alternatives:

Remark 1.13. *Let $\Gamma \subset \mathrm{Aut}(\mathbb{H}^2)$ be a discrete cocompact subgroup acting freely and let $X = \mathbb{H}^2/\Gamma$. Then*

- *Γ is reducible if and only if X is isogenous to a product of curves, i.e., there is a finite group G and two curves of genera at least 2 such that $X \cong C_1 \times C_2/G$. Both cases $q(X) \neq 0$, $q(X) = 0$ can occur here.*
- *Γ is irreducible and $q(X) = 0$ (this result holds in all dimensions and is a well-known result of Matsushima [Ma62]).*

Let us try to explain the main idea of our main result. In order to do this, it is important to make the following

Remark 1.14. *A complex manifold X admits a semi special tensor if and only if it has an unramified cover X' of degree at most two which admits a special tensor.*

Proof. Assume that we have an invertible sheaf η such that $\eta^2 \cong \mathcal{O}_X$, $\eta \not\cong \mathcal{O}_X$. Take the corresponding double connected étale covering $\pi : X' \to X$ and observe that

$$H^0(X', S^n\Omega^1_{X'}(-K_{X'})) \cong H^0(X, S^n\Omega^1_X(-K_X)) \oplus H^0(X, S^n\Omega^1_X(-K_X) \otimes \eta).$$

Whence, there is a special tensor on X' if and only if there is a semi special tensor on X. $\qquad\qquad\square$

In dimension $n = 2$ things are easier, since the existence of a special tensor ω is equivalent to the existence of a trace free endomorphism ϵ of the tangent bundle of X.

Our proof of Theorem 1.7 consists essentially in finding a decomposition of the tangent bundle T_X as a direct sum of two line bundles L_1 and L_2, which are the eigenbundles of an invertible endomorphism $\epsilon \in \mathrm{End}(T_X)$ (see §2 and §3 for details), and then applying the results on surfaces with split tangent bundles as given in [Bea00].

Since the results on manifolds with split tangent bundles hold in dimension $n \geq 3$, one has a characterization of compact manifolds strongly uniformized by the polydisk under a very strong condition on the semi special tensor $\omega \in H^0(X, S^n\Omega^1_X(-K_X) \otimes \eta)$, which essentially corresponds to ask for the local splitting of ω as the product of n 1-forms which are linearly independent at each point. There remains the problem of finding a simpler characterization.

2. Preliminaries and remarks

Notation. X denotes throughout a compact complex surface. We use standard notation of algebraic geometry: Ω^1_X is the cotangent sheaf, T_X is the holomorphic tangent bundle (locally free sheaf), $c_1(X)$, $c_2(X)$ are the Chern classes of X; K_X is the canonical divisor, and $P_n := h^0(X, nK_X)$ is called the n-th plurigenus, in particular for $n = 1$ we have the geometric genus of

X $p_g(X) := h^0(X, K_X)$, while $q := h^1(X, \mathcal{O}_X)$ is classically called the irregularity of X. Finally, $\chi(X) := \chi(\mathcal{O}_X) = 1 + p_g - q$ is the holomorphic Euler characteristic.

With a slight abuse of notation, we do not distinguish between invertible sheaves, line bundles and divisors, while the symbol $\equiv$ denotes linear equivalence of divisors.

First of all let us recall a result of Beauville which characterizes compact complex surfaces whose universal cover is a product of two complex curves (cf. [Bea00, Thm. C]).

Theorem 2.1 (Beauville). *Let X be a compact complex surface. The tangent bundle T_X splits as a direct sum of two line bundles if and only if either X is a special Hopf surface or the universal covering space of X is a product $U \times V$ of two complex curves and the group $\pi_1(X)$ acts diagonally on $U \times V$.*

Given a direct sum decomposition of the cotangent bundle $\Omega_X^1 \cong L_1 \oplus L_2$, Beauville shows that $(L_1)^2 = (L_2)^2 = 0$ (cf. [Bea00, 4.1, 4.2]) hence

$$K_X \equiv L_1 + L_2 \qquad c_1(X)^2 = 2 \cdot (L_1 \cdot L_2) = 2 \cdot c_2(X)$$

The last equality corresponds to $K_X^2 = 8\chi(X)$.

