Stability Results for Scattered Data Interpolation by Trigonometric Polynomials

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Content

- Basics
- 'direct' Problem, matrix vector multiplication, Vandermonde-like, NFFT
- 'inverse' Problem, solving Vandermonde-like systems, INFFT
- Interpolation, Stability
- Numerical examples, MRI

Basics

Geometry

torus, sampling set

$$\mathbb{T} := \mathbb{R}/\mathbb{Z}, \qquad (x_j)_{j=0,\ldots,M-1} =: \mathcal{X} \subset \mathbb{T}$$

separation distance, fill distance

$$h := \min_{j=0,\ldots,M-1} \textsf{dist}\,(x_j, x_{j+1})\,, \qquad \delta := \max_{j=0,\ldots,M-1} \textsf{dist}\,(x_j, x_{j+1})$$

Ansatz

trigonometric polynomials

$$T_N := \textsf{span}\left\{ \mathrm{e}^{2\pi \mathrm{i} k(\cdot)} : \; k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1 \right\}$$

discrete system

$$\boldsymbol{A} := \left(\mathrm{e}^{2\pi \mathrm{i} k x_j}\right)_{j=0,\ldots,M-1;k=-\frac{N}{2},\ldots,\frac{N}{2}-1}, \quad \boldsymbol{f} \in \mathbb{C}^M, \quad \hat{\boldsymbol{f}} \in \mathbb{C}^N$$

Matrix vector multiplication - Vandermonde-like matrix - NFFT

$\hat{\boldsymbol{f}} \in \mathbb{C}^N$ given, compute

$$\boldsymbol{f} = \boldsymbol{A}\hat{\boldsymbol{f}}, \qquad f_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} k x_j}, \quad j = 0, \ldots, M-1$$

FFT for $M = N$ equispaced nodes, $\mathcal{O}\left(N \log N\right)$ operations

FFT for non equispaced nodes (Dutt, Rokhlin; Beylkin; P., Steidl, Tasche), in $\mathcal{O}\left(N \log N + M\right)$ operations

Linear system of equations - iNFFT

,,inverse" problem, $\boldsymbol{f} \in \mathbb{C}^M$ given in

$$\boldsymbol{A}\hat{\boldsymbol{f}} \approx \boldsymbol{f}$$

Moore-Penrose pseudo-inverse solution $\hat{\boldsymbol{f}}^{\dagger} = \boldsymbol{A}^{\dagger}\boldsymbol{f}$ fulfills

$$\|\hat{\boldsymbol{f}}\|_2 \rightarrow \min \quad \text{subject to} \quad \|\boldsymbol{f} - \boldsymbol{A}\hat{\boldsymbol{f}}\|_2 = \min\,.$$

special case IDFT, Gauß quadrature, $M = N,\; x_j = \frac{j}{M} - 0.5$

$$\boldsymbol{A}^{\mathsf{H}} \underbrace{\boldsymbol{W}}_{\frac{1}{M}\boldsymbol{I}} \boldsymbol{A} = \boldsymbol{I} \quad \Rightarrow \quad \hat{\boldsymbol{f}} = \boldsymbol{A}^{\mathsf{H}}\boldsymbol{W}\boldsymbol{f}$$

direct solver: Reichel, Ammar, Gragg; Faßbender

Approximation problem

weighted approximation problem, $\omega_j > 0,\ \boldsymbol{W} = \mathrm{diag}\,(\omega_j)_{j=0}^{M-1}$,

$$\|\boldsymbol{A}\hat{\boldsymbol{f}} - \boldsymbol{f}\|_{\boldsymbol{W}} \xrightarrow{\hat{\boldsymbol{f}}} \min$$

weighted normal equation of first kind

$$\underbrace{\boldsymbol{A}^{\mathsf{H}}\boldsymbol{W}\boldsymbol{A}}_{\text{Toeplitz}}\hat{\boldsymbol{f}} = \boldsymbol{A}^{\mathsf{H}}\boldsymbol{W}\boldsymbol{f}$$

dense sampling set

$$\delta := \max_{j=0,\dots,M-1} \mathrm{dist}\,(x_j, x_{j+1})$$

Feichtinger, Gröchenig, Strohmer
(weighted normal equation of first kind ($N < \frac{1}{\delta}$))

$$\mathrm{cond}_2\left(\boldsymbol{A}^{\mathsf{H}}\boldsymbol{W}\boldsymbol{A}\right) \le \left(\frac{1+\delta N}{1-\delta N}\right)^2$$

