ON 2-VERY AMPLE DIVISORS OF GENUS ≤ 8 ON ALGEBRAIC SURFACES

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Abstract. In this paper we study the k-very ampleness of certain classes of divisors of genus ≤ 8 on rational and ruled surfaces appearing in the papers [La] of A.Lanteri and [An] of M.Andreatta.

Introduction

The purpose of this paper is to extend the techniques used in [Ca-Fra] to the problem of higher order embeddings of surfaces.

To explain what does it mean "*higher order embedding*" M.Beltrametti, P.Francia and A.J.Sommese in [Be-Fr-So] introduced the notion of *k-very ample sheaf*. Essentially an invertible sheaf $\mathcal{L}$ on a projective scheme of dimension m is said to be *k-very ample* if $\Gamma(\mathcal{L})$ separates any 0-*dimensional scheme* of length $\leq k+1$ (cf. Def. 1.1).

Let (S, L) be a pair where S is a smooth complex projective surface and L is a divisor on S. A natural problem is to give the classification of the pairs (S, L) such that $\mathcal{O}_S(L)$ is k-very ample ($k \geq 2$) for low numerical invariants, e.g. for fixed $p_a(L)$ ($p_a(L)$ being the arithmetic genus of L).

For $p_a(L) \leq 5$ the complete classification was carried out in [Be-So 1]. In [La] we can find a list of all the admissible pairs (S, L) such that $p_a(L) = 6$, and in [An] for $p_a(L) = 7, 8$ (notice that in their lists one should add the case $S = K3$ surface as pointed out by S.Di Rocco in [Di]). Furthermore in these papers there are the proofs of the existence of such admissible pairs except for one case if $p_a(L) = 6$ and five cases if $p_a(L) = 8$.

We shall treat here three of these cases: in §2 two cases where S is a Segre-Hirzebruch surface with respectively 9 and 10 points blown up, and $p_a(L) = 6$, respectively 8 (cf. case 8 of Thm.1 in [La], case 8(14) in [An]); in §3 the case where S is a ruled surface over an elliptic curve with 2 points blown up and $p_a(L) = 8$ (cf. case 8 (13) in [An]).

The method we use to analyze the problem whether an invertible sheaf $\mathcal{L}$ on a projective surface S is k-very ample (valid also for variety of dimension ≥ 3) is classical and consists in choosing a linear system of divisors $|\Delta|$ of sufficiently positive dimension and then to consider the restriction of $|\mathcal{L}|$ to each divisor $D \in |\Delta|$. To study the restriction of $\mathcal{L}$ to a curve, possibly reducible or non reduced, we use a criterion developed in [C-F-H-R], valid in all characteristics, which gives sufficient numerical conditions for the k-very ampleness of an invertible sheaf $\mathcal{L}$ on a curve C. It states that if the degree of $\mathcal{L}$ on each generically Gorenstein subcurve B of C is at least $2p_a(B) + k$, then necessarily $\mathcal{L}$ is k-very ample on C.

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Our results are the following (for the notation we refer to §2 and §3).

Theorem A . *Let be $S = \hat{\mathbf{F}}_e(x_1, \dots, x_{9+\nu})$, $\nu \in \{0, 1\}$, $0 \le e \le 2 + \nu$, $L = 4C_0 + (2e + 6 + \nu)F - \sum_{i=1}^{9+\nu} 2E_i$.*

Assume the points x_i be in general position.

Then $\mathcal{O}_S(L)$ is 2-very ample on S.

Theorem B . *Let S be the blow up in two points x_1, x_2 of a ruled surface $\underline{S}$ over an elliptic curve with $e = -1$ and let be $L = 4C_0 + F - 2E_1 - 2E_2$.*

Assume the points x_i be in general position.

Then $\mathcal{O}_S(L)$ is 2-very ample on S.

More precisely, with easy and short proofs, in Thm. 2.1 and Thm. 3.1 we will give the necessary conditions on the position of the points blown up in order that $\mathcal{O}_S(L)$ be 2-very ample and by such conditions we will prove the 2-very ampleness of $\mathcal{O}_S(L)$ (cf. [Ba], [Ca-Fra] for analogue statements).

We believe it could be possible to use these methods also for the other cases appearing in the paper [An] (except possibly in the case "S of general type"), but for the reason of keeping this note not too long we restrict ourselves to illustrate the applications only for the mentioned classes of surfaces.