Let us now consider the bundle $\mathrm{End}(T_X)$ of endomorphisms of the tangent bundle. We can write $\mathrm{End}(T_X) = \Omega_X^1 \otimes T_X$ and from the nondegenerate bilinear map

$$\Omega_X^1 \times \Omega_X^1 \longrightarrow \Omega_X^2 \cong K_X$$

we see that $T_X = (\Omega_X^1)^\vee \cong \Omega_X^1(-K_X)$. This exactly means that we have an isomorphism $\mathrm{End}(T_X) \cong \Omega_X^1 \otimes \Omega_X^1(-K_X)$.

Let us see how this isomorphism works in local coordinates (z_1, z_2). I.e., let us see how an element $\frac{\mathrm{d}\, z_i \otimes \mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}$ in $\Omega_X^1 \otimes \Omega_X^1(-K_X)$ acts on a vector of the form $\frac{\partial}{\partial z_h}$. We have

$$\frac{\mathrm{d}\, z_i \otimes \mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}\Big(\frac{\partial}{\partial z_h}\Big) = \begin{cases} \frac{\mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2} & \text{if } h = i \\ 0 & \text{if } h \neq i \end{cases}$$

In turn, $\dfrac{\mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}$ evaluated on $\mathrm{d}\, z_k$ gives $\dfrac{\mathrm{d}\, z_j \wedge \mathrm{d}\, z_k}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}$.

Therefore a generic element $\sum\limits_{i,j} a_{ij} \dfrac{\mathrm{d}\, z_i \otimes \mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}$ corresponds to an endomorphism, which, with respect to the basis $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\}$ is expressed by the matrix

$$\begin{pmatrix} -a_{12} & -a_{22} \\ a_{11} & a_{21} \end{pmatrix}$$

In particular for the symmetric tensors (i.e., $a_{12} = a_{21}$), respectively for the skewsymmetric tensors (i.e., $a_{12} = -a_{21}, a_{11} = a_{22} = 0$) the following isomorphisms hold:

$$S^2(\Omega_X^1)(-K_X) \cong \left\{ \begin{pmatrix} -a & -a_{22} \\ a_{11} & a \end{pmatrix} \right\}; \qquad \bigwedge^2(\Omega_X^1)(-K_X) \cong \left\{ \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \right\}$$

We can summarize the above discussion in the following

Lemma 2.2. *If X is a complex surface there is a natural isomorphism between the sheaf $S^2(\Omega_X^1)(-K_X)$ and the sheaf of trace zero endomorphisms of the (co)tangent sheaf $\mathrm{End}^0(T_X) \cong \mathrm{End}^0(\Omega_X^1)$.*

A special tensor $\omega \in H^0(S^2(\Omega_X^1)(-K_X))$ with nonzero determinant $det(\omega) \in \mathbb{C}$ yields an eigenbundle splitting $\Omega_X^1 \cong L_1 \bigoplus L_2$ of the cotangent bundle.

If instead $det(\omega) = 0 \in \mathbb{C}$, the corresponding endomorphism ϵ is nilpotent and yields an exact sequence of sheaves

$$0 \to L \to \Omega_X^1 \to \mathcal{I}_Z L(-\Delta) \to 0$$

where $L := \ker(\epsilon)$ is invertible, Δ is an effective divisor, and Z is a 0-dimensional subscheme(which is a local complete intersection).

We have in particular $K_X \equiv 2L - \Delta$ and $c_2(X) = length(Z) + L \cdot (L - \Delta)$.

Proof. We need only to observe that $det(\omega)$ is a constant, since $det(\mathrm{End}(T_X)) = det(\mathrm{End}(\Omega_X^1)) \cong \mathcal{O}_X$.

If $det(\omega) \neq 0$, there is a constant $c \in \mathbb{C} \setminus \{0\}$ such that $det(\omega) = c^2$, hence at every point of X the endomorphism ϵ corresponding to the special tensor ω has two distinct eigenvalues $\pm c$.

Let $\omega \in H^0(S^2\Omega_X^1(-K_X))$, $\omega \neq 0$, be such that $det(\omega) = 0$. Then the corresponding endomorphism ϵ is nilpotent of order 2, and there exists an open nonempty subset $U \subseteq X$ such that $\mathrm{Ker}(\epsilon_{|U}) = \mathrm{Im}(\epsilon_{|U})$. At a point p where $\mathrm{rank}(\epsilon) = 0$, in local coordinates the endomorphism ϵ may be expressed by

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad a, b, c \text{ regular functions such that } a^2 = -b \cdot c$$