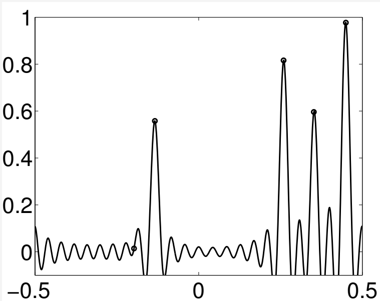
Interpolation problem

vanishing residual, i.e. $\boldsymbol{A}\hat{\boldsymbol{f}} - \boldsymbol{f} = \mathbf{0}$, $\rightsquigarrow N \ge M$

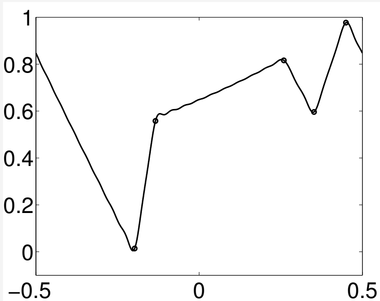
(damped) minimisation problem, $0 < \hat{\boldsymbol{W}} := \operatorname{diag}\left(\hat{\omega}_k\right)_{k=-\frac{N}{2},\ldots,\frac{N}{2}-1}$

$$\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{\omega}_k^{-1} |\hat{f}_k|^2 =: \|\hat{\boldsymbol{f}}\|^2_{\hat{\boldsymbol{W}}^{-1}} \xrightarrow{\hat{\boldsymbol{f}}} \min \quad \text{subject to} \quad \boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$$

example $N = 50$, $M = 5$ nodes, $\|f\|_{L^2}$ and $\|f\|_{L^2} + \|f'\|_{L^2}$ minimal, resp.



$\hat{\omega}_k = 1$



$\hat{\omega}_k^{-1} = 1 + (2\pi k)^2$

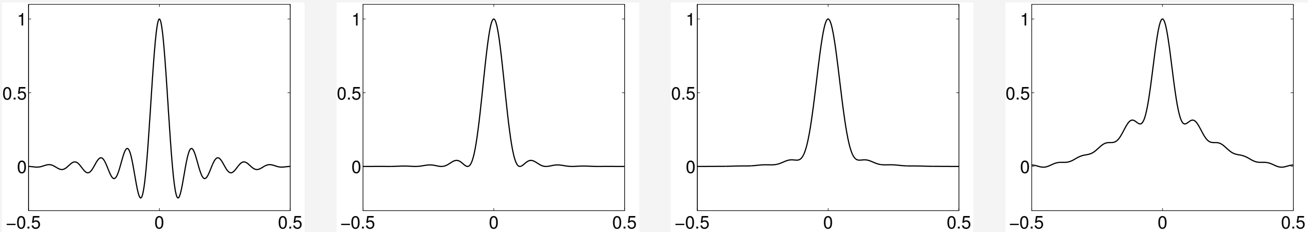
normal equation of second kind

$$\boldsymbol{A}\hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\tilde{\boldsymbol{f}} = \boldsymbol{f}, \qquad \hat{\boldsymbol{f}} = \hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\tilde{\boldsymbol{f}}$$

Interpolation with polynomial kernels

$$K(x-y) := \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \mathrm{e}^{-2\pi \mathrm{i} kx} \hat{\omega}_k \mathrm{e}^{2\pi \mathrm{i} ky},$$

$$f(y) = \sum_{l=0}^{M-1} \alpha_l K(x_l - y)$$



Dirichlet Fejer Cesaro Sobolev

centers of the kernels and nodes for interpolation are equal

$$\left(\boldsymbol{A}\hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\right)_{j,l} = K(x_j - x_l)$$

Stability

aim: find bounds dependent only on N, h for

$$\lambda = \lambda_{\min}\left(\boldsymbol{A}\hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\right), \qquad \Lambda = \lambda_{\max}\left(\boldsymbol{A}\hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\right)$$

norm equivalence

$$\|\boldsymbol{f}\|_2^2 \sim \inf_{f\in T_N, f(x_j)=f_j} \|f\|_{\hat{\boldsymbol{W}}^{-1}}^2$$

Marcinkiewicz-Zygmund-inequality

$$\Lambda^{-1}\|\boldsymbol{f}\|_2^2 \leq \inf_{f\in T_N, f(x_j)=f_j} \|f\|_{\hat{\boldsymbol{W}}^{-1}}^2 \leq \lambda^{-1}\|\boldsymbol{f}\|_2^2.$$

equispaced nodes, circulant interpolation matrix, eigenvalue characterisation

$$\lambda_j\left(\boldsymbol{A}\hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\right) = M \sum_{k=j \mod M} \hat{\omega}_k$$