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Notation.

- X A pure projective scheme of dimension m over an algebraically closed field $\mathbb{K}$ of characteristic $p \ge 0$.
- $p_a(X)$ The arithmetic genus of X, $p_a(X) = (-1)^m(\chi(\mathcal{O}_X) - 1)$, $\chi(\mathcal{O}_X)$ being the Euler characteristic.
- $\equiv$ The linear equivalence of Cartier divisors on X.
- $|D|$ Linear system defined by a Cartier divisor D on X.
- S A smooth projective surface over an algebraically closed field $\mathbb{K}$.
- $D.D'$ Intersection number of divisors D, D' on a smooth projective surface.
- $\sim$ The numerical equivalence of divisors on a smooth projective surface.

1. k-very ampleness on algebraic surfaces

1.1. k-very ampleness.

We recall the notion of *k-very ampleness* as introduced by M.Beltrametti,P.Francia and A.J.Sommese (cf. [Be-Fr-So]).

Definition 1.1. *Let X be a pure projective scheme of dimension m and $\mathcal{L}$ an invertible sheaf on X. $\mathcal{L}$ is said to be* k-very ample *($k \ge 0$) if for any 0-dimensional subscheme $(Z, \mathcal{O}_Z)$ of X with* $\text{length}(\mathcal{O}_Z) = d \le k + 1$ *the map*

$$r_Z : \Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_Z)$$

is surjective.

If $\mathcal{L} = \mathcal{O}_X(L)$, L being a Cartier divisor on X, we say that L is a *k-very ample divisor.*

Note that the notion *0-very ample* corresponds to the classical notion *spanned by global sections*, while the notion *1-very ample* corresponds to the notion *very ample*. For $k = 2$ it means that the global sections of $\mathcal{L}$ define an embedding $\varphi_{\mathcal{L}} : X \to \mathbb{P}^{\mathbb{N}}$

such that $\varphi_{\mathcal{L}}(X)$ has no 0-dimensional subscheme of length 3 on a line. We may say that $\varphi_{\mathcal{L}}(X)$ has no *3-secant lines* (cf. [An]). Note also that if X is a smooth surface and $k \leq 2$, *k-very ampleness* is equivalent to the notion *k-spannedness* introduced in [Be-Fr-So]. In this case we will talk about *k-spanned surface*.

1.2. **The method of restriction to curves.**
Let us firstly consider the case of dimension $m = 1$, i.e., let C be a pure projective curve, possibly reducible and non reduced.

For each subcurve $B \subseteq C$ we denote by ω_B the dualizing sheaf of B (cf. [Ha] Chap. III, §7) and we define the degree of the restriction of the invertible sheaf $\mathcal{L}$ to B by

$$\deg \mathcal{L}_{|B} = \chi(\mathcal{L}_{|B}) - \chi(\mathcal{O}_B).$$

We say that B is *generically Gorenstein* if, outside a finite set, ω_B is locally isomorphic to $\mathcal{O}_B$.

Here is a generalization which appears in [C-F-H-R] of the results of [Ca-Fra] and [Ca-Hu] on very-ample divisors on curves lying on smooth projective surfaces. It is an extension of the case where C is smooth and irreducible.

Theorem 1.2. *Let C be a projective curve and let $\mathcal{L}$ be an invertible sheaf on C such that for each generically Gorenstein subcurve B of C we have*

$$\deg \mathcal{L}_{|B} \geq (2p_a(B) + k).$$

Then $\mathcal{L}$ is k-very ample on C.

(cf. Theorem 1.1 in [C-F-H-R]).

Let us remark that if $C \subset$ a smooth projective surface S then each subcurve B of C is generically Gorenstein.

Having this strong result on curves, to study the behaviour of an invertible sheaf $\mathcal{L}$ on a smooth projective surface S, we can apply the following simple and classical proposition (cf. [Ra], Lemma 3.1, and [Ba], Claim 2.19 for a refinement in the case $k = 1$).

Proposition 1.3. *Let X be an irreducible projective scheme of dimension $m \geq 2$, $\Delta \subset X$ be an effective Cartier divisor and $\mathcal{L}$ be an invertible sheaf on X. Assume $\dim|\Delta| \geq (k+1)$ and $\rho_\Delta : H^0(X, \mathcal{L}) \longrightarrow H^0(\Delta, \mathcal{L}_{|\Delta})$ be surjective.*

If $\forall D \in |\Delta|$ $\mathcal{L}_{|D}$ is k*-very ample on D then $\mathcal{L}$ is* k*-very ample on X.*

(We omit the obvious proof).

