A note on impulse response for continuous, linear, time-invariant, continuous-time systems

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***Abstract*— In his paper "Causality and the impulse response scandal" (IEEE Trans. Circuits Syst., vol. 50, 810–811, 2003), Sandberg proved that, even if a continuous, linear, time-invariant, continuous-time system admits an impulse response, such a response does *not* always give a complete description of the system. In this paper, a Theorem of Schwartz is used to define an impulse response under almost general assumptions, and to understand what we really know about two systems with the same impulse response. These results are applied to a survey of systems (significant by themselves and as leading examples), showing that, apart from three classes of exceptions, all of them are completely described by their impulse response. Concerning the first two classes of exceptions, counterexamples were given by Sandberg; concerning the remaining third class, a counterexample is deduced here from the results of Sandberg.**

***Index Terms*— Continuous-time signals, distributional signals, continuous linear time-invariant systems, impulse response.**

I. Introduction

In SIGNAL PROCESSING theory, a linear, time-invariant (LTI), continuous-time system is a map

$$\mathscr{L} : \mathscr{I} \to \mathscr{O}$$

where: $\mathscr{I}$ (input space) and $\mathscr{O}$ (output space) are linear spaces of signals defined on $\mathbb{R}$, both closed under translation,[1] and $\mathscr{L}$ is a linear map which commutes with translation.[2]

In recent papers [1], [2], [3], Sandberg considered the Banach space $\mathscr{C}$ of bounded uniformly-continuous complex-valued functions defined on $\mathbb{R}$, equipped with the usual sup–norm

$$\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$$

and the class of all *continuous* (with respect to the sup–norm) LTI maps $\mathscr{L} : \mathscr{C} \to \mathscr{C}$ admitting an impulse response, in the sense that there exists a function Δ such that for every sequence f_k of progressively taller and narrower unit-area functions of $\mathscr{C}$, centered at $t = 0$, the sequence $\mathscr{L}(f_k)$ is pointwise convergent on $\mathbb{R}$ to Δ. In this class of maps, Sandberg proved the existence of a non null map (in particular, of a non null *causal* map) with impulse response $\Delta = 0$.

This unexpected result of Sandberg shows in particular that: even if a continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ admits an impulse response, such a response does *not* always give a complete description of $\mathscr{L}$, i.e., there may exist continuous LTI maps $\mathscr{G} : \mathscr{I} \to \mathscr{O}$ different from $\mathscr{L}$ with the same impulse response.

In this paper we consider *continuous* LTI maps and face the two problems arising from the results of Sandberg, namely: how to define an impulse response for a continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ and what we really know about two continuous LTI maps $\mathscr{L}, \mathscr{G} : \mathscr{I} \to \mathscr{O}$ with the same impulse response.

In order to exclude spaces of too generic functions (e.g. the Banach space of all bounded, measurable or not, complex valued functions defined on $\mathbb{R}$) but to allow handling more general signals (e.g. the derivatives, of any order, of bounded measurable complex valued functions defined on $\mathbb{R}$), and to guarantee that the mildest available signals are allowed inputs, we assume:

Assumption 1: $\mathscr{I}$ and $\mathscr{O}$ are subspaces of the set $\mathscr{D}'$ of complex valued distributions on $\mathbb{R}$, both equipped with a notion of convergence and limit (denoted $\mathscr{I}$-lim and $\mathscr{O}$-lim respectively) for sequences, such that if a sequence f_k is convergent in $\mathscr{I}$ (respectively: in $\mathscr{O}$) with limit f, then f_k is also convergent in $\mathscr{D}'$ with the same limit f;

Assumption 2: The space $\mathscr{D}$ of complex valued C^∞ functions defined on $\mathbb{R}$ with compact support, is a subspace of $\mathscr{I}$, and the notion of convergence in $\mathscr{I}$ is such that if a sequence f_k is convergent in $\mathscr{D}$ with limit f, then f_k is also convergent in $\mathscr{I}$ with the same limit f.

Concerning the meaning of continuity, we assume the following

Definition 1: A map $\mathscr{L}$ is said to be *continuous* if it is sequentially continuous, i.e., if for every $f \in \mathscr{I}$ and every sequence $f_k \in \mathscr{I}$ such that $f = \mathscr{I}\text{-}\lim_{k\to\infty} f_k$, it is $\mathscr{L}(f) = \mathscr{O}\text{-}\lim_{k\to\infty} \mathscr{L}(f_k)$.

The preliminary problem of defining an impulse response for every continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ (obviously under Assumptions 1 and 2) is treated in Section II. A Theorem of Schwartz allows us to solve this problem in a way that agrees with the usual impulse response for every $\mathscr{L} : \mathscr{C} \to \mathscr{C}$ admitting one, and with $\mathscr{L}(\delta)$ whenever the Dirac impulse δ is a member of $\mathscr{I}$ and, as usual, there is a sequence $\varphi_k \in \mathscr{D}$ convergent, both in the space $\mathscr{E}'$ of the distributions with compact support and in $\mathscr{I}$, to δ.

The crucial problem of understanding what we really know about two continuous LTI maps $\mathscr{L}, \mathscr{G} : \mathscr{I} \to \mathscr{O}$, with the same impulse response, is treated in Section III. By (transfinite) induction we construct the widest set $\Sigma(\mathscr{D}, \mathscr{I})$

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[1]A space $\mathscr{X}$ is closed under translation when for every $f(t) \in \mathscr{X}$ and $\tau \in \mathbb{R}$, also $f(t-\tau) \in \mathscr{X}$.

[2]That is: for every $f \in \mathscr{I}$ and $\tau \in \mathbb{R}$, defined $y(t) = \mathscr{L}(f(t))$ it is $\mathscr{L}(f(t-\tau)) = y(t-\tau)$.

of members of $\mathscr{I}$ related to $\mathscr{D}$ through limits of sequences, and we prove that $\mathscr{L}(f) = \mathscr{G}(f)$ for at least all $f \in \Sigma(\mathscr{D}, \mathscr{I})$.

As a consequence of this result, whenever $\Sigma(\mathscr{D}, \mathscr{I}) = \mathscr{I}$, any continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is completely described by its impulse response. This result agrees with a recent result of Sandberg concerning a wide and significant class of LTI maps $\mathscr{L} : L^\infty \to L^\infty$ continuous with respect to suitable notions of convergence and limit (see Theorem 1 of [4]).

To better understand the extent of the above result and to give methods to determine $\Sigma(\mathscr{D}, \mathscr{I})$, in Section IV we analyze some (in our opinion particularly significant by themselves and as leading examples) spaces $\mathscr{I}$ of functions or distributions, showing in each case whether it is $\Sigma(\mathscr{D}, \mathscr{I}) = \mathscr{I}$ or not.

To be more precise, we prove that the following input spaces $\mathscr{I}$:

- L^p with $1 \leqslant p < \infty$
- ${\mathscr{D}'}_{L^p}$ (weak convergence) with $1 \leqslant p \leqslant \infty$
- ${\mathscr{D}'}_{L^p}$ (strong convergence) with $1 \leqslant p < \infty$
- $\mathscr{E}'$, $\mathscr{S}'$, $\mathscr{D}'$

(where ${\mathscr{D}'}_{L^p}$ is the natural extension of L^p into $\mathscr{D}'$ and $\mathscr{S}'$ is the space of tempered distributions) all verify $\Sigma(\mathscr{D}, \mathscr{I}) = \mathscr{I}$, so proving that for such $\mathscr{I}$ every continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is completely described by its impulse response.