Arbitrary nodes

if $K(0) = 1$ and

$$|K(x)| \leq \frac{C_\beta}{N^\beta |x|^\beta}$$

for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, then the interpolation matrix $\left(K(x_j - x_l)\right)_{j,l=0,\ldots,M-1}$ has bounded eigenvalues

$$1 - \frac{2\,\zeta(\beta)\,C_\beta}{N^\beta h^\beta} \leq \lambda \leq 1 \leq \Lambda \leq 1 + \frac{2\,\zeta(\beta)\,C_\beta}{N^\beta h^\beta}$$

where ζ denotes the Riemann-zeta-function

for $\hat{\omega}_k \approx g\left(\frac{k}{N}\right)$ and under mild assumptions on g,

$$C_\beta \leq \frac{\left(\zeta(\beta) + 1\right)\left|g^{(\beta-1)}\right|_V}{(2\pi)^\beta - 2^{1-\beta}\zeta(\beta)\left|g^{(\beta-1)}\right|_V}.$$

explicit estimates for

Dirichlet's kernel

$$1 - \frac{(1 - \ln 2h)}{Nh} \leq \lambda \leq 1 \leq \Lambda \leq 1 + \frac{(1 - \ln 2h)}{Nh}$$

Fejer's kernel

$$1 - \frac{\pi^2}{3N^2h^2} \leq \lambda \leq 1 \leq \Lambda \leq 1 + \frac{\pi^2}{3N^2h^2}$$

Jackson's kernel

$$1 - \frac{16\pi^4}{45N^4h^4} \leq \lambda \leq 1 \leq \Lambda \leq 1 + \frac{16\pi^4}{45N^4h^4}$$

condition number	equispaced	arbitrary
Dirichlet	$\frac{Nh+1}{Nh-1}$	$\frac{Nh+(1-\ln 2h)}{Nh-(1-\ln 2h)}$
Fejer	$\frac{N^2h^2+1}{N^2h^2-1}$	$\frac{N^2h^2+\frac{\pi^2}{3}}{N^2h^2-\frac{\pi^2}{3}}$

Multivariate setting

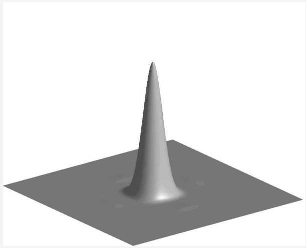
torus, metric

$$\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d, \qquad \text{dist}\,(\boldsymbol{x}, \boldsymbol{y}) := \min_{\boldsymbol{j}\in\mathbb{Z}^d} \|(\boldsymbol{x}+\boldsymbol{j})-\boldsymbol{y}\|_\infty$$

normal equation, kernels

$$\left(\boldsymbol{A}\hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\right)_{j,l} = K\left(\boldsymbol{x}_j - \boldsymbol{x}_l\right)$$

example, Jackson's kernel, $N = 22$



if $K(\mathbf{0}) = 1$ and $|K_N(\boldsymbol{x})| \le \frac{C_\beta}{N^\beta \|\boldsymbol{x}\|_\infty^\beta}$ for $\boldsymbol{x} \in \mathbb{T}^d$, then

$$1 - \frac{2d\zeta(\beta)C_\beta}{N^\beta h^{\beta+d-1}} \le \lambda \le 1 \le \Lambda \le 1 + \frac{2d\zeta(\beta)C_\beta}{N^\beta h^{\beta+d-1}}$$

Iterative methods

Landweber iteration

$$\hat{\boldsymbol{f}}_{l+1} = \hat{\boldsymbol{f}}_l + \alpha \hat{\boldsymbol{W}} \hat{\boldsymbol{z}}_l$$

steepest descent

$$\alpha_l = \frac{\hat{\boldsymbol{z}}_l^{\mathrm{H}} \hat{\boldsymbol{W}} \hat{\boldsymbol{z}}_l}{\boldsymbol{v}_l^{\mathrm{H}} \boldsymbol{W} \boldsymbol{v}_l}$$

conjugated gradient

$$\mathcal{K}_l\left(\boldsymbol{A}, \hat{\boldsymbol{r}}_0\right) := \text{span}\left(\hat{\boldsymbol{W}} \boldsymbol{A}^{\mathrm{H}} \boldsymbol{W} \boldsymbol{r}_0, \hat{\boldsymbol{W}} \boldsymbol{A}^{\mathrm{H}} \boldsymbol{W} \boldsymbol{A} \hat{\boldsymbol{W}} \boldsymbol{A}^{\mathrm{H}} \boldsymbol{W} \boldsymbol{r}_0, \ldots\right)$$

