Multigrid preconditioning for anisotropic positive semidefinite block Toeplitz systems

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Outline

- Multigrid for symmetric positive definite Toeplitz and multilevel Toeplitz systems
- Problems caused by anisotropic BTTB systems
- Anisotropy along coordinate axes
- Anisotropy in other directions

Toeplitz matrices and generating functions

The 2π-periodic generating function

$$f(x) = \sum_{k=-\infty}^{\infty} t_k e^{kix}$$

corresponds to a series of Toeplitz matrices $(T_n)_{n\in\mathbb{N}}$ with

$$T_n = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & & t_{2-n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ t_{n-2} & & t_1 & t_0 & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix} = (t_{i-j})_{i,j=1}^{n}$$

The coefficients are computed by

$$t_k = \frac{1}{2\pi} \int_{\pi}^{\pi} f(x) e^{-ikx} dx$$

BTTB matrices and generating functions

In two dimensions, the generating function

$$f(x,y) = \sum_{j,k=-\infty}^{\infty} t_{j,k} e^{ijx+iky}$$

corresponds to a series of BTTB matrices $(T_{mn})_{mn\in\mathbb{N}}$ with

$$T_{mn} = \begin{pmatrix} T_0 & T_{-1} & \cdots & T_{2-n} & T_{1-n} \\ T_1 & T_0 & T_{-1} & & T_{2-n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ T_{n-2} & & T_1 & T_0 & T_{-1} \\ T_{n-1} & T_{n-2} & \cdots & T_1 & T_0 \end{pmatrix}$$

where each matrix T_j is itself a Toeplitz matrix. The coefficients are computed by

$$t_{j,k} = \frac{1}{4\pi^2} \int_{\pi}^{\pi} \int_{\pi}^{\pi} f(x,y) e^{-ijx-iky} dx$$

Example

$$\begin{aligned} f(x,y) &= 4 - 2\cos(x) - 2\cos(y) \\ &= 4 - e^{ix} - e^{-ix} - e^{iy} - e^{-iy} \end{aligned}$$

$$T_{mn} = \begin{pmatrix} T_0 & -I & 0 & \cdots & 0 \\ -I & T_0 & -I & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -I & T_0 & -I \\ 0 & \cdots & 0 & -I & T_0 \end{pmatrix}$$

with the identity matrix I and $T_0 = tridiag(-1, 4, -1)$

Multigrid for the BTTB matrix T_{mn}

Let T_{mn} be an spd and ill-conditioned BTTB matrix, corresponding to a generating function $f(x,y) \geq 0$ with $f(x_0, y_0) = 0$.

The prolongation is defined by $P = B \cdot E$

B: defined to deal with the zero (x_0, y_0) of f

E: elementary projection matrix $E_{mn} = E_m \otimes E_n$

with $E_n = I(:, 2:2:n)$

The coarse grid matrix T_C is computed with the Galerkin approach:

$$T_C = P^T T P = E^T (BTB) E$$

Multigrid for the generating function f

- Product $\hat{T} = B \cdot T \cdot B$ translated into functions

$$\hat{f}(x,y) = f(x,y) \cdot b(x,y)^2$$

- Elementary projection E: picks every second row/column and every second block row / block column. Then, $T_C = E^T \cdot \hat{T} \cdot E$ corresponds to

$$f_2(x,y) = \frac{1}{4} \cdot [\, \hat{f}(\frac{x}{2}, \frac{y}{2}) + \hat{f}(\frac{x}{2}+\pi, \frac{y}{2}) + \hat{f}(\frac{x}{2}, \frac{y}{2}+\pi) + \hat{f}(\frac{x}{2}+\pi, \frac{y}{2}+\pi) \,]$$

  f_2 is obtained by picking every second coefficient in x and every second coefficient in y .

- Possible choices for B , i.e. $b(x,y)$:

$$b(x,y) = (1 + \cos(x - x_0)) \cdot (1 + \cos(y - y_0))$$

$$b(x,y) = f(x, y+\pi) \cdot f(x+\pi, y) \cdot f(x+\pi, y+\pi)$$

Convergence of multigrid for BTTB

- If $b(x, y)$ is spd and if it has three zeros at
$$(x_0 + \pi, y_0)\,,\ (x_0, y_0 + \pi)\,,\ (x_0 + \pi, y_0 + \pi)\ ,$$
then $f_2(x, y) \geq 0$ with $f_2(2x_0, 2y_0) = 0$.

- Convergence proofs by S. Serra-Capizzano and R. Chan

- Multigrid fails if $f(x, y)$ has another zero at $(x_0 + \pi, y_0)\,, (x_0, y_0 + \pi)$, or $(x_0 + \pi, y_0 + \pi)$. Even if f is close to zero at one of these points, convergence is extremely slow.

