On Property N_p for algebraic curves

Edoardo Ballico, Marco Franciosi

Abstract

Let $C = C_1 \cup \ldots \cup C_s$ be a reduced, but reducible, curve and let $\mathcal{L}$ in $\mathrm{Pic}(C)$ be very ample. Here we give conditions on $\deg(\mathcal{L}_{|C_i})$ insuring that the embedding of C induced by $\mathcal{L}$ satisfies Property N_p.

We also study the minimal free resolution for general projections of smooth curves.

Introduction

Let $X \subset \mathbb{P}^N$ be a projective scheme and let $\mathcal{I} = \mathcal{I}_{X/\mathbb{P}^n}$ denote the homogeneous ideal of X. We recall the definition of Property N_p as introduced in [G].

Let $E_\bullet$ be a minimal graded free resolution of $\mathcal{I}$ over the homogeneous coordinate ring S of $\mathbb{P}^N$:

$$0 \to E_{N+1} \to E_N \to \ldots \to E_1 \to \mathcal{I} \to 0$$

where $E_i = \oplus S(-a_{ij})$. $X \subset \mathbb{P}^N$ satisfies Property N_p if

$$E_i = \oplus S(-i-1) \quad \text{for} \quad 1 \leq i \leq p.$$

Property N_0 holds if and only if X is projectively Cohen–Macaulay. Property N_1 holds if Property N_0 does and the ideal $\mathcal{I}$ is generated by quadrics.

If $\mathcal{L}$ is an invertible sheaf on a projective scheme X', we will say that $\mathcal{L}$ satisfies Property N_p if $\mathcal{L}$ is very ample and $\varphi_{|\mathcal{L}|}(X') := X \subset \mathbb{P}^N$ satisfies Property N_p.

For a smooth projective curve of genus g in the papers [G] and [G-L] it is proved that an invertible sheaf $\mathcal{L}$ of degree $\deg \mathcal{L} \geq 2g+1+p$ satisfies Property N_p.

In section 1 we study the case where C is a reduced curves, under some numerical conditions. Our first result is the following (for the definitions we refer to the next section)

Theorem A *Let C be a connected reduced curve, and let $\mathcal{L}$ be an invertible sheaf on C.*

Assume there exists a decomposition $C = C_1 \cup \ldots \cup C_s$ (C_i irreducible components of arithmetic genus $g(C_i)$) such that, if we set $Y_1 = C_1$ and for $i = 2, \ldots, s$ $Y_i := Y_{i-1} \cup C_i$ then

Y_i is connected; $\deg \mathcal{L}_{|Y_i} \geq 2g(Y_i) + 1 + p$
$\deg \mathcal{L}_{|C_i} = d_i \geq \max\{2g(C_i) + Y_{i-1}.C_i + p,\ 2g(C_i) + C_i.(C - C_i) - 1\}$
where for $i = 1$ we let $Y_{i-1}.C_i = 1$ by definition

Then $\mathcal{L}$ satisfies Property N_p.

By theorem 1.1 of [CFHR] the linear system $|\mathcal{L}|$ is very ample and defines an embedding $\varphi_{|\mathcal{L}|} : C \hookrightarrow \mathbb{P}^N$. Furthermore from theorem A of [F] $\mathcal{L}$ is normally generated, that is, $\varphi_{|\mathcal{L}|}(C) \subset \mathbb{P}^N$ is projectively Cohen–Macaulay.

Thus, cf. remark 1.2, to prove Property N_p for $\varphi_{|\mathcal{L}|}(C)$ it will suffices to consider a generic hyperplane section.

Notice that with only the condition

$$\deg \mathcal{L}_{|C} \geq 2g(C) + 1 + p$$

but without any further assumption on the irreducible components the theorem is no longer true. An easy example is the case where $C = C_1 \cup C_2$, with C_1 an irreducible curve of genus 1 and C_2 an irreducible curve of genus 0, such that their intersection $C_1.C_2 = 1$. If $\mathcal{L}$ is an invertible sheaf such that $\deg \mathcal{L}_{|C_1} = 3$, $\deg \mathcal{L}_{|C_2} = 1$, then $\mathcal{L}$ is very ample and $\varphi_{|\mathcal{L}|}(C) \subset \mathbb{P}^3$ consists of a cubic plane curve plus a line (not contained in the plane of the curve) which intersects the cubic in exactly one point. Then it is easy to see that $\varphi_{|\mathcal{L}|}(C)$ satisfies Property N_0 (cf. also [F]), but obviously $\varphi_{|\mathcal{L}|}(C)$ is not cut out by quadrics!