On the other hand, we prove that the following input spaces $\mathscr{I}$:

$$\mathscr{C},\ L^\infty,\ {\mathscr{D}'}_{L^\infty}\ \text{(strong convergence)}$$

all verify $\Sigma(\mathscr{D}, \mathscr{I}) \neq \mathscr{I}$. As a consequence, when $\mathscr{I}$ is one of these last spaces, there may exist different continuous LTI maps $\mathscr{L}, \mathscr{G} : \mathscr{I} \to \mathscr{O}$ with the same impulse response.

Concerning $\mathscr{C}$, the result of Sandberg proves that there really exist different causal continuous LTI maps $\mathscr{L}, \mathscr{G} : \mathscr{C} \to \mathscr{C}$ with the same impulse response. Concerning L^∞ an analogous result has been proved by Sandberg in [4]. In Section V, as corollaries of the result of Sandberg on $\mathscr{C}$, we give a new proof that there exist different causal continuous LTI maps $\mathscr{L}, \mathscr{G} : L^\infty \to L^\infty$ with the same impulse response, and we prove that, whenever ${\mathscr{D}'}_{L^\infty}$ is considered with the strong convergence, there exist different causal continuous LTI maps $\mathscr{L}, \mathscr{G} : {\mathscr{D}'}_{L^\infty} \to {\mathscr{D}'}_{L^\infty}$ with the same impulse response.

II. Impulse response

In this section we recall some basic notions on distributions then, applying a theorem of Schwartz, we give a definition of distributional impulse response and we show its first relation with convolution. Finally we analyze the relation of this impulse response with the one adopted by Sandberg for continuous LTI maps $\mathscr{L} : \mathscr{C} \to \mathscr{C}$ and with the response to the Dirac impulse.

Let $\mathscr{D}$ be the linear space of C^∞ complex-valued functions defined on $\mathbb{R}$ with compact support. Let φ_k be a sequence of members of $\mathscr{D}$, and let $\varphi \in \mathscr{D}$; if there is a compact subset K of $\mathbb{R}$ such that $\operatorname{supp} \varphi_k \subseteq K$ for every k, and moreover for every $h \in \mathbb{N}$ the sequence $D^h \varphi_k$ converges to $D^h \varphi$ uniformly on $\mathbb{R}$, then we write $\mathscr{D}\text{-}\lim_{k\to\infty} \varphi_k = \varphi$.

Let $\mathscr{D}'$ be the linear space of distributions on $\mathbb{R}$, i.e., the space of the linear and continuous functionals from $\mathscr{D}$ into $\mathbb{C}$. As usual, for every $f \in \mathscr{D}'$ and every $\varphi \in \mathscr{D}$, the complex number $f(\varphi)$ is denoted by $\langle f, \varphi \rangle$, and, whenever f is a locally integrable function, it is

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(t)\varphi(t)\mathrm{d}t$$

Let f_k be a sequence of members of $\mathscr{D}'$, and let $f \in \mathscr{D}'$; if for every $\varphi \in \mathscr{D}$ it is

$$\lim_{k\to\infty} \langle f_k, \varphi \rangle = \langle f, \varphi \rangle$$

then we write $\mathscr{D}'\text{-}\lim_{k\to\infty} f_k = f$.

In this Section we need also the space $\mathscr{E}'$ of distributions with compact support. It is well known that $\mathscr{E}'$ can be introduced via a duality pairing as a space of functionals as follows (see Theorem XXV of Chapter 3 of [5]).

Let $\mathscr{E}$ be the linear space of C^∞ complex-valued functions defined on $\mathbb{R}$. Let φ_k be a sequence of members of $\mathscr{E}$, and let $\varphi \in \mathscr{E}$; if for every $h \in \mathbb{N}$ the sequence $D^h \varphi_k$ converges to $D^h \varphi$ uniformly on every compact subset of $\mathbb{R}$, then we write $\mathscr{E}\text{-}\lim_{k\to\infty} \varphi_k = \varphi$.

$\mathscr{E}'$ is the set of the linear and continuous functionals from $\mathscr{E}$ into $\mathbb{C}$. As usual, for every $f \in \mathscr{E}'$ and every $\varphi \in \mathscr{E}$, the complex number $f(\varphi)$ is denoted by $\langle f, \varphi \rangle$, and, whenever f is an integrable function with compact support, it is

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(t)\varphi(t)\mathrm{d}t$$

Let f_k be a sequence of members of $\mathscr{E}'$, and let $f \in \mathscr{E}'$; if for every $\varphi \in \mathscr{E}$ it is

$$\lim_{k\to\infty} \langle f_k, \varphi \rangle = \langle f, \varphi \rangle$$

then we write $\mathscr{E}'\text{-}\lim_{k\to\infty} f_k = f$; moreover it is well known that this condition is verified if and only if there exists a compact subset K of $\mathbb{R}$ such that $\operatorname{supp} f_k \subseteq K$ for every k, and $\mathscr{D}'\text{-}\lim_{k\to\infty} f_k = f$.

In order to handle linear changes of variables for distributions, we agree to denote an element $f \in \mathscr{D}'$ by a function-like symbol $f(t)$, so that the name "t" of the current variable is pointed out. In this way, for every $\lambda, a \in \mathbb{R}$ such that $\lambda \neq 0$, we denote by $f(\lambda t + a) = f(a + \lambda t)$ the distribution defined by

$$\langle f(\lambda t + a), \varphi(t) \rangle = |\lambda|^{-1} \left\langle f(t), \varphi\left(\lambda^{-1}(t - a)\right) \right\rangle$$

for every $\varphi \in \mathscr{D}$. In particular, for $\lambda = 1, a = -\tau$, we obtain $f(t - \tau)$ defined by

$$\langle f(t - \tau), \varphi(t) \rangle = \langle f(t), \varphi(t + \tau) \rangle$$

and, for $\lambda = -1, a = \tau$, we obtain $f(\tau - t)$ defined by

$$\langle f(\tau - t), \varphi(t) \rangle = \langle f(t), \varphi(\tau - t) \rangle$$

For every $f(t) \in \mathscr{D}', \varphi(t) \in \mathscr{D}$, the convolution $f * \varphi$ is the C^∞ function defined, for every $t \in \mathbb{R}$, by

$$(f * \varphi)(t) = \langle f(\tau), \varphi(t - \tau) \rangle = \langle f(t - \tau), \varphi(\tau) \rangle$$

(see Theorem XI of Chapter 6 of [5]). Observe that, whenever $f(t)$ is a locally integrable function, this definition agrees with the usual definition

$$(f * \varphi)(t) = \int_{-\infty}^{+\infty} f(\tau)\varphi(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{+\infty} f(t-\tau)\varphi(\tau)\mathrm{d}\tau$$

Concerning continuous LTI maps, the remark following Theorem XXIII of Chapter 6 of [5] can be rewritten as:

Theorem 2.1 (Schwartz): Let $\mathscr{L} : \mathscr{D} \to \mathscr{D}'$ be a continuous LTI map. Then the following statements hold:

a) for every sequence $\varphi_k \in \mathscr{D}$ with $\mathscr{E}'\text{-}\lim_{k\to\infty} \varphi_k = \delta$, the sequence $\mathscr{L}(\varphi_k)$ is convergent in $\mathscr{D}'$, and $\Delta = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}(\varphi_k)$ is independent of the particular sequence φ_k;
b) for every $\varphi \in \mathscr{D}$ it is $\mathscr{L}(\varphi) = \Delta * \varphi$.