Moreover we can use the curves $\subset S$ to get some numerical necessary conditions in order that a divisor $L \subset S$ be 2-very ample (cf. [Be-So 1], 0.5 and [La], Lemma 0.3 for the smooth case).

Proposition 1.4. *Let S be a smooth projective surface and L be a 2-very ample divisor on S. Let $C \subset S$ be an effective divisor. Then:*

1) $C.L \geq 2$
2) $p_a(C) \geq 1 \Rightarrow C.L \geq 4$

Proof. Assume L be 2-very ample and $L.C \leq 1$. Since L is in particular very ample, then $L.C = 1$ and $(C, \mathcal{O}_C(L)) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. But then $h^0(C, \mathcal{O}_C(L)) = 2$, whence the restriction map r_Z is obviously not onto.

Concerning the other inequality, since L is very ample we can observe that $p_a(C) = 1$ implies $C.L \geq 3$ by [Ca-Fra] prop. 5.2, and moreover if equality occurs then by [Ca-Fra] prop. 6.1 C is isomorphic to a plane cubic (which of course has many trisecants !) $\square$

2. ON 2-SPANNED RATIONAL SURFACES

The first cases that we consider are about Segre-Hirzebruch surfaces $\mathbf{F}_e$ blown up in 9 and 10 points.

For the definition of $\mathbf{F}_e$ and the main results on these surfaces we refer e.g. to [Ha], pp. 379-383, or [Bea], pp. 33-43.

We recall the standard notation and some classical descriptions of divisors and linear systems on Segre-Hirzebruch surfaces.

By C_0 and F we denote respectively the fundamental section (i.e., the section with minimal self-intersection) and the fibre of S so that the surface is determined by the invariant $e = -{C_0}^2$ (cf. e.g. [Ha], p.373).

Any divisor $D \in \mathrm{Pic}(\mathbf{F}_e)$ is linearly equivalent to $aC_0 + bF$ with $a, b \in \mathbb{Z}$ (cf. e.g. [Bea] p. 42), and given $D \equiv aC_0 + bF$, $D' \equiv a'C_0 + b'F$, their intersection product $D.D' = a'b + ab' - eaa'$. The canonical divisor on $\mathbf{F}_e$ is linearly equivalent to $-2C_0 - (e+2)F$ (cf. [Ha] p. 374), then, given an effective divisor $D \equiv aC_0 + bF$, by the adjunction formula we get

$$p_a(D) = (a-1)(b-1) - ea(a-1)/2. \tag{1}$$

Moreover from the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to \mathcal{O}_D(D) \to 0$$

since the surface S is rational (which implies $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$), taking cohomology we get

$$\begin{aligned} &h^0(\mathcal{O}_S(D)) = h^0(\mathcal{O}_D(D)) + 1 \geq D^2 + 2 - p_a(D) \text{ (by R-R on } D), \\ &h^1(\mathcal{O}_S(D)) = h^1(\mathcal{O}_D(D)). \end{aligned} \tag{2}$$

Let $\pi : S = \hat{\mathbf{F}}_e(x_1, \ldots, x_{9+\nu}) \to \mathbf{F}_e$ be the blow up of $\mathbf{F}_e$ in $9+\nu$ distinct points, $(\nu \in \{0,1\}, 0 \leq e \leq 2+\nu)$. We denote by E_i the total transform $\pi^{-1}(x_i)$, and we maintain the notation C_0, F for the pull-backs of C_0, respectively F. Furthermore by $|\alpha C_0 + \beta F - \sum c_i E_i|$, $c_i \in \mathbb{N}$, we denote the pull-back of the linear system of effective divisors in $|\alpha C_0 + \beta F|$ ($\subset \mathbf{F}_e$) with multiplicity at least c_i in x_i.