In section 2 we introduce the notion of *Weak Property N_p* for projective scheme of dimension ≤ 1 and we prove a theorem on general projections of smooth curves.

Definition. *Let $X \subset \mathbb{P}^n$ be a projective scheme of dimension ≤ 1 (we allow X to be not equidimensional or with embedded points.)*

We say that X satisfies the Weak Property N_0 if for all $t \geq 2$ the restriction map

$$\rho_t : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(X, \mathcal{O}_X(t))$$

is surjective.

We say that X satisfies the Weak Property N_1 if it satisfies the Weak Property N_0 and the homogeneous ideal of X is generated by quadrics. For $2 \leq p \leq n-2$ we define inductively that X satisfies the Weak Property N_p if it satisfies the Weak Property N_{p-1} and the $p-$ syzygies of the homogeneous ideal of X are generated by linear forms.

Our result is the following

Theorem B *Fix integers g, d, p, n with $3 \leq n < d - g$ and assume $d \geq 2g + 1 + p + 3(d - g - n)$.*

Let C be a smooth connected projective curve of genus g and $\mathcal{L} \in \mathrm{Pic}^d(C)$. Let $\varphi_{|\mathcal{L}|} : C \hookrightarrow \mathbb{P}(H^0(C, \mathcal{L})^\vee)$ be the complete embedding associated to $|\mathcal{L}|$ and let $X \subset \mathbb{P}^n$ be a general projection of $\varphi_{|\mathcal{L}|}(C)$. Then X satisfies the Weak Property N_p.

The proof of the above theorem will follow by a degeneration argument and the study of the Weak Property N_p for a curve with embedded points.

The method we use to prove both the theorems is based on the analysis of the 0-dimensional scheme obtained taking a sufficiently general hyperplane section.

For this case the following remark turns out to be fundamental.

Remark C *Fix integers p, r, d with $0 \leq p \leq r-2$ and $d \leq 2r+1-p$.*

Let $Z \subset \mathbb{P}^r$ be a 0-dimensional scheme of length d for which there exists a partition $Z = \Sigma \cup \Gamma$ into disjoint subschemes with the following properties:

- *(a)* $\mathrm{length}(\Sigma) = r+1$, *$\Sigma$ is reduced and Σ spans $\mathbb{P}^r$;*
- *(b)* $\mathrm{length}(\Gamma) \leq r-p$, $\dim(\langle\Gamma\rangle) = \mathrm{length}(\Gamma) - 1$ *(i.e., Γ is in linearly general position) and for any $\Sigma' \subset \Sigma$ with* $\mathrm{card}(\Sigma') = p$ *and every $Q \in \Sigma \setminus \Sigma'$ there exists a hyperplane H of $\mathbb{P}^r$ with $\Sigma' \cup \Gamma \subset H$ and $Q \notin H$.*

Then the method of [G-L], theorem 2.1, gives that Z satisfies the Weak Property N_p.

Notation

For all the paper we will assume C to be a reduced curve (a pure projective scheme of pure dimension 1 such that for every point $P \in C$ the local ring $\mathcal{O}_{C,P}$ has no nilpotent elements) over an algebraically closed field $\mathbb{K}$ of characteristic 0.

For the positive characteristic case see Remark 1.4.

- $\mathcal{L}$ An invertible sheaf on C.
- $|\mathcal{L}|$ Linear system of divisors of sections of $H^0(C, \mathcal{L})$.
- $\deg \mathcal{L}_{|C}$ The degree of $\mathcal{L}$ on C; it can be defined for every torsion free sheaf of rank 1 by

$$\deg \mathcal{L}_{|C} = \chi(\mathcal{L}) - \chi(\mathcal{O}_C).$$

- $g(C)$ The arithmetic genus of C, $g(C) = 1 - \chi(\mathcal{O}_C)$.

If $C = A \cup B$ scheme theoretically with $\dim A \cap B = 0$ and $x \in A \cap B$, we can define (cf. [Ca], p. 54)

$$(A.B)_x = \mathrm{length}\, \mathcal{O}_{A\cap B,x}; \quad A.B = \sum_{x \in A\cap B} \mathrm{length}\, \mathcal{O}_{A\cap B,x}$$

Notice that if $C = A \cup B$, with $\dim A \cap B = 0$, then we recover the classical formula

$$g(C) = g(A) + g(B) + A.B - 1$$

Sometimes, with abuse of notation, we will denote the curve B as $C - A$.