Let $\delta := \mathrm{G.\,C.\,D.}(a, b, c)$. After dividing by δ, every prime factor of a is either not in b, or not in c, thus we can write

$$-b = \beta^2 \qquad c = \gamma^2 \qquad a = \beta \cdot \gamma$$

Therefore we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathrm{Ker}\,\epsilon \iff \begin{cases} a \cdot u + b \cdot v = 0 \\ c \cdot u - a \cdot v = 0 \end{cases} \iff \gamma \cdot u - \beta \cdot v = 0 \iff \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \beta \cdot f \\ \gamma \cdot f \end{pmatrix}$$

and, writing our endomorphism ϵ as $\epsilon = \delta \cdot \alpha$, we have

$$\mathrm{Im}(\alpha) = \begin{cases} \beta \cdot \gamma \cdot u - \beta^2 \cdot v = \beta \cdot (\gamma \cdot u - \beta \cdot v) \\ \gamma^2 \cdot u - \gamma \cdot \beta \cdot v = \gamma \cdot (\gamma \cdot u - \beta \cdot v) \end{cases}$$

Let Z be the 0-dimensional scheme defined by $\{\beta = \gamma = 0\}$ and Δ be the Cartier divisor defined by $\{\delta = 0\}$.

From the above description we deduce that the kernel of ϵ is a line bundle L which fits in the following exact sequence:

$$0 \to L \to \Omega_X^1 \to \mathcal{I}_Z L(-\Delta) \to 0.$$

Taking the total Chern classes we infer that: $K_X \equiv 2L - \Delta$ as divisors on X and $c_2(X) = length(Z) + L \cdot (L - \Delta)$. $\qquad\square$

Lemma 2.3. *Let X be a complex surface and let X' be the blow up of X at a point p. Then a special tensor ω' on X' induces a special tensor*

ω on X, and the converse only holds if and only if ω vanishes at p (in particular, it must hold : $det(\omega) = 0$).

Proof. First of all, ω' induces a special tensor on $X \setminus \{p\}$, and by Hartogs' theorem the latter extends to a special tensor ω on X.

Conversely, choose local coordinates (x, y) for X around p and take a local chart of the blow up with coordinates (x, u) where $y = ux$. Locally around p we can write

$$\omega = \frac{a(\mathrm{d}\,x)^2 + b(\mathrm{d}\,y)^2 + c(\mathrm{d}\,x\,\mathrm{d}\,y)}{\mathrm{d}\,x \wedge \mathrm{d}\,y}.$$

The pull back ω' of ω is given by the following expression:

$$\frac{a(\mathrm{d}\,x)^2 + b(u\,\mathrm{d}\,x + x\,\mathrm{d}\,u)^2 + c(u\,\mathrm{d}\,x + x\,\mathrm{d}\,u)\,\mathrm{d}\,x}{x\,\mathrm{d}\,x \wedge \mathrm{d}\,u} =$$

$$= \frac{\mathrm{d}\,x^2(a + bu^2 + cu) + bx^2\,\mathrm{d}\,u^2 + (2bux + cx)\,\mathrm{d}\,x\,\mathrm{d}\,u}{x\,\mathrm{d}\,x \wedge \mathrm{d}\,u},$$

hence ω' is regular if and only if $\frac{a+bu^2+cu}{x}$ is a regular function.

This is obvious if a, b, c vanish at p, since then their pull back is divisible by x. Assume on the other side that a, b, c are constant: then we get a rational function which is only regular if $a = b = c = 0$.

$\square$

Lemma 2.4. *Let X be a compact minimal rational surface admitting a special tensor ω. Then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ if $det(\omega) \neq 0$.*

Proof. Assume that X is a $\mathbb{P}^1$ bundle over a curve $B \cong \mathbb{P}^1$, i.e., a ruled surface $\mathbf{F}_n$ with $n \geq 0$. Let $\pi \colon X \to B$ the projection.

By the exact sequence

$$0 \to \pi^*\Omega^1_B \to \Omega^1_X \to \Omega^1_{X|B} \to 0$$

and since on a general fibre F the subsheaf $\pi^*\Omega^1_B$ is trivial, while the quotient sheaf $\Omega^1_{X|B}$ is negative, we conclude that any endomorphism ϵ carries $\pi^*\Omega^1_B$ to itself. If it has non zero determinant we can conclude by Theorem 2.1 that $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Otherwise, ϵ is nilpotent and we have a nonzero element in $\mathrm{Hom}(\Omega^1_{X|B}, \pi^*\Omega^1_B)$.