This theorem allows us to define an impulse response for every continuous LTI map (obviously under Assumptions 1 and 2). *Indeed*: let

$$\mathscr{L} : \mathscr{I} \to \mathscr{O}$$

be a continuous LTI map.Then the map

$$\widetilde{\mathscr{L}} : \mathscr{D} \to \mathscr{D}'$$

defined by $\widetilde{\mathscr{L}}(\varphi) = \mathscr{L}(\varphi)$ for every $\varphi \in \mathscr{D}$, is a continuous LTI map. By Theorem 2.1 applied to $\widetilde{\mathscr{L}}$ we obtain that:

a) there exists a unique $\Delta \in \mathscr{D}'$ such that, for every sequence $\varphi_k \in \mathscr{D}$ with $\mathscr{E}'\text{-}\lim_{k\to\infty} \varphi_k = \delta$, it is $\Delta = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}(\varphi_k)$

and that

b) for every $\varphi \in \mathscr{D}$ it is $\mathscr{L}(\varphi) = \Delta * \varphi$

The distribution Δ will be called the *impulse response* of $\mathscr{L}$.

The following theorems relate the above defined impulse response to the one adopted by Sandberg for continuous LTI maps $\mathscr{L} : \mathscr{C} \to \mathscr{C}$ and to the response to the Dirac impulse.

Let $\mathscr{I} = \mathscr{O} = \mathscr{C}$. Since Assumptions 1 and 2 are verified, every continuous LTI map $\mathscr{L} : \mathscr{C} \to \mathscr{C}$ has an impulse response $\Delta \in \mathscr{D}'$. As proven by the identity map, it may be that $\mathscr{L}$ does not have an impulse response in the usual sense. The following theorem proves that, whenever $\mathscr{L}$ has an impulse response in the usual sense, here denoted by Δ_{us}, then it is $\Delta_{us} = \Delta$.

Theorem 2.2: Let $\mathscr{L} : \mathscr{C} \to \mathscr{C}$ be a continuous LTI map, admitting an impulse response Δ_{us} in the usual sense, and let $f_k \in \mathscr{C}$ be a sequence such that for every k it is:

$$f_k(t) \geqslant 0 \text{ for every } t \in \mathbb{R}$$
$$\operatorname{supp} f_k \subseteq [-1/k, 1/k]$$
$$\int_{-\infty}^{+\infty} f_k(t)\mathrm{d}t = 1$$

The following statements hold:

a) the sequence $\mathscr{L}(f_k)$ converges to Δ_{us} uniformly on every compact subset of $\mathbb{R}$;
b) Δ_{us} is a continuous (not necessarily bounded and uniformly continuous) function;
c) $\Delta_{us} = \Delta$.

Proof: For $k = 1, 2, \ldots$, let T_k be the set of all $f \in \mathscr{C}$ such that

$$f(t) \geqslant 0 \text{ for every } t \in \mathbb{R}$$
$$\operatorname{supp} f \subseteq [-1/k, 1/k]$$
$$\int_{-\infty}^{+\infty} f(t)\mathrm{d}t = 1$$

Obviously it is $T_k \supseteq T_{k+1}$.

For every $k = 1, 2, \ldots$, and every $\tau \in \mathbb{R}$, let $D_k(\tau)$ be the diameter of

$$\left\{ (\mathscr{L}f)(\tau) : f \in T_k \right\} \subseteq \mathbb{C}$$

Obviously it is $D_k(\tau) \geqslant D_{k+1}(\tau)$.

Observe that for every $\tau \in \mathbb{R}$ it is $\lim_{k\to\infty} D_k(\tau) = 0$. *Otherwise*, there would exist $\rho > 0$ and positive integers $k_1 < k_2 < \cdots$ such that every $D_{k_j}(\tau) \geqslant 2\rho$. Hence there would exist $f_{11}, f_{12}, \ldots$ and $f_{21}, f_{22}, \ldots$ such that, for every j, it would be $f_{1j}, f_{2j} \in T_{k_j}, |(\mathscr{L}f_{1j})(\tau) - (\mathscr{L}f_{2j})(\tau)| > \rho$. *This is absurd*, since it is $\lim_{j\to\infty} (\mathscr{L}f_{1j})(\tau) = \Delta_{us}(\tau)$, $\lim_{j\to\infty} (\mathscr{L}f_{2j})(\tau) = \Delta_{us}(\tau)$.

As a consequence of the time invariance of $\mathscr{L}$, it is easily proven that, for every $k = 1, 2, \ldots$, every $\eta \in \mathbb{R}$ such that $|\eta| < 1/k - 1/(k+1) = 1/(k(k+1))$, and every $\tau \in \mathbb{R}$, it is $\left\{ (\mathscr{L}f)(\tau) : f \in T_{k+1} \right\} \subseteq \left\{ (\mathscr{L}g)(\tau+\eta) : g \in T_k \right\}$ and hence it is $D_{k+1}(\tau) \leqslant D_k(\tau+\eta)$.

Let $2d_k = 1/(k(k+1))$; the previous result proves that, for every k, and every τ, it is

$$D_{k+1}(\tau) \leqslant \inf_{t\in(\tau-2d_k, \tau+2d_k)} D_k(t)$$

as a consequence

$$\sup_{t\in(\tau-d_k, \tau+d_k)} D_{k+1}(t) \leqslant \inf_{t\in(\tau-d_k, \tau+d_k)} D_k(t)$$

Let K be a compact subset of $\mathbb{R}$, and let $\varepsilon > 0$. For every $\tau \in K$, there exists k_τ such that $D_{k_\tau}(\tau) < \varepsilon$; hence, for every $t \in (\tau - d_{k_\tau}, \tau + d_{k_\tau})$ it is $D_{k_\tau+1}(t) < \varepsilon$. An usual trick, depending on the compactness of K, proves that there exists k_K such that $\sup_{t\in K} D_{k_K}(t) < \varepsilon$.

Let f_k be as in the text of the theorem. Let K be a compact subset of $\mathbb{R}$, and let $\varepsilon > 0$. Since every $f_k \in T_k$, then for every $k_1, k_2 \geqslant k_K$ and every $\tau \in K$ it is

$$|(\mathscr{L}f_{k_2})(\tau) - (\mathscr{L}f_{k_1})(\tau)| \leqslant D_{k_K}(\tau) < \varepsilon$$

This result proves that the sequence $\mathscr{L}(f_k)$ is uniformly convergent on K, and hence proves statements a) and b).