1 Property N_p on reduced curves

Let C, $\mathcal{L}$ be as in theorem A. Then we have $\deg \mathcal{L}_{|B} \geq 2g(B)+1$ for all subcurve B of C. Thus from theorem 1.1 of [CFHR] the linear system $|\mathcal{L}|$ is very ample and it defines an embedding $\varphi_{|\mathcal{L}|} : C \hookrightarrow \mathbb{P}^N$.

Let us consider the sequence of theorem A

$$C_1 = Y_1 \subset Y_2 \subset \ldots \subset Y_s = C$$

where the Y_i's are still connected. The following restriction lemma holds

Lemma 1.1 *Let C and $\mathcal{L}$ be as in theorem A.*

Let $C_1 = Y_1 \subset Y_2 \subset \ldots \subset Y_s = C$ be the sequence of theorem A. Then the following restriction maps are onto

$$\begin{array}{lll} (a) & H^0(Y_{i+1}, \mathcal{L}) \to H^0(Y_i, \mathcal{L}) & \forall i \in \{1, \ldots, s-1\} \\ (b) & H^0(Y_{i+1}, \mathcal{L}) \to H^0(C_{i+1}, \mathcal{L}) & \forall i \in \{1, \ldots, s-1\} \end{array}$$

Proof. Cosidering the exact sequences of coherent sheaves

$$0 \to \mathcal{O}_{C_{i+1}}(\mathcal{L}) \otimes \mathcal{I}_{Y_i} \to \mathcal{O}_{C_{i+1}}(\mathcal{L}) \to \mathcal{O}_{Y_i}(\mathcal{L}) \to 0$$

$$0 \to \mathcal{O}_{Y_i}(\mathcal{L}) \otimes \mathcal{I}_{C_{i+1}} \to \mathcal{O}_{C_{i+1}}(\mathcal{L}) \to \mathcal{O}_{C_{i+1}}(\mathcal{L}) \to 0$$

the lemma will follow if

$$\begin{array}{lll} (a') & H^1(C_{i+1}, \mathcal{O}_{C_{i+1}}(\mathcal{L}) \otimes \mathcal{I}_{Y_i}) = 0 & \forall i \in \{1, \ldots, s-1\} \\ (b') & H^1(Y_i, \mathcal{O}_{Y_i}(\mathcal{L}) \otimes \mathcal{I}_{C_{i+1}} = 0 & \forall i \in \{1, \ldots, s-1\} \end{array}$$

(a') We apply the foundamental argument of [CFHR],thm. 1.1 to C_{i+1}.

Indeed, C_{i+1} is reduced and irreducible and by our numerical hypothesis we have $\deg \mathcal{L}_{|C_{i+1}} \geq 2g(C_{i+1}) + Y_i.C_{i+1} - 1$.

Furthermore $Y_i \cap C_{i+1}$ is a 0-dimensional subscheme of C_{i+1} of length $Y_i.C_{i+1}$. Call it ζ.

We have $\mathcal{O}_{C_{i+1}}(\mathcal{L}) \otimes \mathcal{I}_{Y_i} \cong \mathcal{O}_{C_{i+1}}(\mathcal{L}) \otimes \mathcal{I}_{\zeta}$ and

$$H^1(C_{i+1}, \mathcal{O}_{Y_{i+1}}(\mathcal{L}) \otimes \mathcal{I}_{Y_i}) = H^1(C_{i+1}, \mathcal{O}_{Y_{i+1}}(\mathcal{L}) \otimes \mathcal{I}_{\zeta}) \stackrel{\text{d}}{=} \mathrm{Hom}(\mathcal{L} \otimes \mathcal{I}_{\zeta}, \omega_{C_{i+1}})$$

where $\stackrel{\text{d}}{=}$ means duality of vector spaces and $\omega_{C_{i+1}}$ is the dualizing sheaf. Now thm.1.1 of [CFHR] yields $\mathrm{Hom}(\mathcal{L} \otimes \mathcal{I}_{\zeta}, \omega_{C_{i+1}}) = 0$ since $\deg \mathcal{L}_{|C_{i+1}} \geq 2g(C_{i+1}) - 1 + \mathrm{length}(\zeta)$, which concludes the proof.

(b') Let $Y_{i+1} := Y_i \cup C_{i+1}$. Then $\mathcal{O}_{Y_i} \otimes \mathcal{I}_{C_{i+1}}$ defines on Y_i a 0-dimensional scheme of length $Y_i.C_{i+1}$. Call it ξ.