Our result is the following

Theorem 2.1. *Let be $S = \hat{\mathbf{F}}_e(x_1, \ldots, x_{9+\nu})$, $\nu \in \{0,1\}$, $0 \leq e \leq 2+\nu$, $L = 4C_0 + (2e+6+\nu)F - \sum_{i=1}^{9+\nu} 2E_i$.*

Then L is 2-very ample on S if and only if

(i) $h^0(E_i - E_j) = 0 \quad \forall i,j \in \{1,\ldots,9+\nu\}, i \neq j$;
(ii) $h^0(F - E_i - E_j) = 0 \quad \forall i,j \in \{1,\ldots,9+\nu\}, i \neq j$;
(iii) $h^0(C_0 + bF - \sum_{i\in\Lambda} E_i) = 0 \quad \forall \Lambda \subseteq \{1,\ldots,9+\nu\}$ s.t. $\#\Lambda \geq 2b+3-e$;
(iv) $h^0(2C_0 + (e+1)F - \sum_{i\in\Lambda} E_i) = 0 \quad \forall \Lambda \subseteq \{1,\ldots,9+\nu\}$ s.t. $\#\Lambda \geq 8$;
(v) $h^0(2C_0 + (e+2)F - \sum_{i\in\Lambda} E_i) = 0 \quad \forall \Lambda \subseteq \{1,\ldots,9+\nu\}$ s.t. $\#\Lambda \geq 9$.

Proof. Remember that

$C_0^2 = -e,\ F^2 = 0,\ C_0.F = 1,\ E_i^2 = -1,\ C_0.E_i = F.E_i = 0\ \ \forall i \in \{1,\ldots,9+\nu\}$;
in particular: $L.E_i = 2, L.C_0 = -2e+6, L.F = 4$.

Necessity of the above conditions. The necessity of the above conditions $(i), ..., (iv)$ follows by considering the intersection product with L.

For (v) it is enough to notice that $A \in \mid 2C_0 + (e+2)F - \sum_{i \in \Lambda} E_i \mid$ has arithmetic genus 1 while $A.L \leq 3$, against prop. 1.4.

Sufficiency of the above conditions. Let $\mathcal{L} = \mathcal{O}_S(L)$. We look for a divisor Δ satisfying the hypotheses of prop. 1.3.

To simplify the computations and to consider at the same time the different surfaces corresponding to the different values of e we consider a divisor Δ close to L for which we have $H^1(S, \mathcal{O}_S(L - \Delta)) = 0$, and for which the value e does not appear in $p_a(\Delta)$, $L.\Delta$.

Our chosen Δ is

$$\Delta = 4C_0 + (2e+5)F - \sum_{i=1}^{8} 2E_i - \sum_{j=9}^{9+\nu} E_j.$$

If we set $\underline{\Delta} = 4C_0 + (2e+5)F \subset \mathbf{F}_e$, we have by (1)

$$p_a(\underline{\Delta}) = 3(2e + 4 +) - e(4 \cdot 3)/2 = 12.$$

Moreover $(\underline{\Delta})^2 = 2 \cdot 4(2e+5) - 16e = 40$, whence $h^0(\mathbf{F}_e, \mathcal{O}_{\mathbf{F}_e}(\underline{\Delta})) \geq 40 - 12 + 2 = 30$ and then

- $h^0(S, \mathcal{O}_S(\Delta)) \geq 30 - 3 \cdot 8 - (1 + \nu) \geq 4$;
- $H^1(S, \mathcal{O}_S(L - \Delta)) = H^1(L - \Delta, \mathcal{O}_{L-\Delta}(L - \Delta)) = 0$
  since $L - \Delta = \begin{cases} F - E_9 & \text{if } \nu = 0 \\ F - E_9 + F - E_{10} & \text{if } \nu = 1 \end{cases}$
  and $F - E_j \cong \mathbb{P}^{1}$, $(F - E_j)^2 = -1$, $(F - E_9).(F - E_{10}) = 0$.

Thus we obtain $\dim |\Delta| \geq 3$ and that the map $\rho_\Delta : H^0(X, \mathcal{L}) \to H^0(\Delta, \mathcal{L}_{|\Delta})$ is surjective. Our claim is then

Claim . $\forall D \in |\Delta|, \mathcal{L}_{|D}$ *is 2-very ample.*

Now $L.\Delta = 4(2e + 6 + \nu) + 4(2e + 5) - 16e - 8 \cdot 4 - 2 - 2\nu = 10 + 2\nu$ and $p_a(\Delta) = 12 - 8 = 4$. Our purpose is to verify the assumptions of Theorem 1.2.