Since these are invertible sheaves, it suffices to see when

$$H^0(\mathcal{O}_X(2\pi^*K_B - K_X)) \neq 0.$$

But, letting Σ be the section with selfintersection $\Sigma^2 = -n$, our vector space equals $H^0(\mathcal{O}_X(2\Sigma - (n+2)F))$. Intersecting this divisor with Σ we see that (since each time the intersection number with Σ is negative) $H^0(\mathcal{O}_X(2\Sigma - (n+2)F)) = H^0(\mathcal{O}_X(\Sigma - (n+2)F)) = H^0(\mathcal{O}_X(-(n+2)F)) = 0$.

There remains the case where X is $\mathbb{P}^2$.

In this case ϵ must be a nilpotent endomorphism by Theorem 2.1, and it cannot vanish at any point by our previous result on $\mathbf{F}_1$. Therefore the rank

of ϵ equals 1 at each point. By lemma 2.2 it follows that there is a divisor L such that $K_X = 2L$, a contradiction.

□

3. Proof of Theorems 1.7 and 1.9

Proof. If X is strongly uniformized by the bidisk, then K_X is ample, in particular $K_X^2 \geq 1$ and, since by Castelnuovo's theorem $\chi(X) \geq 1$, by the vanishing theorem of Kodaira and Mumford it follows that $P_2(X) \geq 2$ (see [Bom73]).

Thus one direction follows from proposition 1.5, except that we shall show only later that (1*) holds.

Assume conversely that $(1), (2)$ hold. Without loss of generality we may assume by lemma 2.3 that X is minimal, since K_X^2 can only decrease via a blowup and the bigenus is a birational invariant.

$K_X^2 \geq 1$ implies that either the surface X is of general type, or it is a rational surface. In the latter case we conclude by lemma 2.4.

Observe that the further hypothesis (3) (obviously implied by (3*)) guarantees that X is of general type.

Thus, from now on, we may assume that X is of general type and, passing to an étale double cover if necessary, that X admits a special tensor.

By the cited Theorem 2.1 of [Bea00] it suffices to find a decomposition of the cotangent bundle Ω_X^1 as a direct sum of two line bundles L_1 and L_2.

The two line bundles L_1, L_2 will be given as eigenbundles of a diagonizable endomorphism $\epsilon \in \mathrm{End}(\Omega_X^1)$.

Our previous discussion shows then that it is sufficient to show that any special tensor cannot yield a nilpotent endomorphism.

Otherwise, by lemma 2.2, we can write $2L \equiv K_X + \Delta$ and then deduce that L is a big divisor since Δ is effective by construction and K_X is big because X is of general type. This assertion gives the required contradiction since by the Bogomolov-Castelnuovo-de Franchis Theorem (cf. [Bog77]) for an invertible subsheaf L of Ω_X^1 it is $h^0(X, mL) \leq O(m)$, contradicting the bigness of L.

There remains to show (1*). But if $h^0(X, S^2\Omega_X^1(-K_X)) \geq 2$ then, given a point $p \in X$, there is a special tensor which is not invertible in p, hence a special tensor with vanishing determinant, a contradiction.

□

4. Proof of proposition 1.11

In this section we consider surfaces X with bigenus $P_2(X) \geq 2$ (property (3*)), therefore their Kodaira dimension equals 1 or 2, hence either they are properly (canonically) elliptic, or they are of general type.

Since we took already care of the latter case in the main theorems 1.7 and 1.9, we restrict our attention here to the former case, and try to see when does a properly elliptic surface admit a special tensor (we can reduce to this

situation in view of remark 1.14). We can moreover assume that the associated endomorphism ϵ is nilpotent by theorem 2.1.

Again without loss of generality we may assume that X is minimal by virtue of lemma 2.3.

Proof. Let X be a minimal properly elliptic surface and let $f : X \to B$ be its (multi)canonical elliptic fibration. Write any fibre $f^{-1}(p)$ as $F_p = \sum_{i=1}^{h_p} m_i C_i$ and, setting $n_p := G.C.D.(m_i)$, $F_p = n_p F'_p$, we say that a fibre is multiple if $n_p > 1$. By Kodaira's classification ([Kod60]) of the singular fibres we know that in this case $m_i = n_p, \forall i$.