Concerning c), let f_k be as above, but with the ulterior condition that every $f_k \in \mathscr{D}$; observe that $\mathscr{E}'\text{-}\lim_{k\to\infty} f_k = \delta$. By the definition of impulse response, it is $\Delta = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}(f_k)$. Since Δ_{us} is continuous, it is $\Delta_{us} \in \mathscr{D}'$. By a) it is also $\Delta_{us} = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}(f_k)$. Hence $\Delta = \Delta_{us}$. ∎

Let the Dirac impulse δ be an allowed input (i.e., $\delta \in \mathscr{I}$), let $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ be a continuous LTI map, and let Δ be its impulse response. Then we may expect the impulse response of the system being the response of the system to the Dirac impulse δ, i.e., we may expect that $\Delta = \mathscr{L}(\delta)$.

The following unusual example proves that it may be $\Delta \neq \mathscr{L}(\delta)$. Let $\mathscr{K}$ be the subspace of $\mathscr{D}'$ spanned by L^∞ and by the family $\delta(t-\tau)$, with $\tau \in \mathbb{R}$, of the translated of $\delta(t)$. It is easily seen that every $g(t) \in \mathscr{K}$ may be uniquely written, apart from zero summands, in the form

$$g(t) = f(t) + \sum_{h=1}^{\nu} c_h \delta\left(t - \tau_h\right)$$

with $f(t) \in L^\infty$, $\nu \in \mathbb{N}$, every $c_h \in \mathbb{C}$ and every $\tau_h \in \mathbb{R}$. A sequence $g_k(t) \in \mathscr{K}$ will be called convergent in $\mathscr{K}$ if there exist $\nu, \tau_1, \ldots, \tau_\nu$ such that every

$$g_k(t) = f_k(t) + \sum_{h=1}^{\nu} c_{kh} \delta\left(t - \tau_h\right)$$

with the sequence $f_k(t)$ convergent in L^∞, and the ν sequences $c_{k1}, \ldots, c_{k\nu}$ convergent in $\mathbb{C}$; in that case the $\mathscr{K}$-limit of the sequence $g_k(t)$ is defined by

$$\mathscr{K}\text{-}\lim_{k\to\infty} g_k(t) = L^\infty\text{-}\lim_{k\to\infty} f_k(t) + \sum_{h=1}^{\nu}\left(\lim_{k\to\infty} c_{kh}\right)\delta\left(t-\tau_h\right)$$

Let $d \in \mathbb{C}$, and let $\mathscr{L} : \mathscr{K} \to \mathscr{K}$ be the continuous LTI causal map defined by

$$\mathscr{L}\left(f(t) + \sum_{h=1}^{\nu} c_h \delta\left(t-\tau_h\right)\right) = f(t) + \sum_{h=1}^{\nu} c_h d \delta\left(t-\tau_h\right)$$

It is easily seen that

a) the impulse response of $\mathscr{L}$ is $\Delta = \delta$,
b) the response of $\mathscr{L}$ to δ is $\mathscr{L}(\delta) = d\delta$;

so, choosing $d = 1$ it is $\mathscr{L}(\delta) = \delta = \Delta$ but, choosing for instance $d = 0$ it is $\mathscr{L}(\delta) = 0 \neq \Delta$.

The following theorem gives a usually verified sufficient condition on $\mathscr{I}$ in order that $\Delta = \mathscr{L}(\delta)$.

Theorem 2.3: Let $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ be a continuous LTI map, and let Δ be the impulse response of $\mathscr{L}$. If $\delta \in \mathscr{I}$, and there exists a sequence $\varphi_k \in \mathscr{D}$ such that $\mathscr{E}'\text{-}\lim_{k\to\infty} \varphi_k = \delta$ and $\mathscr{I}\text{-}\lim_{k\to\infty} \varphi_k = \delta$, then it is $\mathscr{L}(\delta) = \Delta$.

Proof: Let $\varphi_k \in \mathscr{D}$ be a sequence such that $\mathscr{E}'\text{-}\lim_{k\to\infty} \varphi_k = \delta$ and $\mathscr{I}\text{-}\lim_{k\to\infty} \varphi_k = \delta$. By definition it is $\Delta = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}\left(\varphi_k\right)$. By assumption it is $\mathscr{I}\text{-}\lim_{k\to\infty} \varphi_k = \delta$; hence by continuity it is $\mathscr{O}\text{-}\lim_{k\to\infty} \mathscr{L}\left(\varphi_k\right) = \mathscr{L}(\delta)$. Since, by Assumption 1, $\mathscr{O}\text{-}\lim_{k\to\infty} \mathscr{L}\left(\varphi_k\right) = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}\left(\varphi_k\right)$, then it is $\mathscr{L}(\delta) = \Delta$. ■

III. Continuous LTI maps with the same impulse response. Sequential closure

Let $\mathscr{I}$ be an input space. In this section we find a sufficient condition on $\mathscr{I}$ in order that every continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is completely described by its impulse response.

To this aim, for every finite or transfinite ordinal i, with $1 \leqslant i$, let $\Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$ be the subset of $\mathscr{I}$ inductively defined by

a) $\Sigma_1\left(\mathscr{D}, \mathscr{I}\right)$ is the set of the $f \in \mathscr{I}$ such that there exists a sequence $\varphi_k \in \mathscr{D}$ with $f = \mathscr{I}\text{-}\lim_{k\to\infty} \varphi_k$
b) for $1 < i$, $\Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$ is the set of the $f \in \mathscr{I}$ such that there exists a sequence

$$f_k \in \bigcup_{1 \leq j < i} \Sigma_j\left(\mathscr{D}, \mathscr{I}\right)$$

with $f = \mathscr{I}\text{-}\lim_{k\to\infty} f_k$

It is immediately seen that

i) every $\Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$ is a subspace of $\mathscr{I}$
ii) for every $j < i$ it is $\Sigma_j\left(\mathscr{D}, \mathscr{I}\right) \subseteq \Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$
iii) if, for $j < i$ it is $\Sigma_j\left(\mathscr{D}, \mathscr{I}\right) = \Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$, then, for every h such that $j < h$ it is $\Sigma_j\left(\mathscr{D}, \mathscr{I}\right) = \Sigma_h\left(\mathscr{D}, \mathscr{I}\right)$

As a consequence, since the cardinality of every $\Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$ is bounded by the cardinality of $\mathscr{I}$, there exists a unique finite or transfinite ordinal $\omega \geqslant 1$ such that

iv) for every $1 \leqslant j < i \leqslant \omega$, it is $\Sigma_j\left(\mathscr{D}, \mathscr{I}\right) \subsetneqq \Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$
v) for every $\omega < h$, it is $\Sigma_\omega\left(\mathscr{D}, \mathscr{I}\right) = \Sigma_h\left(\mathscr{D}, \mathscr{I}\right)$

The space $\Sigma_\omega\left(\mathscr{D}, \mathscr{I}\right)$ will be denoted by $\Sigma\left(\mathscr{D}, \mathscr{I}\right)$ and will be called the *sequential closure* of $\mathscr{D}$ in $\mathscr{I}$.

The significance of $\Sigma\left(\mathscr{D}, \mathscr{I}\right)$ rests on the following two results.