Since

$$\deg \mathcal{L}_{|C_j} \geq 2g(C_j) + C_j.(C - C_j) - 1 \quad \text{for all } C_j$$

by the genus formula for a reduced curve our numerical hypotheses imply

$$\deg \mathcal{L}_{|B} \geq 2g(B) + B.C_{i+1} - 1 \text{ for all } B \subseteq Y_i$$

which is equivalent to

$$\deg \mathcal{L}_{|B} \geq 2g(B) - 1 + \mathrm{length}(\xi \cap B) \text{ for all } B \subseteq Y_i.$$

As in (i) we can apply thm. 1.1 of [CFHR] to the 0-dimensional scheme ξ_i and the invertible sheaf $\mathcal{L}_{|Y_i}$ to get $H^1(Y_i, \mathcal{L} \otimes \mathcal{I}_\xi) = H^1(Y_i, \mathcal{L} \otimes \mathcal{I}_{C_{i+1}}) = 0$.

□

Remark 1.2 *By an induction argument using the above lemma and the classical results on normal generation (cf. e.g. [G]), or applying theorem A of [F], we see that $\mathcal{L}$ is normally generated, that is, $\varphi_{|\mathcal{L}|}(C) \subset \mathbb{P}^N$ is projectively Cohen–Macaulay.*

Thus, as shown in [G-L] *prop. 3.2, if $s_0 \in |\mathcal{L}|$ is a generic hyperplane section and $Z := \varphi_{|\mathcal{L}|}(C) \cap \{s_0 = 0\}$, a minimal free resolution of $\mathcal{I}_{\varphi_{|\mathcal{L}|}(C)/\mathbb{P}^N}$ restricts to one of $\mathcal{I}_{Z/\mathbb{P}^{N-1}}$, which implies in particular that $\varphi_{|\mathcal{L}|}(C)$ satisfies Property N_p if Z satisfies the Weak property N_p.*

Therefore theorem A will follow if we prove that Property N_p holds for a generic section of $|\mathcal{L}|$.

For simplicity, from now on, we will identify C with $\varphi_{|\mathcal{L}|}(C) \subset \mathbb{P}^N$ and, similarly, its subcurves with their images. Note that each $C_i \subset W_i$, W_i linear subspace of dimension $N_i := H^0(C_i, \mathcal{L}) - 1$, satisfies Property N_p relative to W_i and furthermore a general section of $|\mathcal{L}_{|C_i}|$ cuts on C_i d_i points for which the uniform position principle holds (relative to W_i).

Proposition 1.3 *Let $\mathcal{L}$ be as in theorem A and let $C \subset \mathbb{P}^N$ be the image of the embedding $\varphi_{|\mathcal{L}|}$.*

Let $s_0 \in |\mathcal{L}|$ be a generic hyperplane section and $H := \{s_0 = 0\} \cong \mathbb{P}^{N-1}$ the corresponding hyperplane.

Then $Z := C \cap H \subset H$ satisfies the Weak Property N_p.

Proof. Let $s_0 \in |\mathcal{L}|$ be a generic hyperplane section, $H := \{s_0 = 0\} \cong \mathbb{P}^{N-1}$ the corresponding hyperplane and $Z := C \cap H \subset H \cong \mathbb{P}^{N-1}$.

We want to apply remark , finding a decomposition $Z = \Sigma \cup \Gamma$ such that

$\deg(\Sigma) = N$ and Σ spans $H \cong \mathbb{P}^{N-1}$.

$\deg(\Gamma) \leq N - p$ and for all $Q \in \Sigma$, for all $\Sigma' \subset \Sigma \setminus \{Q\}$ of degree p there exists an hyperplane H' in $\mathbb{P}^N$ such that $Q \notin H'$, but $\Sigma' \cup \Gamma \subset H'$.

Related to the sequence $C_1 = Y_1 \subset Y_2 \subset \ldots \subset Y_s = C$ of theorem A we have a sequence of linear subspaces $V_1 \subset V_2 \subset \ldots \subset V_s = \mathbb{P}^N$ where $\forall\ i = 1, \ldots, s$ $Y_i \subset V_i \cong H^0(Y_i, \mathcal{L})$.

We construct $\Sigma \subset Z$, inductively, in the following way:

- we start from $\Sigma_1 := \Sigma_{|C_1}$ on C_1 of degree $\deg \Sigma_1 = h^0(C_1, \mathcal{L})$ such that the linear span $\langle \Sigma_1 \rangle = H \cap V_1$ and the points of Σ_1 are in linear general position;
- by induction we may assume $\deg \Sigma_{|Y_i} = h^0(Y_i, \mathcal{L})$ and that $\Sigma_{|Y_i}$ spans $V_i \cap H$; on C_{i+1} we let $\Sigma_{i+1} := \Sigma_{|C_{i+1}}$ of degree $= h^0(C_{i+1}, \mathcal{L} \otimes \mathcal{I}_{Y_i})$ so that $\langle \Sigma_{i+1} \cup \Sigma_{|Y_i} \rangle = V_{i+1}$.