Assume that the multiple fibres of the elliptic fibration are $n_1 F'_1, \ldots, n_r F'_r$, and consider the divisorial part of the critical locus

$$\mathcal{S}_p := \sum_{i=1}^{h_p} (m_i - 1) C_i, \quad \mathcal{S} := \sum_{p \in B} \mathcal{S}_p$$

so that we have then the exact sequence

$$0 \to f^* \Omega_B^1(\mathcal{S}) \to \Omega_X^1 \to \mathcal{I}_{\mathcal{C}} \, \omega_{X|B} \to 0,$$

where $\mathcal{C}$ is a 0-dimensional (l.c.i.) subscheme.

For further calculations we separate the divisorial part of the critical locus as the sum of two disjoint effective divisors, the multiple fibre contribution and the rest:

$$\mathcal{S}_m := \sum_{i=1}^{r} (n_i - 1) F'_i, \ \hat{\mathcal{S}} := \mathcal{S} - \mathcal{S}_m.$$

Let us assume that we have a nilpotent endomorphism corresponding to another exact sequence

$$0 \to L \to \Omega_X^1 \to \mathcal{I}_Z L(-\Delta) \to 0,$$

in turn determined by a homomorphism

$$\epsilon' : \mathcal{I}_Z L(-\Delta) \to L,$$

i.e., by a section

$$s \in H^0(\mathcal{O}_X(\Delta)) =$$
$$= H^0(\mathcal{O}_X(2L - K_X)) = H^0(S^2(L)(-K_X)) \subset H^0(S^2(\Omega_X^1)(-K_X)).$$

We observe that, since $2L \equiv K_X + \Delta$, it follows that, if F is a general fibre, then

$$L \cdot F = \Delta \cdot F = 0,$$

hence the effective divisor Δ is contained in a finite union of fibres.

The first candidate to try with is the choice of $L = L'$, where we set $L' := f^* \Omega_B^1(\mathcal{S})$.

To this purpose we recall Kodaira's canonical bundle formula:

$$K_X \equiv \mathcal{S}_m + f^*(\delta) = \sum_{i=1}^{r} (n_i - 1) F'_i + f^*(\delta), \ deg(\delta) = \chi(X) - 2 + 2b,$$

where b is the genus of the base curve B.

Then $H^0(\mathcal{O}_X(2L' - K_X)) = H^0(\mathcal{O}_X(f^*(2K_B - \delta) + 2\mathcal{S} - \mathcal{S}_m))$, and we search for an effective divisor linearly equivalent to

$$f^*(2K_B - \delta) + 2\mathcal{S} - \mathcal{S}_m = f^*(2K_B - \delta) + 2\hat{\mathcal{S}} + \mathcal{S}_m.$$

We claim that $H^0(\mathcal{O}_X(2L' - K_X)) = H^0(\mathcal{O}_X(f^*(2K_B - \delta)))$: it will then suffice to have examples where $|2K_B - \delta| \neq \emptyset$.

Proof of the claim

It suffices to show that $f_*\mathcal{O}_X(2\hat{\mathcal{S}} + \mathcal{S}_m) = \mathcal{O}_B$. Since the divisor $2\hat{\mathcal{S}} + \mathcal{S}_m$ is supported on the singular fibres, and it is effective, we have to show that, for each singular fibre $F_p = \sum_{i=1}^{h_p} m_i C_i$, neither $2\hat{\mathcal{S}}_p \geq F_p$ nor $\mathcal{S}_{m,p} \geq F_p$.

The latter case is obvious since $\mathcal{S}_{m,p} = (n_p - 1)F'_p < F_p = n_p F'_p$.

In the former case, $2\hat{\mathcal{S}}_p = \sum_{i=1}^{h_p} 2(m_i - 1)C_i$, but it is not possible that $\forall i$ one has $2(m_i - 1) \geq m_i$, since there is always an irreducible curve C_i with multiplicity $m_i = 1$.

$$Q.E.D.\text{for the claim}$$

Assume that the elliptic fibration is not a product (in this case there is no special tensor with vanishing determinant): then the irregularity of X equals the genus of B, whence our divisor on the curve B has degree equal to $2b - 2 - (1 - b + p_g(X)) = 3b - 3 - p_g$.

Since $\chi(X) \geq 1$, $p_g := p_g(X) \geq b$, and there exist an elliptic surface X with any $p_g \geq b$ ([Cat07]).

Since any divisor on B of degree $\geq b$ is effective, it suffices to choose $b \leq p_g \leq 2b - 3$ and we get a special tensor with trivial determinant, provided that $b \geq 3$.