Theorem 3.1: Let $\mathscr{L}, \mathscr{G} : \mathscr{I} \to \mathscr{O}$ be two continuous LTI maps with the same impulse response Δ. Then, for every $f \in \Sigma\left(\mathscr{D}, \mathscr{I}\right)$, it is $\mathscr{L}(f) = \mathscr{G}(f)$.

Proof: By transfinite induction on $1 \leqslant i$, we prove that for every $f \in \Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$ it is $\mathscr{L}(f) = \mathscr{G}(f)$.

Let $i = 1$, and let $f \in \Sigma_1\left(\mathscr{D}, \mathscr{I}\right)$. Since there exists a sequence $\varphi_k \in \mathscr{D}$ such that $f = \mathscr{I}\text{-}\lim_{k\to\infty} \varphi_k$, then by Theorem 2.1 it is

$$\mathscr{L}(f) = \mathscr{O}\text{-}\lim_{k\to\infty} \mathscr{L}\left(\varphi_k\right) = \mathscr{O}\text{-}\lim_{k\to\infty} \Delta * \varphi_k$$
$$\mathscr{G}(f) = \mathscr{O}\text{-}\lim_{k\to\infty} \mathscr{G}\left(\varphi_k\right) = \mathscr{O}\text{-}\lim_{k\to\infty} \Delta * \varphi_k$$

Hence $\mathscr{L}(f) = \mathscr{G}(f)$.

Let $1 < i$, and assume that the inductive statement holds for every $j < i$. Let $f \in \Sigma_i\left(\mathscr{D}, \mathscr{I}\right)$. By definition, there exists a sequence $f_1, f_2, \ldots$ such that every $f_k \in \Sigma_{j_k}\left(\mathscr{D}, \mathscr{I}\right)$, with a suitable $j_k < i$, and that $f = \mathscr{I}\text{-}\lim_{k\to\infty} f_k$; then it is

$$\mathscr{L}(f) = \mathscr{O}\text{-}\lim_{k\to\infty} \mathscr{L}\left(f_k\right) \quad , \quad \mathscr{G}(f) = \mathscr{O}\text{-}\lim_{k\to\infty} \mathscr{G}\left(f_k\right)$$

Since $f_k \in \Sigma_{j_k}\left(\mathscr{D}, \mathscr{I}\right)$ and $j_k < i$, by the inductive assumption it is $\mathscr{L}(f_k) = \mathscr{G}(f_k)$.

Hence $\mathscr{L}(f) = \mathscr{G}(f)$. ■

Theorem 3.2: Let $\Sigma\left(\mathscr{D}, \mathscr{I}\right) = \mathscr{I}$. Let $\mathscr{L}, \mathscr{G} : \mathscr{I} \to \mathscr{O}$ be two continuous LTI maps with the same impulse response Δ. Then it is $\mathscr{L} = \mathscr{G}$.

Proof: The statement is a straightforward consequence of Theorem 3.1. ■

The above theorem proves that, whenever $\Sigma\left(\mathscr{D}, \mathscr{I}\right) = \mathscr{I}$, every continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is completely described by its impulse response.

This result agrees with a recent one obtained by Sandberg (see Theorem 1 of [4]) concerning a wide and significant class of continuous LTI maps $\mathscr{L} : \mathscr{I} \to \mathscr{O}$, where:

- $\mathscr{I}$ is L^∞ equipped with the following notion of convergence and limit for sequences: $f = \mathscr{I}\text{-}\lim_{k\to\infty} f_k$ if either $f \in L^\infty \cap L^1$, $f_k \in L^\infty \cap L^1$ and $f = L^1\text{-}\lim_{k\to\infty} f_k$, or

$f \in L^\infty$ and $f_k = f w_k$, where w_k is the characteristic function of the interval $[-k, k]$;

- $\mathscr{O}$ is again L^∞ but equipped with the following notion of convergence and limit for sequences: $f = \mathscr{O}\text{-}\lim_{k\to\infty} f_k$ if $f = L^1_{\mathrm{loc}}\text{-}\lim_{k\to\infty} f_k$.

Indeed we have:

a) $\mathscr{I}$ and $\mathscr{O}$ verify Assumptions 1 and 2, hence every continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ admits an impulse response Δ (in $\mathscr{D}'$);

b) whenever an $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is in the class considered by Sandberg and moreover admits an impulse response $h \in L^1$ in the sense specified in (a) of Theorem 1 of [4], then $h = \Delta$;

c) since for all $f \in \mathscr{I}$ it is $f = \mathscr{I}\text{-}\lim_{k\to\infty} f w_k$ and for all $f w_k$ there exists a sequence $\varphi_{kj} \in \mathscr{D}$ such that $f w_k = \mathscr{I}\text{-}\lim_{j\to\infty} \varphi_{kj}$ then it is

$$\Sigma(\mathscr{D}, \mathscr{I}) = \Sigma_2(\mathscr{D}, \mathscr{I}) = \mathscr{I}$$

hence every continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is completely described by its impulse response; in particular, given any sequence $\varphi_k \in \mathscr{D}$ with $\mathscr{E}'\text{-}\lim_{k\to\infty} \varphi_k = \delta$, since $\Delta = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}(\varphi_k)$, then the behavior of the sequence $\mathscr{L}(\varphi_k)$ uniquely determines the behavior of $\mathscr{L}$;

d) whenever an $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is in the class considered by Sandberg, the representation of $\mathscr{L}(f)$ for every $f \in \mathscr{I}$ given in (8) of Theorem 1 of [4] agrees with the above remark c) and makes apparent how the behavior of $\mathscr{L}$ on particular sequences converging to δ in $\mathscr{E}'$, uniquely determines the behavior of $\mathscr{L}$ on the whole input space $\mathscr{I}$.

IV. A SURVEY OF SEQUENTIAL CLOSURES AND CONTINUOUS LTI MAPS COMPLETELY DESCRIBED BY THEIR IMPULSE RESPONSE

In this section, in order to apply Theorem 3.2, we test the condition $\Sigma(\mathscr{D}, \mathscr{I}) = \mathscr{I}$ where $\mathscr{I}$ is one of the following spaces: $\mathscr{C}, L^p, \mathscr{E}', \mathscr{S}', \mathscr{D}', \mathscr{D}'_{L^p}$. In our opinion these spaces are particularly significant both as spaces of signals and as leading examples to determine $\Sigma(\mathscr{D}, \mathscr{I})$ for other possible choices of $\mathscr{I}$.

Concerning $\mathscr{C}$, since it is a Banach space, we have

$$\Sigma(\mathscr{D}, \mathscr{C}) = \Sigma_1(\mathscr{D}, \mathscr{C})$$

Obviously, $\Sigma(\mathscr{D}, \mathscr{C}) \subseteq \mathscr{C}_0$, where $\mathscr{C}_0$ is the space of the $f \in \mathscr{C}$ such that $\lim_{|t|\to\infty} f(t) = 0$; moreover, it is well known that every $f \in \mathscr{C}_0$ is the limit, with respect to the sup norm, of a sequence of members of $\mathscr{D}$. Hence it is $\Sigma(\mathscr{D}, \mathscr{C}) = \mathscr{C}_0$ and then

$$\Sigma(\mathscr{D}, \mathscr{C}) = \mathscr{C}_0 \neq \mathscr{C}$$

Concerning the spaces L^p we distinguish between the case $1 \leqslant p < \infty$ and the case $p = \infty$.