CGNR: $\|\boldsymbol{r}_l - \boldsymbol{r}^\dagger\|_{\boldsymbol{W}} \rightarrow \min$

CGNE: $\|\hat{\boldsymbol{f}}_l - \hat{\boldsymbol{f}}^\dagger\|_{\hat{\boldsymbol{W}}^{-1}} \rightarrow \min$

residuals

$$\boldsymbol{r}_l = \boldsymbol{f} - \boldsymbol{A}\hat{\boldsymbol{f}}_l, \quad \hat{\boldsymbol{z}}_l = \boldsymbol{A}^{\mathrm{H}} \boldsymbol{W} \boldsymbol{r}_l, \quad \boldsymbol{v}_l = \boldsymbol{A} \hat{\boldsymbol{W}} \hat{\boldsymbol{z}}_l$$

approximation problem, $N \leq \delta^{-1}$

$$\boldsymbol{A}^{\mathrm{H}}\boldsymbol{W}\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{A}^{\mathrm{H}}\boldsymbol{W}\boldsymbol{f}$$

ACT, CGNR (Feichtinger, Gröchenig, Strohmer)

$$\|\boldsymbol{r}_l - \boldsymbol{r}^{\dagger}\|_{\boldsymbol{W}} \leq 2\,(N\delta)^l\,\|\boldsymbol{r}_0 - \boldsymbol{r}^{\dagger}\|_{\boldsymbol{W}}$$

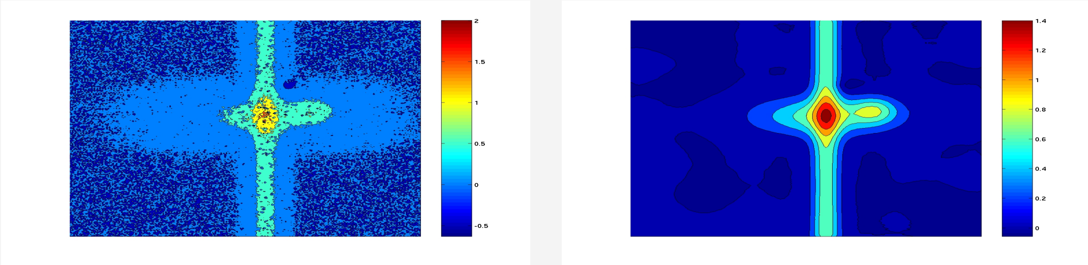
interpolation problem, $N \geq ch^{-1}$

$$\boldsymbol{A}\hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\tilde{\boldsymbol{f}} = \boldsymbol{f}, \qquad \hat{\boldsymbol{f}} = \hat{\boldsymbol{W}}\boldsymbol{A}^{\mathrm{H}}\tilde{\boldsymbol{f}}$$

CGNE (P., Kunis)

$$\|\hat{\boldsymbol{f}}_l - \hat{\boldsymbol{f}}^{\dagger}\|_{\hat{\boldsymbol{W}}^{-1}} \leq 2\left(\frac{c_\beta}{N^\beta h^\beta}\right)^l\|\hat{\boldsymbol{f}}_0 - \hat{\boldsymbol{f}}^{\dagger}\|_{\hat{\boldsymbol{W}}^{-1}}$$