Indeed, by lemma 1.1 we have $= h^1(C_{i+1}, \mathcal{L} \otimes \mathcal{I}_{Y_i}) = 0$, which from one hand means that $Y_i \cap C_{i+1}$ imposes independent conditions to the system $|\mathcal{L}_{|C_{i+1}}|$ and from the other implies the exactness of the following sequence

$$0 \to H^0(C_{i+1}, \mathcal{O}_{C_{i+1}}(\mathcal{L}) \otimes \mathcal{I}_{Y_i}) \to H^0(Y_{i+1}, \mathcal{O}_{Y_{i+1}}(\mathcal{L})) \to H^0(Y_i, \mathcal{O}_{Y_i}(\mathcal{L})) \to 0$$

Thus we can conclude by induction. Furthermore, since each Y_i is connected it is easy to see that for each point $S \in \Sigma$ we have

$$\mathbb{K} \cong H^0(C, \mathcal{L} \otimes \mathcal{I}_{\Sigma \setminus \{S\}}) \hookrightarrow H^0(C, \mathcal{L}) \twoheadrightarrow H^0(\Sigma \setminus \{S\}, \mathcal{L}) \cong \mathcal{O}_{\Sigma \setminus \{S\}}$$

that is, the points of Σ are in linear general position and $\langle \Sigma \rangle = H \cong \mathbb{P}^{N-1}$.

Let $\Gamma = Z \setminus \Sigma$ and $\Gamma_i := \Gamma \cap C_i$.

Since $\deg \mathcal{L}_{|C} \geq 2g(C) + 1 + p$ and $\deg \Sigma = N = h^0(C, \mathcal{L})$ we have $\deg \Gamma \leq N - p$.

It remains to prove that for all $Q \in \Sigma$ and for all $\Sigma' \subset \Sigma \setminus \{Q\}$ of degree p there exists an hyperplane H' in $\mathbb{P}^N$ such that $Q \notin H'$, but $\Sigma' \cup \Gamma \subset H'$.

We will prove it by an induction argument, making use of our numerical conditions.

For $Y_1 = C_1$ the proposition follows by the standard arguments of [G-L] p. 309.

For Y_{i+1} let us consider the decomposition $Y_{i+1} = Y_i \cup C_{i+1}$. We recall that the restriction maps $H^0(Y_{i+1}, \mathcal{L}) \to H^0(C_i, \mathcal{L})$, $H^0(Y_{i+1}, \mathcal{L}) \to H^0(Y_i, \mathcal{L})$ are onto.

If $Q \in Y_i$, by induction hypothesis, for all $\Sigma' \cap Y_i \subset \Sigma \cap Y_i$ of degree $\leq p$ there exists an hyperplane $H'_i \subset \langle Y_i \rangle$ such that $\Sigma' \cap Y_i \subset H'_i$ but $Q \notin H'_i$. Then we simply take H' such that $H' \cap \langle Y_i \rangle = H'_i$ and $\langle \Sigma_{i+1} \rangle \subset H'$. Indeed, let $s'_i \in |\mathcal{L}_{|Y_i}|$ be a section such that $s'_i(Q) \neq 0$, $s'_i(\Sigma' \cap Y_i) = 0$. By our construction of Σ and the surjectivity of the restriction maps then there exists $s' \in |\mathcal{L}|$ such that $s' \mapsto s'_i$ and $s'(\Sigma_{i+1}) = 0$.

If $Q \in C_{i+1}$, $\deg \mathcal{L}_{|C_{i+1}} \geq 2g(C_{i+1}) + Y_i.C_{i+1} + p$ implies

$$\begin{cases} \deg \Sigma_{i+1} = d_{i+1} - g(C_{i+1}) - Y_i.C_{i+1} + 1 \\ \deg \Gamma_{i+1} = g(C_{i+1}) + Y_i.C_{i+1} - 1 \leq \dim(W_{i+1}) - 1 - p \end{cases}$$

This means that on C_{i+1} there exists an hyperplane H''_{i+1} which contains $\Gamma_{i+1} \cup \Sigma'_{i+1}$ but not Q and then we can proceed exactly as in the above case taking H' such that $H' \cap \langle C_{i+1} \rangle = H''_{i+1}$ and $\langle \Sigma_1 \cup \ldots \cup \Sigma_i \rangle \subset H'$.