Let $1 \leqslant p < \infty$, and let L^p be the usual Banach space with the norm defined by

$$\|f\|_p = \left(\int_{-\infty}^{+\infty} |f(t)|^p \mathrm{d}t \right)^{1/p}$$

It is well known that every $f \in L^p$ is the limit of a sequence of members of $\mathscr{D}$, which implies as a direct consequence that $\Sigma_1(\mathscr{D}, L^p) = L^p$, hence

$$\Sigma(\mathscr{D}, L^p) = L^p$$

Let $p = \infty$, and let L^∞ be the usual Banach space with the norm defined by

$$\|f\|_\infty = \text{essential} \sup_{t\in\mathbb{R}} |f(t)|$$

The argument adopted for $\mathscr{C}$ prove that

$$\Sigma(\mathscr{D}, L^\infty) = \Sigma_1(\mathscr{D}, L^\infty) = \mathscr{C}_0 \neq L^\infty$$

Concerning the spaces of distributions $\mathscr{D}'$, $\mathscr{E}'$, $\mathscr{S}'$, by the proof of Theorem 1.20, by Corollary 1.5 and by the proof of Theorem 1.31 of [6] it is $\Sigma_2(\mathscr{D}, \mathscr{D}') = \mathscr{D}'$, $\Sigma_2(\mathscr{D}, \mathscr{E}') = \mathscr{E}'$ and $\Sigma_2(\mathscr{D}, \mathscr{S}') = \mathscr{S}'$; hence

$$\Sigma(\mathscr{D}, \mathscr{D}') = \mathscr{D}' \ , \ \Sigma(\mathscr{D}, \mathscr{E}') = \mathscr{E}' \ , \ \Sigma(\mathscr{D}, \mathscr{S}') = \mathscr{S}'$$

Last, we consider the $\mathscr{D}'_{L^p}$ spaces, which we are going to illustrate.

For each $1 \leqslant p \leqslant \infty$ the space $\mathscr{D}'_{L^p}$ is the natural extension of L^p in the space $\mathscr{D}'$ of distributions, namely the subspace of $\mathscr{D}'$ spanned by L^p itself and by the distributional derivatives (of any order) of its elements. Despite their easy definition and the obvious reasons of their introduction—e.g. an L^p voltage across a capacitor results in a $\mathscr{D}'_{L^p}$ current—, in these spaces plays a fundamental role a phenomenon which is not perceived working with $\mathscr{D}'$, $\mathscr{E}'$, $\mathscr{S}'$.

As $\mathscr{D}', \mathscr{E}'$ and $\mathscr{S}'$ (for this last, see Section 4 of Chapter 7 of [5]), the $\mathscr{D}'_{L^p}$ spaces can be introduced via a duality pairing, as spaces of functionals, as follows (see Theorem XXV, Chapter 6 of [5]).

Let $\mathscr{D}_{L^p}$ be the space of all C^∞ complex-valued functions φ defined on $\mathbb{R}$, such that $D^h\varphi \in L^p$ for any $h \in \mathbb{N}$. Given a sequence φ_k of members of $\mathscr{D}_{L^p}$, and $\varphi \in \mathscr{D}_{L^p}$ we will write $\mathscr{D}_{L^p}\text{-}\lim_{k\to\infty} \varphi_k = \varphi$ if for every $h \in \mathbb{N}$, the sequence $D^h\varphi_k$ converges to $D^h\varphi$ in L^p. For $p = \infty$, $\dot{\mathscr{D}}_{L^\infty}$ denotes the subspace of $\mathscr{D}_{L^\infty}$, whose elements are the φ such that $\lim_{|t|\to\infty} D^h\varphi(t) = 0$ for every $h \in \mathbb{N}$, equipped with a similar notion of convergence and $\dot{\mathscr{D}}_{L^\infty}\text{-}\lim$ for sequences.

For $1 < p \leqslant \infty$, $\mathscr{D}'_{L^p}$ is the space of linear and continuous functional f from $\mathscr{D}_{L^{p'}}$ into $\mathbb{C}$, where p' is defined by $1/p' + 1/p = 1$. As usual, for every $f \in \mathscr{D}'_{L^p}, \varphi \in \mathscr{D}_{L^{p'}}$, the complex number $f(\varphi)$ is denoted by $\langle f, \varphi \rangle$, and whenever $f \in L^p$ it is

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(t)\varphi(t)\mathrm{d}t$$

For $p = 1$, $\mathscr{D}'_{L^1}$ is the space of linear and continuous functionals from $\dot{\mathscr{D}}_{L^\infty}$ into $\mathbb{C}$.

Working with spaces of distributions introduced via a duality pairing, two notions of convergence need to be considered: a strong convergence and a weak one. As far as we are concerned with $\mathscr{D}', \mathscr{E}'$ and $\mathscr{S}'$ and only sequences are taken into account, there is no distinction: a sequence is strongly convergent if and only if it is weakly convergent (see Theorem XIII of Chapter 3, Section 7 of Chapter 3 and Section 4 of Chapter 7 of [5]). But this is no longer true for $\mathscr{D}'_{L^p}$ spaces (see Section 8 of Chapter 6 of [5]). Thus we first introduce the notion of bounded set in $\mathscr{D}_{L^{p'}}$ with $1 \leqslant p' < \infty$ and in $\dot{\mathscr{D}}_{L^\infty}$, and then we explain *strong* and *weak* convergence for sequences in $\mathscr{D}'_{L^p}$.

Let $1 \leqslant p' < \infty$, and let $B \subseteq \mathscr{D}_{L^{p'}}$; B is called a bounded subset of $\mathscr{D}_{L^{p'}}$ if there exist positive real numbers $M_0, M_1, \ldots$ such that, for every $h \in \mathbb{N}$, it is

$$\sup\left\{\|D^h\varphi\|_{p'} : \varphi \in B\right\} \leqslant M_h$$

Bounded subsets of $\dot{\mathscr{D}}_{L^\infty}$ have a similar definition.

Let $1 < p \leq \infty$. Let f_k be a sequence of members of $\mathscr{D}'_{L^p}$, and let $f \in \mathscr{D}'_{L^p}$. If, for every bounded subset B of $\mathscr{D}_{L^{p'}}$, it is

$$\lim_{k\to\infty}\langle f_k, \varphi\rangle = \langle f, \varphi\rangle$$

uniformly with respect to $\varphi \in B$, then we say that the sequence f_k *strongly converges* to f, and we write

$$s\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty} f_k = f$$

If, for every $\varphi \in \mathscr{D}_{L^{p'}}$, it is

$$\lim_{k\to\infty}\langle f_k, \varphi\rangle = \langle f, \varphi\rangle$$

then we say that the sequence f_k *weakly converges* to f, and write

$$w\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty} f_k = f$$

Strong and weak convergence for sequences in $\mathscr{D}'_{L^1}$ have similar definitions by using bounded subsets in $\dot{\mathscr{D}}_{L^\infty}$ and $\varphi \in \dot{\mathscr{D}}_{L^\infty}$.