Take now a Jacobian elliptic surface in Weierstrass normal form

$$ZY^2 - 4X^3 - g_2 XZ^2 - g_3 Z^3 = 0,$$

where $g_2 \in H^0(\mathcal{O}_B(4M))$, $g_3 \in H^0(\mathcal{O}_B(6M))$, and assume that all the fibres are irreducible.

Then the space of special tensors corresponding to our choice of L corresponds to the vector space $H^0(\mathcal{O}_B(2K_B - \delta)) = H^0(\mathcal{O}_B(K_B - 6M))$. It suffices to take a hyperelliptic curve B of genus $b = 6h + 1$, and, denoting by H the hyperelliptic divisor, set $M := hH$, so that $K_B - 6M \equiv 0$ and we have $h^0(\mathcal{O}_X(2L - K_X)) = 1$. We leave aside for the time being the question whether the surface X admits a unique special tensor.

$\square$

We would like to thank Eckart Viehweg and Kang Zuo for pointing out some aspects of Yau's uniformization theorem that we had not properly credited in the first version.

References

[BPV84] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces.* **Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag**, Berlin,(1984).

[Bea00] A. Beauville, *Complex manifolds with split tangent bundle* Complex analysis and algebraic geometry, de Gruyter, Berlin, 2000, 61–70.

[Bog77] F. A. Bogomolov *Families of curves on a surface of general type* (Russian) **Dokl. Akad. Nauk SSSR, 236** no. 5, 1977 , p.1041–1044.

[Bom73] E. Bombieri, *Canonical models of surfaces of general type*, **Publ. Math. I.H.E.S., 42** (1973), 173–219.

[Cat07] F. Catanese, "*Q.E.D. for algebraic varieties*", **Jour. Diff. Geom. 77** no. 1 (2007) 43–75

[Kod60] K. Kodaira, *On compact complex analytic surfaces, I*, **Ann. of Math. 71** (1960), 111–152.

[K-M71] Kodaira, K. Morrow, J. *Complex manifolds* Holt, Rinehart and Winston, Inc., New-York-Montreal, Que.-London, 1971

[Ma62] Y . Matsushima, *On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces* **Ann. of Math. (2) 75**, 1962, 312–330

[MT87] B.Moishezon, M. Teicher, *Simply-connected algebraic surfaces of positive index* **Invent. Math. 89** (1987), no. 3, 601–643.

[Miy77] Y. Miyaoka, *On the Chern numbers of surfaces of general type* **Invent. Math. 42** (1977), 225–237.

[Miy82] Y. Miyaoka, *Algebraic surfaces with positive indices* Classification of algebraic and analytic manifolds (Katata, 1982), **Progr. Math. 39** Birkhuser Boston, Boston, MA, (1983) 281–301

[Mu79] D. Mumford, *An algebraic surface with K ample, $K^2 = 9$, $p_g = q = 0$* **Amer. J. Math. 101**, no. 1, (1979), 233-244.

[Sieg73] C.L. Siegel, *Topics in complex function theory. Vol. III. Abelian functions and modular functions of several variables.* Translated from the German by E. Gottschling and M. Tretkoff. With a preface by Wilhelm Magnus. Reprint of the 1973 original. **Wiley Classics Library. A Wiley-Interscience Publication. John Wiley and Sons, Inc.**, New York (1989), x+244 pp.

[V-Z05] E. Viehweg, K. Zuo, *Arakelov inequalities and the uniformization of certain rigid Shimura varieties*, arXiv:math/0503339.

[Yau77] S. T. Yau, *Calabi' s conjecture and some new results in algebraic geometry* **Proc. Nat. Acad. Sci. USA 74** (1977), 1798–1799

[Yau88] S. T. Yau, *Uniformization of geometric structures.* The mathematical heritage of Hermann Weyl (Durham, NC, 1987), **Proc. Sympos. Pure Math., 48, Amer. Math. Soc., Providence, RI** (1988), 265–274.

[Yau93] S. T. Yau, *A splitting theorem and an algebraic geometric characterization of locally hermitian symmetric spaces* **Comm. in Analysis and Geometry 1** (1993), 473–486

Authors' addresses:

Prof. Fabrizio Catanese

Lehrstuhl Mathematik VIII, Universität Bayreuth, NWII

D-95440 Bayreuth, Germany

e-mail: Fabrizio.Catanese@uni-bayreuth.de

Marco Franciosi
Dipartimento di Matemativa Applicata "U. Dini", Università di Pisa
via Buonarroti 1C, I-56127, Pisa, Italy
e-mail: franciosi@dma.unipi.it