Lemma 2.2. *Let S and L be as in Theorem 2.1. Let $D \in |4C_0 + (2e+5)F - \sum_{i=1}^{8} 2E_i - \sum_{j=9}^{9+\nu} E_j|$. Then:*

$$\forall B, 0 < B \leq D, \quad B.L \geq (2p_a(B) + 2).$$

Proof. Let $B \equiv aC_0 + bF - \sum_{i=1}^{9+\nu} c_i E_i$, with $a, b, c_i \in \mathbb{Z}$, $0 < B \leq D$.

Let us recall that if $B \subset \mathbf{F}_e$ is an effective divisor, $\underline{B} \equiv \alpha C_0 + \beta F$, then $\alpha \geq 0$, $\beta \geq 0$ (cf. e.g. [Ha] p.380); whence, since we have assumed $B > 0$ and $D - B \geq 0$, there must be $0 \leq a \leq 4$, $0 \leq b \leq 2e + 5$.

We have

$$L.B = a(2e + 6 + \nu) + 4b - 4ae - \sum 2c_i = (6 + \nu)a + 4b - 2ae - \sum 2c_i,$$

and moreover, if B is the strict transform of an effective $\underline{B} \subset \mathbf{F}_e$,

$$p_a(B) = (a-1)(b-1) - ea(a-1)/2 - \sum c_i(c_i - 1)/2.$$

We prove the lemma studying separately the different values of a. Before we start, let us point out this obvious remark:

Remark 2.3. $\sum c_i(3 - c_i)/2 \leq \#\{i | c_i \neq 0\}$ *since* $c_i(3 - c_i) \leq 2\ \forall c_i \in \mathbb{Z}$.

Note that if $p_a(B) \leq 0$, it suffices to show $B'.L \geq 2$ for B' reduced and irreducible, $0 < B' \leq B$, since it can be easily seen that L is ample.

Moreover, recall that if $b < ae$ then C_0 is a component of B (cf. [Ha] p.380).

a=0,1. For $a = 0, 1$, $p_a(B) \leq 0$, hence we have to prove $B'.L \geq 2$ for B' reduced and irreducible, $0 < B' \leq B$. Setting $B' \equiv a'C_0 + b'F - \sum_{i=1}^{9+\nu} c'_i E_i$ we necessarily get:

$a' = 0, b' = 0$ and there exists an index i s.t. $c'_i = -1$; or

$a' = 0, b' = 1$ and $0 \leq c'_i \leq 1$ $\forall i \in \{1, ..., 9+\nu\}$; or

$a' = 1$ and $0 \leq c'_i \leq 1$ $\forall i \in \{1, ..., 9+\nu\}$.

Whence we have $B'.L \geq 2$ by conditions $(i), (ii), (iii)$.

a=2. If $a = 2$ we have $p_a(B) = b - 1 - e - \sum c_i(c_i - 1)/2$. Whence $p_a(B) \leq 0$ if $b \leq e + 1$, while if $b < 2e$, B has C_0 as a fixed component; thus we can reduce to the cases $a \leq 1$ except if

$b = e + 1$, $e \leq 1$, or

$b \geq e + 2$.

But then

$$\begin{aligned} L.B \geq 2p_a(B) + 2 \quad & \Leftrightarrow 12 + 2\nu + 4b - 4e - \sum 2c_i \geq 2b - 2e - \sum c_i(c_i - 1) \\ & \Leftrightarrow b \geq -6 - \nu + e + \sum c_i(3 - c_i)/2. \end{aligned}$$

By remark 2.3, if $b \geq e+3$ the inequality is satisfied just because $\#\{i | c_i \neq 0\} \leq 9+\nu$; in the case $b = e + 2$, if such a divisor exists, then $\#\{i | c_i \neq 0\} \leq 8$ by condition (v), while in the case $b = e + 1$, condition (iv) implies $\#\{i | c_i \neq 0\} \leq 7$.

a=3. In this case

$$\begin{aligned} L.B \geq 2p_a(B) + 2 \quad & \Leftrightarrow 18 + 3\nu + 4b - 6e - \sum 2c_i \geq 4b - 2 - 6e - \sum c_i(c_i - 1) \\ & \Leftrightarrow 20 + 3\nu \geq \sum c_i(3 - c_i) \end{aligned}$$

which is always true since $\#\{i | c_i \neq 0\} \leq 9 + \nu$.