The sequential closure of $\mathscr{D}$ in $\mathscr{D}'_{L^p}$ with respect to the strong and the weak convergence will be denoted respectively by

$$\Sigma\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^p}\right) \quad , \quad \Sigma\left(\mathscr{D}, w\text{-}\mathscr{D}'_{L^p}\right)$$

In order to analyze the above sets, we need some lemmas.

Lemma 4.1: Let $1 \leqslant p \leqslant \infty$. For every $f \in \mathscr{D}'_{L^p}$ there exists a sequence $\varphi_k \in \mathscr{D}_{L^p}$ such that $f = s\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty}\varphi_k$, and hence such that $f = w\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty}\varphi_k$.

Proof: Let $f \in \mathscr{D}'_{L^p}$. Let $\psi \in \mathscr{D}$ be such that $\psi(t) \geq 0$ for every $t \in \mathbb{R}$, supp $\psi = [-1, 1]$ and $\int \psi = 1$, and consider the sequence $\psi_k(t) = k\psi(kt) \in \mathscr{D}$. It is easily seen that $s\text{-}\mathscr{D}'_{L^1}\text{-}\lim_{k\to\infty}\psi_k = \delta$. Hence, since $f \in \mathscr{D}'_{L^p}$, then $\varphi_k = f * \psi_k$ is a sequence of members of $\mathscr{D}_{L^p}$ which strongly converges to f in $\mathscr{D}'_{L^p}$ (see the results on regularization in Section 8 of Chapter 6 of [5]). ■

Lemma 4.2: Let $1 \leqslant p < \infty$. For every $\varphi \in \mathscr{D}_{L^p}$ there exists a sequence $\varphi_k \in \mathscr{D}$ such that $\varphi = s\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty}\varphi_k$, and hence such that $\varphi = w\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty}\varphi_k$.

Proof: Let $\varphi \in \mathscr{D}_{L^p}$. Since, by Section 8 of Chapter 6 of [5] there exists a sequence φ_k of members of $\mathscr{D}$ such that $\varphi = \mathscr{D}_{L^p}\text{-}\lim_{k\to\infty}\varphi_k$, and it is easily seen that convergence in $\mathscr{D}_{L^p}$ implies strong convergence in $\mathscr{D}'_{L^p}$, then it is $\varphi = s\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty}\varphi_k$. ■

Lemma 4.3: For every $\varphi \in \mathscr{D}_{L^\infty}$ there exists a sequence $\varphi_k \in \mathscr{D}$ such that $\varphi = w\text{-}\mathscr{D}'_{L^\infty}\text{-}\lim_{k\to\infty}\varphi_k$.

Proof: Let $\varphi \in \mathscr{D}_{L^\infty}$. For $k = 1, 2, \ldots$, let $\eta_k \in \mathscr{D}$ be such that $|\eta_k(t)| \leqslant 1$ for every $t \in \mathbb{R}$, and that $\eta_k(t) = 1$ for every $t \in [-k, k]$. Let $\varphi_k = \eta_k\varphi \in \mathscr{D}$.

Let $\sigma \in \mathscr{D}_{L^1}$. Every $\varphi_k\sigma \in L^1$; for all $\tau \in \mathbb{R}$ it is $\lim_{k\to\infty}\varphi_k(\tau)\sigma(\tau) = \varphi(\tau)\sigma(\tau)$; moreover $\varphi\sigma \in L^1$ and, for every k and every $\tau \in \mathbb{R}$ it is $|\varphi_k(\tau)\sigma(\tau)| \leqslant |\varphi(\tau)\sigma(\tau)|$. As a consequence of Lebesgue Theorem it is

$$\lim_{k\to\infty}\int_{-\infty}^{+\infty}\varphi_k(t)\sigma(t)\mathrm{d}t = \int_{-\infty}^{+\infty}\varphi(t)\sigma(t)\mathrm{d}t$$

i.e., $\lim_{k\to\infty}\langle\varphi_k, \sigma\rangle = \langle\varphi, \sigma\rangle$. ■

As a consequence of Lemma 4.1, 4.2 and 4.3, we obtain that

for $1 \leqslant p \leqslant \infty$ it is $\Sigma\left(\mathscr{D}, w\text{-}\mathscr{D}'_{L^p}\right) = \Sigma_2\left(\mathscr{D}, w\text{-}\mathscr{D}'_{L^p}\right) = \mathscr{D}'_{L^p}$

for $1 \leqslant p < \infty$ it is $\Sigma\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^p}\right) = \Sigma_2\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^p}\right) = \mathscr{D}'_{L^p}$

The analysis of $\Sigma\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^\infty}\right)$ is slightly more difficult. Let $\dot{\mathscr{D}}'_{L^\infty}$ denote the space of distributions f convergent to 0 at infinity, i.e., of the distributions f such that $\mathscr{D}'\text{-}\lim_{|\tau|\to\infty} f(t-\tau) = 0$.

Lemma 4.4: It is $\Sigma\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^\infty}\right) \subseteq \dot{\mathscr{D}}'_{L^\infty}$

Proof: The statement follows from the second and the last subsections of Section 8, Chapter 6 of [5]. ■

Lemma 4.5: For every $f \in \dot{\mathscr{D}}'_{L^\infty}$ there exists a sequence $\varphi_k \in \dot{\mathscr{D}}_{L^\infty}$ such that $f = s\text{-}\mathscr{D}'_{L^\infty}\text{-}\lim_{k\to\infty}\varphi_k$.

Proof: Let $f \in \dot{\mathscr{D}}'_{L^\infty}$. Let $\psi_k \in \mathscr{D}$ and $\varphi_k = f * \psi_k \in \mathscr{D}_{L^\infty}$ be as in the Proof of Lemma 4.1, so that $f = s\text{-}\mathscr{D}'_{L^\infty}\text{-}\lim_{k\to\infty}\varphi_k$. Since $\mathscr{D}'\text{-}\lim_{|\tau|\to\infty} f(t-\tau) = 0$, then for every k and h it is $\lim_{|t|\to\infty} D^h(f * \psi_k)(t) = \lim_{|t|\to\infty}(f * D^h\psi_k)(t) = \lim_{|t|\to\infty}\left\langle f(t-\tau), D^h\psi_k(\tau)\right\rangle = 0$; hence every $\varphi_k = f * \psi_k \in \dot{\mathscr{D}}_{L^\infty}$. ■

Lemma 4.6: For every $\varphi \in \dot{\mathscr{D}}_{L^\infty}$ there exists a sequence $\varphi_k \in \mathscr{D}$ such that $\varphi = s\text{-}\mathscr{D}'_{L^\infty}\text{-}\lim_{k\to\infty}\varphi_k$.