Examples

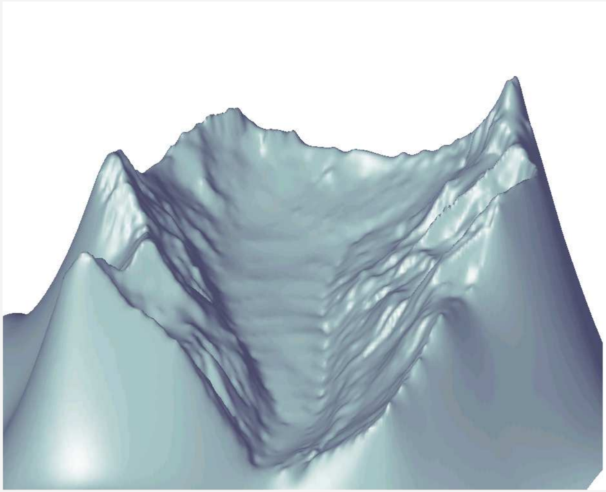


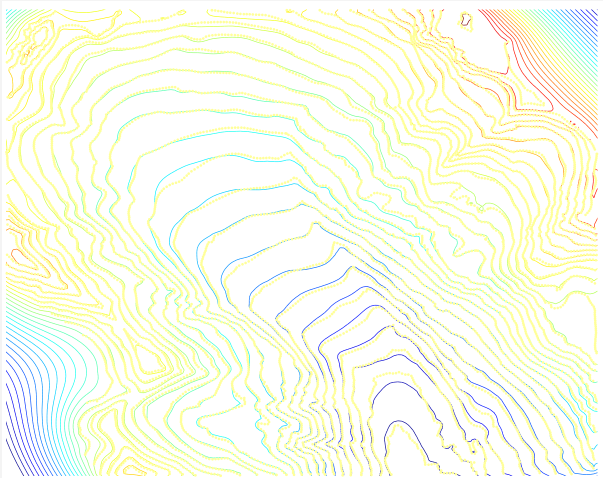
Franke function, $M = 100000$ random nodes, $N = 512$, L^2 and Sobolev-type, CGNE

image reconstruction, $M = 30000$ random nodes, $N = 256$, multiquadric-type, CGNR

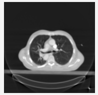
Glacier contour data

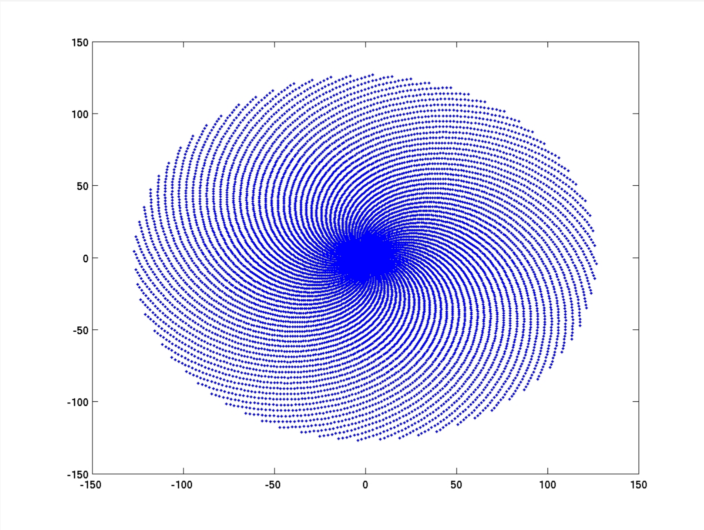
$M = 8345$ points, $N = 256$, multiquadric-type, CGNE

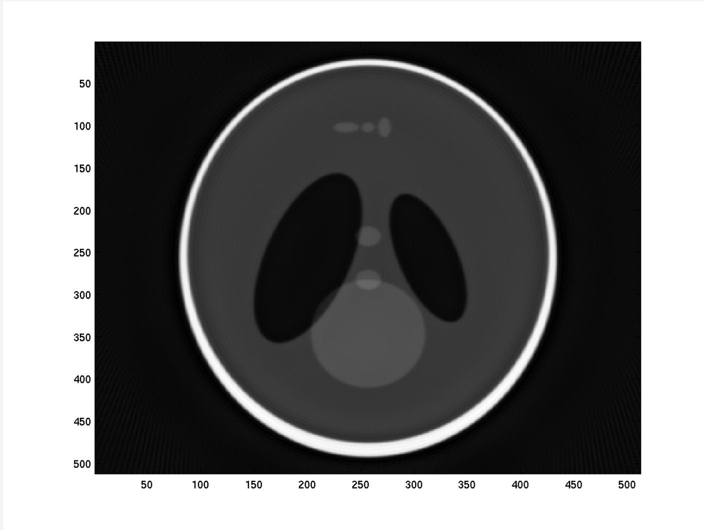




spiral MRI, reconstruction







data points

INFFT

Software available:

NFFT – C subroutine library (Kunis, P. 2002–2004)

http://www.math.uni-luebeck.de/potts/nfft

Features

– Implemented transforms for d dimensions
– Arbitrary-size transforms
– Works on any platform with a C compiler and the FFTW package
– iterative solution of the inverse transform (LANDWEBER, STEEPEST-DESCENT, CGNR, CGNE)

NFFT 2.0 manual online

Fast Fourier transform at nonequispaced knots,
A user's guide to a C-library (Kunis, P.)

Conclusions

- NFFT
  fast computation of NFFT
- iterative method for solving Vandermonde-like systems,
- Applications
  - MRI
  - Radon transform (P., Steidl 2002)
  - polar FFT, polar IFFT
  - next step

$$\|\boldsymbol{f} - \boldsymbol{A}\hat{\boldsymbol{f}}\|_{\boldsymbol{W}} + \lambda \|\hat{\boldsymbol{f}}\|_{\hat{\boldsymbol{W}}} \to \min$$