a=4. Now we have

$$\begin{aligned} L.B \geq 2p_a(B) + 2 \quad & \Leftrightarrow 24 + 4\nu + 4b - 8e - \sum 2c_i \geq 6b - 4 - 12e - \sum c_i(c_i - 1) \\ & \Leftrightarrow b \leq 14 + 2\nu + 2e - \sum c_i(3 - c_i)/2 \end{aligned}$$

Since the R.H.S. is $\geq 2e + 5$ because $\#\{i | c_i \neq 0\} \leq 9 + \nu$, and $0 \leq b \leq 2e + 5$ because $0 < B \leq D$, the inequality follows. $\square$

Q.E.D. for Thm. 2.1

3. On 2-spanned ruled surfaces

In this section S will be the blow up of a ruled surface $\underline{S}$ over an elliptic curve (cf. case 8 (13) in [An]).

We know from the theory that $\underline{S} = \mathbb{P}(\mathcal{E})$ for some rank 2 locally free sheaf $\mathcal{E}$ on a nonsingular curve C of genus 1 (cf. e.g. [Ha], p.372). Moreover we can choose $\mathcal{E}$ in such a way that there exists a section C_0 of the projection $p : \underline{S} \to C$, so that C_0^2 is minimal and $\mathrm{Num}(\underline{S}) = \mathbb{Z}\mathbb{C}_{\not{0}} \oplus \mathbb{Z}\mathbb{F}$, F being the class of a fibre (cf. [Ha], p.370). C_0 is said a fundamental section and $e = -C_0^2 = -\deg(\mathcal{E})$ the invariant of S.

Furthermore if $\mathcal{E}$ is an indecomposable locally free sheaf over the smooth elliptic curve C,then $e = 0, -1$ (cf. [Ha], p.377).

We shall treat here the case where $\underline{S}$ is a ruled surface over an elliptic curve with $e = -1$.

Any divisor $D \in \mathrm{Pic}(\underline{S})$ is numerically equivalent to a divisor $aC_0 + bF$ with $a, b \in \mathbb{Z}$ and given $D \sim aC_0 + bF, D' \sim a'C_0 + b'F$, their intersection product is given by $D.D' = a'b + ab' + aa'$. The canonical divisor of $\underline{S}$ is numerically equivalent to $-2C_0 + F$ (cf. [Ha], p. 374), thus if D is an effective divisor, $D \sim aC_0 + bF$, by the adjunction formula we get

$$p_a(D) = 1 + (a-1)b + (a-1)a/2. \tag{3}$$

Moreover, since $H^1(\underline{S}, \mathcal{O}_{\underline{S}}) \cong H^1(C, \mathcal{O}_C) \cong \mathbb{C}$ (cf. [Ha] p.371), by the exact sequence

$$0 \to \mathcal{O}_{\underline{S}} \to \mathcal{O}_{\underline{S}}(D) \to \mathcal{O}_D(D) \to 0,$$

and R-R on C we get the following inequalities:

$$h^0(\mathcal{O}_S(D)) \geq h^0(\mathcal{O}_D(D)) \geq D^2 + 1 - p_a(D). \tag{4}$$

Let $\pi : S \to \underline{S}$ be the blow up of $\underline{S}$ in two distinct points x_1, x_2.

We will show that the divisor $L = 4C_0 + F - 2E_1 - 2E_2$ is 2-very ample on S, where E_i is the total transform $\pi^{-1}(x_i)$, and keeping the same notation C_0, F for the pull-backs of C_0, respectively F.

Theorem 3.1. *Let S be the blow up in two points x_1, x_2 of a ruled surface $\underline{S}$ over an elliptic curve C with invariant $e = -1$ and let be $L = 4C_0 + F - 2E_1 - 2E_2$.*

Then L is 2-very ample on S if and only if

$$\begin{array}{rll} (i) & h^0(E_i - E_j) = 0 & \{i,j\} = \{1,2\}; \\ (ii) & h^0(F - E_1 - E_2) = 0; & \\ (iii) & h^0(C_0 - E_i) = 0 & i = 1,2; \\ (iv) & h^0(2C_0 - F - E_1 - E_2) = 0. & \end{array}$$

Proof. We recall that in this case $L.E_i = 2, L.C_0 = 5, L.F = 4$.

Necessity of the above conditions. The necessity of the conditions (i), (ii) follows by taking the intersection product with L.