$\square$

Remark 1.4 *The results in [G-L] and [CFHR] are stated and proved in arbitrary characteristic. In the proof of the above proposition we used* $\mathrm{char}(\mathbb{K}) = 0$ *to ensure that a general hyperplane section of* C_i *(for all* i*) is in linear general position in its linear span.*

Now $\deg(\mathcal{L}_{|C_i}) \geq 2g(C_i) + 1 + p$*. If* $p \geq 1$ *then the general tangent line to* C_i *has order of contact 2 with* C_i *and indeed this holds for every tangent line at a smooth point of* C_i*; if* $p = 0$ *a general tangent line to* C_i *has order of contact at most 3.*

Applying the theory of duality of projective varieties (see e.g. [H-K] for details) we can see that if either $\mathrm{char}(\mathbb{K}) \geq 5$ *or* $\mathrm{char}(\mathbb{K}) = 3$ *and* $p > 0$*, then each* C_i *is reflexive (cf. [H-K] thm. 3.5) and in particular it is not strange (in the sense of [Ha], IV, §3).*

Under these assumptions on $\mathrm{char}(\mathbb{K})$*, then* C_i *is not strange in its span, which implies that a general hyperplane section of* C_i *is in linear general position (cf. [Ra], Lemma 1.1 or Cor. 2.2).*

2 Weak Property N_p for generic projections of curves

In this section we prove theorem B.

The proof will be based on a degeneration argument and on the analysis of the property N_p for the degenerated curve ???, which will turn out to be non reduced and with embedded points.

First we introduce the notion of *planar fat points*

Definition 2.1 *Let* $Z \subset \mathbb{P}^r$*,* $r \geq 2$*, be a 0-dimensional scheme with* $\mathrm{length}(Z) = 3$*,* $P \in \mathbb{P}^r$ *and* M *a plane with* $P \in M \subset \mathbb{P}^r$*. We say that* Z *is a planar fat point supported by* P *and contained in* M *if* $Z_{red} = \{P\}$ *and* $Z \subset M$*, i.e., if* Z *is the first infinitesimal neighborhood of* P *in* M*.*

Proof of Theorem B. Let d be the degree of $\mathcal{L}$ and g be the genus of C, $X \subset \mathbb{P}^n$ a generic projection of $\varphi_{|\mathcal{L}|}(C)$.

If $d \leq 2n - 1 - p$ then by [G-L], theorem 2.1 a general hyperplane section of X satisfies the property N_p, which implies that X satisfies the Weak property N_p.

From now on we may assume $d \geq 2n - p$.

We identify $\mathbb{P}(H^0(C, \mathcal{L})^\vee)$ with $\mathbb{P}^{d-g}$. After this identification the morphism $\varphi_{|\mathcal{L}|}$ depends on the choice of a basis of $H^0(C, \mathcal{L})$. If we change the basis, the new curve will differ by an element of $\mathrm{Aut}(\mathbb{P}^{d-g})$.

As in [B-E 2] we define $Pr(\mathcal{L}, d-g)$

$$Pr(\mathcal{L}, d-g) := \overline{\{f(\varphi_{|\mathcal{L}|}(C)) | f \in \mathrm{Aut}(\mathbb{P}^{d-g})\}} \subset \mathrm{Hilb}(\mathbb{P}^{d-g}).$$

Thus $Pr(\mathcal{L}, d-g)$ is an irreducible closed subset of $\mathrm{Hilb}(\mathbb{P}^{d-g})$ and we will see it with the reduced structure. Hence $Pr(\mathcal{L}, d-g)$ is a complete variety.

$Pr(\mathcal{L}, d-g)$ contains the reducible curves T defined as follows:

> fix an effective divisor $\mathcal{D}$ on C with $\deg(\mathcal{D}) = d-g-n$, say $\mathcal{D} = P_1 + \ldots + P_{d-g-n}$; set $\mathcal{M} := \mathcal{L}(-\mathcal{D})$; since $\deg(\mathcal{M}) \geq 2g+1$, $\mathcal{M}$ is very ample; let $\varphi_{|\mathcal{M}|} : C \hookrightarrow \mathbb{P}^n \cong W \subset \mathbb{P}^{d-g}$ be the complete embedding induced by $\mathcal{M}$; for every integer i with $1 \leq i \leq d-g-n$, let $D_i \subset \mathbb{P}^{d-g}$ be a general line which intersects the curve $\varphi_{|\mathcal{M}|}(C)$ in $\varphi_{|\mathcal{M}|}(P_i)$; set $T := \varphi_{|\mathcal{M}|}(C) \cup D_1 \cup \ldots \cup D_{d-g-n}$.

Indeed, iterating [B-E 1], Prop. I.1 and 2.5, (or use [B-E 2], Th. 0 for a full statement) we can see that $T \in Pr(\mathcal{L}, d-g)$.