Proof: Let $\varphi \in \dot{\mathscr{D}}_{L^\infty}$, and let $\eta_k \in \mathscr{D}$, $\varphi_k = \eta_k\varphi \in \mathscr{D}$ be as in the Proof of Lemma 4.3. Let B be a bounded subset of $\mathscr{D}_{L^1}$, and let M_0 be a positive real such that $\sup\{\,\|\sigma\|_1 : \sigma \in B\,\} \leq M_0$. For every $\sigma \in B$ it is

$$|\langle\varphi_k, \sigma\rangle - \langle\varphi, \sigma\rangle| \leq \textstyle\int_{-\infty}^{+\infty}|\eta_k(t)\varphi(t) - \varphi(t)|\,|\sigma(t)|\mathrm{d}t \leq$$
$$\leq \|\eta_k\varphi - \varphi\|_\infty\,\|\sigma\|_1 \leq M_0\|\eta_k\varphi - \varphi\|_\infty$$

Since $\lim_{|t|\to\infty}\varphi(t) = 0$, then it is $s\text{-}\mathscr{D}'_{L^\infty}\text{-}\lim_{k\to\infty}\varphi_k = \varphi$. ■

By Lemma 4.6 it is

$$\Sigma_1\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^\infty}\right) \supseteq \dot{\mathscr{D}}_{L^\infty}$$

then, by Lemma 4.5 it is

$$\Sigma_2\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^\infty}\right) \supseteq \dot{\mathscr{D}}'_{L^\infty}$$

so that

$$\Sigma\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^\infty}\right) \supseteq \dot{\mathscr{D}}'_{L^\infty}$$

Hence, by Lemma 4.4 it is

$$\Sigma\left(\mathscr{D}, s\text{-}\mathscr{D}'_{L^\infty}\right) = \dot{\mathscr{D}}'_{L^\infty} \neq \mathscr{D}'_{L^\infty}$$

The above results can be summarized in the following

Theorem 4.1: (a) Let $\mathscr{I}$ be one of the following spaces

L^p	$1 \leqslant p < \infty$
$\mathscr{D}'_{L^p}$ (strong convergence)	$1 \leqslant p < \infty$
$\mathscr{D}'_{L^p}$ (weak convergence)	$1 \leqslant p \leqslant \infty$
$\mathscr{E}', \mathscr{S}', \mathscr{D}'$	

then $\Sigma(\mathscr{D}, \mathscr{I}) = \mathscr{I}$.

(b) Let $\mathscr{I}$ be one of the following spaces

$$\mathscr{C}, L^\infty, \mathscr{D}'_{L^\infty} \text{ (strong convergence)}$$

then $\Sigma(\mathscr{D}, \mathscr{I}) \neq \mathscr{I}$.

As a consequence of Theorem 3.2 and 4.1, we obtain the following

Corollary 4.1: Let $\mathscr{I}$ be one of the spaces listed in part (a) of Theorem 4.1, and let $\mathscr{O}$ be any output space (verifying Assumption 1). Then any continuous LTI map $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is completely described by its impulse response.

The next section proves that Corollary 4.1 cannot be extended to any of the three spaces listed in part (b) of Theorem 4.1.

V. Three pathological input spaces

In [2], Sandberg showed that there exist non null causal continuous LTI maps

$$\mathscr{L} : \mathscr{C} \to \mathscr{C}$$

with impulse response $\Delta = 0$. We recall that all the $\mathscr{L}$ in his proof verify the conditions: $\mathscr{L}(\varphi) = 0$ for every $\varphi \in \mathscr{D}$, and $\mathscr{L}(1)$ is a non zero constant function, where 1 is the function with constant value 1.

The following Propositions 5.1 and 5.2 are corollaries of this result. Propositions 5.1, with a different proof, is due to Sandberg (see Theorem 2 and related remarks of [4]).

Proposition 5.1: There exist non null causal continuous LTI maps

$$\mathscr{L} : L^\infty \to L^\infty$$

with impulse response $\Delta = 0$.

Proof: Let $\varphi_0 \in \mathscr{D}$ be such that

$$\text{supp } \varphi_0 \subseteq [0, +\infty) \quad , \quad \int_{-\infty}^{+\infty} \varphi_0(t) \mathrm{d}t = 1$$

By Theorem XXV of Chapter 6 of [5], for every $f \in L^\infty$ it is $f * \varphi_0 \in \mathscr{D}_{L^\infty}$, and, by the results on regularization in Section 8, Chapter VI of [5], the linear map

$$\mathscr{L}_0 : L^\infty \to \mathscr{D}_{L^\infty}$$

defined by $\mathscr{L}_0(f) = f * \varphi_0$ is continuous. Since $\text{supp } \varphi_0 \subseteq [0, +\infty)$, $\mathscr{L}_0$ is causal. Hence $\mathscr{L}_0$ is a causal continuous LTI map.

Obviously $\mathscr{D}_{L^\infty} \subseteq \mathscr{C} \subseteq L^\infty$, and the maps

$$\mathscr{L}_1 : \mathscr{D}_{L^\infty} \to \mathscr{C} \quad , \quad \mathscr{L}_3 : \mathscr{C} \to L^\infty$$

defined by $\mathscr{L}_1(f) = f$ for every $f \in \mathscr{D}_{L^\infty}$, and $\mathscr{L}_3(f) = f$ for every $f \in \mathscr{C}$, are causal continuous LTI maps.

Now let $\mathscr{L}_2 : \mathscr{C} \to \mathscr{C}$ be a causal continuous LTI map such that: $\mathscr{L}_2(\varphi) = 0$ for every $\varphi \in \mathscr{D}$, and $\mathscr{L}_2(1) = 1$—existing by the result of Sandberg—, and consider the causal continuous LTI map

$$\mathscr{L} = \mathscr{L}_3 \mathscr{L}_2 \mathscr{L}_1 \mathscr{L}_0 : L^\infty \to L^\infty$$

For every $\varphi \in \mathscr{D}$, it is $\varphi * \varphi_0 \in \mathscr{D}$; hence $\mathscr{L}(\varphi) = 0$. Since $\int_{-\infty}^{+\infty} \varphi_0(t) \mathrm{d}t = 1$, it is $1 * \varphi_0 = 1$; hence $\mathscr{L}(1) = 1$, which concludes the proof. ■

Proposition 5.2: Consider $\mathscr{D}'_{L^\infty}$ with the strong convergence. There exist non null causal continuous LTI maps

$$\mathscr{L} : \mathscr{D}'_{L^\infty} \to \mathscr{D}'_{L^\infty}$$

with impulse response $\Delta = 0$.

Proof: The proof is identical to the proof of Proposition 5.1, merely substituting the space L^∞ with the space $\mathscr{D}'_{L^\infty}$ (strong convergence). ■

To understand why the argument of the proof of Proposition 5.2 cannot be applied to LTI maps continuous with respect to the weak convergence, observe that the map

$$\mathscr{L}_0 : \mathscr{D}'_{L^\infty} \to \mathscr{D}_{L^\infty}$$

which must be used in the proof of Proposition 5.2 is continuous with respect to the strong convergence in $\mathscr{D}'_{L^\infty}$ but is *not* continuous with respect to the weak convergence in $\mathscr{D}'_{L^\infty}$. Indeed, the sequence $\delta(t-1), \delta(t-2), \ldots$ is weakly convergent to 0 in $\mathscr{D}'_{L^\infty}$, but the sequence $\delta(t-1) * \varphi_0(t) = \varphi_0(t-1), \delta(t-2) * \varphi_0(t) = \varphi_0(t-2), \ldots$ is not convergent in $\mathscr{D}_{L^\infty}$.

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