For (iii) and (iv) it is enough to notice that $A \in |C_0 - E_i|$ or $|\ 2C_0 - F - E_1 - E_2\ |$ has arithmetic genus 1 while $A.L \leq 3$, contradicting prop. 1.4.

Sufficiency of the above conditions. Following the strategy adopted in Thm.2.1 we choose $\Delta \equiv 4C_0 - 2E_1 - 2E_2$.

Considering the divisor $\underline{\Delta} = 4C_0 \subset \underline{S}$ we have $\underline{\Delta}^2 = 16$, $p_a(\underline{\Delta}) = 7$, whence

$$h^0(S, \mathcal{O}_S(\Delta)) \geq h^0(\underline{S}, \mathcal{O}_{\underline{S}}(\underline{\Delta})) - 2 \cdot 3 \geq 16 - 7 + 1 - 6 = 4.$$

Moreover, considering the Leray spectral sequence related to $\pi : S \to \underline{S}$ we have

$$H^1(S, \mathcal{O}_S(L - \Delta)) = H^1(S, \mathcal{O}_S(F)) \cong H^1(\underline{S}, \mathcal{O}_{\underline{S}}(F)) = 0$$

since by Lemma 2.4, p.371 in [Ha], $H^1(\underline{S}, \mathcal{O}_{\underline{S}}(F)) \cong H^1(C, p_*(\mathcal{O}_{\underline{S}}(F)))$ and on the elliptic curve C $\deg(p_*(\mathcal{O}_{\underline{S}}(F))) = 1$.

To prove the theorem it remains to verify the assumptions of Theorem 1.2 for each $D \in |\Delta|$.

Lemma 3.2. *Let S and L be as in Theorem 3.1. Let be $D \in |4C_0 - 2E_1 - 2E_2|$. Then:*

$$\forall B, 0 < B \leq D, \quad B.L \geq (2p_a(B) + 2).$$

Proof. Let be $B \sim aC_0 + bF - c_1E_1 - c_2E_2$, with $a, b, c_1, c_2 \in \mathbb{Z}$.

If B is π-exceptional we have $p_a(B) \leq 0$; whence, since (as it is easy to see) L is ample, it suffices to show $B'.L \geq 2$ for B' reduced and irreducible, $0 < B' \leq B$. But then, condition (i) implies $B' = E_i$ and thus $B'.L \geq 2$.

Now consider the case where B is not π-exceptional, i.e., $\pi_*(B)$ is an effective divisor $\subset \underline{S}$.

Since, as it is well known, F is nef and C_0 is ample we get that if $\alpha C_0 + \beta F \subset \underline{S}$ is an effective divisor then necessarily $\alpha \geq 0, \beta > -\alpha$.

Whence $B > 0$, $\Delta - B \geq 0$ ($\Delta - B$ possibly π-exceptional) imply the following possibilities for a, b:

$$\begin{array}{ll} a = 0 & 3 \geq b \geq 1; \\ a = 1 & 2 \geq b \geq 0; \\ a = 2 & 1 \geq b \geq -1; \\ a = 3 & 0 \geq b \geq -2; \\ a = 4 & 0 \geq b \geq -3. \end{array}$$

We have

$$L.B = aL.C_0 + bL.F - 2c_1 - 2c_2 = 5a + 4b - 2c_1 - 2c_2,$$
$$p_a(B) = 1 + a(a-1)/2 + (a-1)b - \sum c_i(c_i - 1)/2.$$

Thus $B.L \geq (2p_a(B) + 2) \Leftrightarrow$

$$\Leftrightarrow 5a + 4b - \sum 2c_i \geq 4 + a(a-1) + 2(a-1)b - \sum c_i(c_i - 1)$$
$$\Leftrightarrow a(6-a) + 2b(3-a) \geq 4 + \sum c_i(3 - c_i).$$

Since $(c_1(3-c_1) + c_2(3-c_2)) \leq 4$, in the cases

$$a \geq 3; \quad a = 2, b \geq 0; \quad a = 1, b \geq 1; \quad a = 0, b \geq 2;$$

the inequalities are satisfied.

For the remaining cases, if $(a, b) = (0, 1)$ or $(a, b) = (2, -1)$, we have $(\sum c_i(3 - c_i)) \leq 2$ by conditions (ii),(iv) respectively, while if $(a, b) = (1, 0)$ condition (iii) implies $c_1 = c_2 = 0$. □

Q.E.D. for Thm. 3.1

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