Fix $P_1, \ldots, P_{d-g-n}$ general points and let $\mathcal{M}$ be as above. From now on we will take $W = \mathbb{P}(H^0(C, \mathcal{M})^\vee)$ as our ambient space $\mathbb{P}^n$, so that $(C, \mathcal{M}) \cong (\varphi_{|\mathcal{M}|}(C), \mathcal{O}(1))$.

We define $Pr(\mathcal{L}, \mathcal{M})$ to be the closure in $\mathrm{Hilb}(\mathbb{P}^n)$ of the set of all curves obtained from a general projection of a curve in $Pr(\mathcal{L}, d-g)$. Notice that by the irreducibility of $Pr(\mathcal{L}, d-g)$ and of the Grassmannian of (d-g-n-1)-linear subspaces of $\mathbb{P}^{d-g}$, $Pr(\mathcal{L}, \mathcal{M})$ is irreducible.

Thus to prove the theorem it will suffices to show that for a general $B \in Pr(\mathcal{L}, \mathcal{M})$ the Weak Property N_p holds.

To this aim we will apply a degeneration argument to find a curve $A \in Pr(\mathcal{L}, \mathcal{M})$ with embedded points for which the Weak Property N_p holds and then we will simply apply semicontinuity.

Notice that for every curve $A' \in Pr(\mathcal{L}, \mathcal{M})$ (even with embedded points) the curve $\varphi_{|\mathcal{M}|}(C)$ is an irreducible component of A'_{red} and that by semicontinuity $h^0(A', \mathcal{O}_{A'}(1)) \geq d+1-g$.

Construction of A.

Let us take $Q_1, \ldots, Q_{d-g-n} \in C$ more general points and for every integer i with $1 \leq i \leq d-g-n$ let S_i be the tangent line to $\varphi_{|\mathcal{M}|}(C)$ at $\varphi_{|\mathcal{M}|}(Q_i)$, R_i be the line of $\mathbb{P}^n$ spanned by $\varphi_{|\mathcal{M}|}(P_i)$ and $\varphi_{|\mathcal{M}|}(Q_i)$.

First, we fix an integer i with $1 \leq i \leq d-g-n$ and we consider the points P_i and Q_i).

Since $d \geq 2g+1+p+3(d-g-n)$, we have for all i $\deg(\mathcal{M} - P_i - Q_i)) \geq 2g$ and hence $\mathcal{M}(-P_i - Q_i)$ has no base point. Since $n \geq 3$ this means that the line R_i of $\mathbb{P}^n$ spanned by $\varphi_{|\mathcal{M}|}(P_i)$ and $\varphi_{|\mathcal{M}|}(Q_i)$ intersects $\varphi_{|\mathcal{M}|}(C)$ only at $\{\varphi_{|\mathcal{M}|}(P_i), \varphi_{|\mathcal{M}|}(Q_i)\}$ and quasi-transversally (i.e. both the curves are smooth at the two points and have distinct tangents).

Fix an integer i and take a flat family of lines $\{R_i(t)\}_{t \in \Delta}$ (Δ smooth irreducible affine curve) of $\mathbb{P}^n$ with $\varphi_{|\mathcal{M}|}(P_i) \in R_i(t)$ for every t and $R_i(0) = R_i$ for some $0 \in \Delta$. Since $\mathrm{Hilb}(\mathbb{P}^n)$ is complete, the flat family $\{\varphi_{|\mathcal{M}|}(C) \cup R_i(t)\}_{t \in \Delta}$ has a flat limit for t going to 0.

It is easy to check that this flat limit is the union of $\varphi_{|\mathcal{M}|}(C) \cup R_i$ and a certain unreduced scheme χ_i. We have length$(\chi_i) = 3$, $\chi_{red} = \varphi_{|\mathcal{M}|}(Q_i)$ and χ_i is contained in a 3-dimensional linear space, V_i, containing $R_i \cup S_i$ (see [H], III.9.8.4 and Fig. 11 p. 260, for a similar case, or [B-E 1], fig. 2).

$R_i \cup S_i \cup \chi_i$ contains the first infinitesimal neighborhood, ξ_i, of $\varphi_{|\mathcal{M}|}(Q_i)$ in V_i and it is just the scheme-theoretic union $R_i \cup S_i \cup \xi_i$. Moreover since the linear space V_i depends on the flat family of lines we chose, varying this family we may take as V_i a general 3-dimensional linear subspace of $\mathbb{P}^n$ containing $R_i \cup S_i$ (see [B-E 1], fig. 1 and fig. 2).

Notice that for a general hyperplane H of $\mathbb{P}^n$ $\chi_i \cap H$ is a flat planar point contained in $V_i \cap H$ and it has Q_i as associated reduced scheme.

Repeating the above argument for all indices i, $1 \leq i \leq d-g-n$, we obtain a non-reduced curve

$$A := \varphi_{|\mathcal{M}|}(C) \cup R_1 \cup \ldots \cup R_{d-g-n} \cup \chi_1 \cup \ldots \cup \chi_{d-g-n}$$

of degree d and with $d-g-n$ embedded points.

Weak Property N_p for A.

Fix a general hyperplane H of $\mathbb{P}^n$ with $\varphi_{|\mathcal{M}|}(Q_i) \in H$ for every i and set $Z := A \cap H$.

Our aim is to apply Remark C to Z.

$Z \subset H \cong \mathbb{P}^{n-1}$ is a 0-dimensional scheme of length $d+(d-g-n)$ formed by $d-g-n$ planar fat points $\{\gamma_1, \ldots, \gamma_{d-g-n}\}$ and $\delta = d-2(d-g-n)$ reduced points $\{T_1, \ldots, T_\delta\}$.

By our hypothesys on d we have $\delta = d-2(d-g-n) \geq n$. Indeed, since we have assumed $d \geq 2n-p$ we get

$$d-2(d-g-n) \geq 2g+1+p+(d-g-n) \geq 2g+1+p+(2n-p-g-n) = g+1+n > n.$$

Furthermore $\delta \geq n$ implies that we can find a splitting $Z = \Sigma \cup \Gamma$, where Σ consists of n general points and Γ is the union of the $d-g-n$ planar fat points and $d-2(d-g-n)-n$ reduced points. By our choiche of d we have

$$\text{length}(\Gamma) = d+(d-g-n)-n \leq n-p-1$$

Indeed, the above inequality is equivalent to $d \leq 2n-p-1-(d-g-n)$. Writing $n = d-g-(d-g-n)$ this is equivalent to $d \leq 2(d-g)-p-1-3(d-g-n)$ which follows since $d \geq 2g+p+1+3(d-g-n)$ by assumption.

Since the V_i's are general 3-dimensional linear subspace of $\mathbb{P}^n$ containing $\varphi_{|\mathcal{M}|}(Q_i)$ we may assume H to be transversal to each V_i and furthermore the linear span $U := \langle, V_1 \cap H, \ldots, V_{d-g-n} \cap H\rangle \subset H$ is "general" and have maximal dimension.

This means that the span $\langle \gamma_1 \cup \ldots \gamma_{d-g-n} \subset H$ has maximal dimension. Furthermore, taking the projection π_U with center U we can apply the "uniform position principle" to the image of $\varphi_{|\mathcal{M}|}(C)$. This corresponds to say that the linear span of $\langle \gamma_1 \cup \ldots \gamma_{d-g-n}$ and any h points $T_1, \ldots, T_h$ $(h \leq n - 3(d-g-n))$ has maximal dimension.

Thus we may apply Remark C to $Z = \Sigma \cup \Gamma$ as above to show that Z satisfies the Weak Property N_p.

Notice that $\deg(A) = d = \deg(Z) - (d-g-n) = deg(Z) + n + 1 - h^0(C, \mathcal{L})$. Hence we see that every quadric hypersurface of H containing Z lifts to a unique quadric hypersurface of $\mathbb{P}^n$ containing A. Thus A has the Weak Property N_p.

End of the proof of theorem B.

Since $h^0(A, \mathcal{O}_A(t)) = h^0(C, \mathcal{L}^{\otimes t})$ for every $t > 0$, we may apply semicontinuity to conclude that a general $B \in Pr(\mathcal{L}, \mathcal{M})$ has Weak Property N_p. By semicontinuity a general projection of $\varphi_{|\mathcal{L}|}(C)$ into $\mathbb{P}^n$ has Weak Property N_p, proving the theorem.

Q.E.D. for Theorem B

Remark 2.2 *The above theorem holds also for C irreducible curve of arithmetic genus g (possibly singular), since by [CFHR] an invertible sheaf of degree $\geq 2g$ is non-special and base point free and if the degree is $\geq 2g+1$ then it is very ample.*

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Marco Franciosi,
Dipartimento di Matematica Applicata "U. Dini",
Università degli Studi di Pisa
via Bonanno 25, I–56126 Pisa (Italy)
E-mail address: francios@dm.unipi.it

Edoardo Ballico,
Dipartimento di Matematica
Università degli studi di Trento
Povo I-38050 Trento (Italy)
E-mail address: ballico@science.unitn.it