Anisotropic problems: Examples

Anisotropic model PDE:

$$-\alpha \cdot u_{xx} - u_{yy} = r \quad (\alpha \ll 1)$$

Discretization leads to an anisotropic BTTB system with generating function

$$f(x,y) = \alpha \cdot (1 - \cos(x)) + (1 - \cos(y)) \quad (\alpha \ll 1)$$

The following function belongs to a denser matrix:

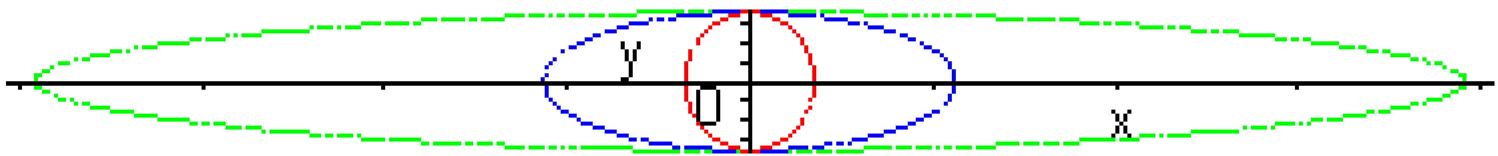
$$f(x,y) = \alpha \cdot x^2 + y^2 \, .$$

Its Fourier expansion is

$$f(x,y) = (1+\alpha)\frac{\pi^2}{3} + 4 \cdot \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} (\alpha \cos(jx) + \cos(jy))$$

Problems with multigrid methods

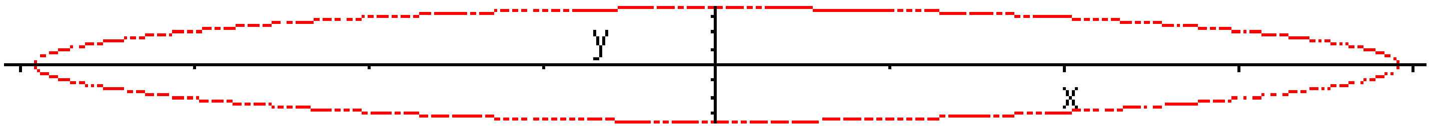
- If the anisotropy is strong, i.e. if $\alpha \ll 1$,
  f(x,0) becomes close to zero for all $x \in [0,2\pi]$
  $\Rightarrow$ standard multigrid fails
- Weak coupling in x-direction, i.e. level curves are extremely flat.
  Example: f(x,y)=0.01 for α = 1, 0.1, 0.01



Anisotropic problems and solution methods

- o Different types of anisotropies:
  - along coordinate axes, e.g.
    $f(x,y)=\alpha\cdot(1-\cos(x))+(1-\cos(y))$
  - along other directions, e.g.
    $f(x,y)=\alpha\cdot(1-\cos(x+y))+(1-\cos(x-y))$
- o Strategies for solving anisotropic systems:
  - Semicoarsening
  - Use of line smoothers, e.g. block-GS

Anisotropy along coordinate axes



Semicoarsening, a two-level method

T_{mn} : spd, ill-cond., anisotropic along x , e.g.

$$f(x,y) = \alpha \cdot (1 - \cos(x)) + (1 - \cos(y))$$

Functions for prolongation and coarse grid representation:

$$b(x,y) = 1 + \cos(y) \quad or$$
$$b(x,y) = f(x, y + \pi)$$

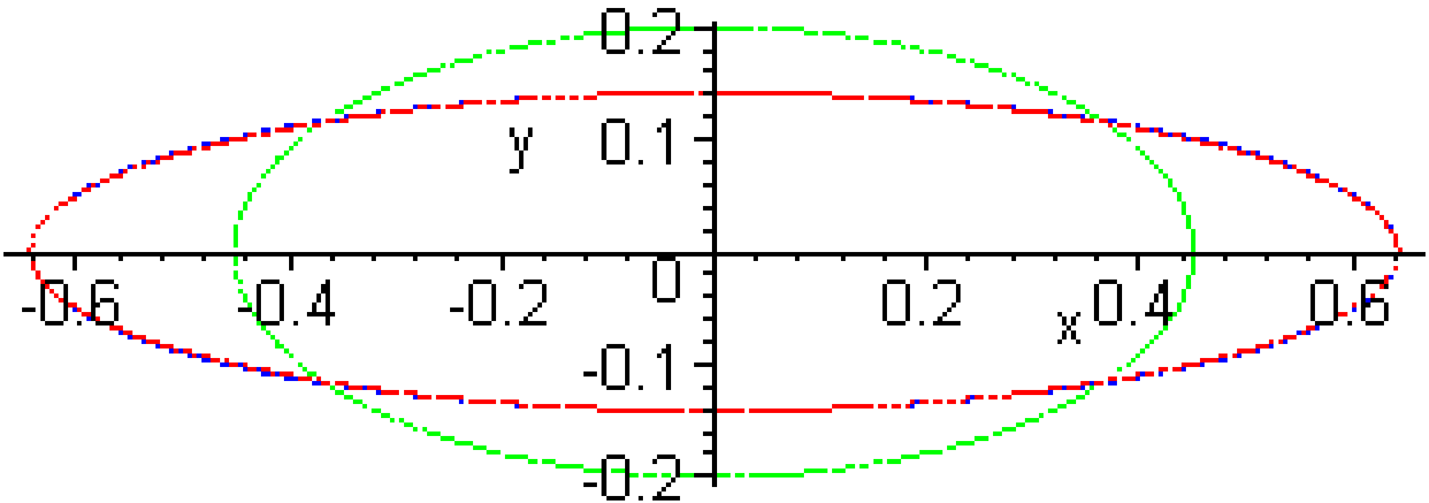
$$\hat{f}(x,y) = f(x,y) \cdot b(x,y)^2$$

$$f_2(x,y) = \frac{1}{2}\left(\hat{f}(x, \frac{y}{2}) + \hat{f}(x, \frac{y}{2} + \pi)\right)$$

T_C , which belongs to f_2 , has $n/2$ blocks of size n .

Semicoarsening and level curves

$f(x,y)=0.05\cdot(1-\cos(x))+(1-\cos(y))$



A two-level convergence result

Theorem:

Let T_{mn} be a spd BTTB matrix whose generating function is real-valued even and satisfies

$$\min_{(x,y)\in[-\pi,\pi]^2} \frac{f(x,y)}{1-\cos(y)} = C > 0 \ .$$

Furthermore, let the prolongation matrix correspond to the generating function

$$b(x,y) = 1 + \cos(y) \, .$$

Then the two-level method converges.

Semicoarsening, a multilevel method

Heuristic:

Apply semicoarsening steps until the system is not anisotropic anymore, i.e. until level curves are circles. Then switch to full coarsening.

Example:

$f(x,y)=(1-\cos(x))+0.01\cdot(1-\cos(y))$

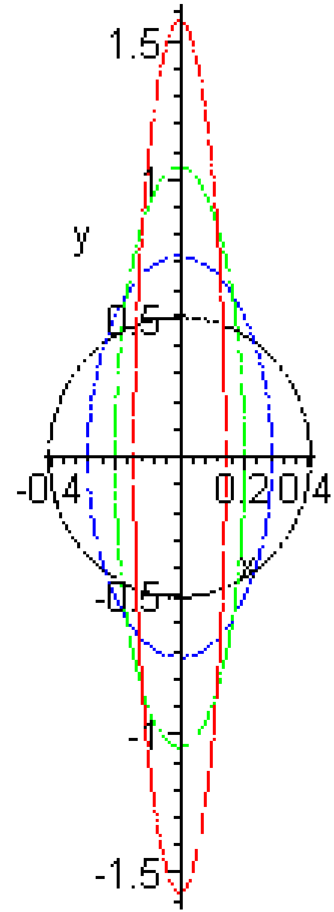
$b(x,y)=1+\cos(x)$ three times

Level curves: $f(x,y)=0.01$

$f_2(x,y)=0.01$

$f_3(x,y)=0.01$

$f_4(x,y)=0.01$



Numerical Results

Iteration numbers for the matrix T_{nn} corresponding to the generating functions

$$f(x,y) = (1-\cos(x)) + 0.01 \cdot (1-\cos(y)) \quad .$$

coarsening	$n{=}2^6{-}1$	$n{=}2^7{-}1$	$n{=}2^8{-}1$
x,xy,xy	63	65	66
x,x,x,xy	16	17	17
x,x,x,x,x	20	20	19

$$f(x,y) = (1-\cos(x)) + 0.001 \cdot (1-\cos(y)) \quad .$$

coarsening	$n{=}2^6{-}1$	$n{=}2^7{-}1$	$n{=}2^8{-}1$
x,xy,xy	125	181	> 200
x,x,x,xy	32	45	50
x,x,x,x,x	15	15	15

Size of T_C on the coarsest level: $\frac{n^2}{32}$-by-$\frac{n^2}{32}$

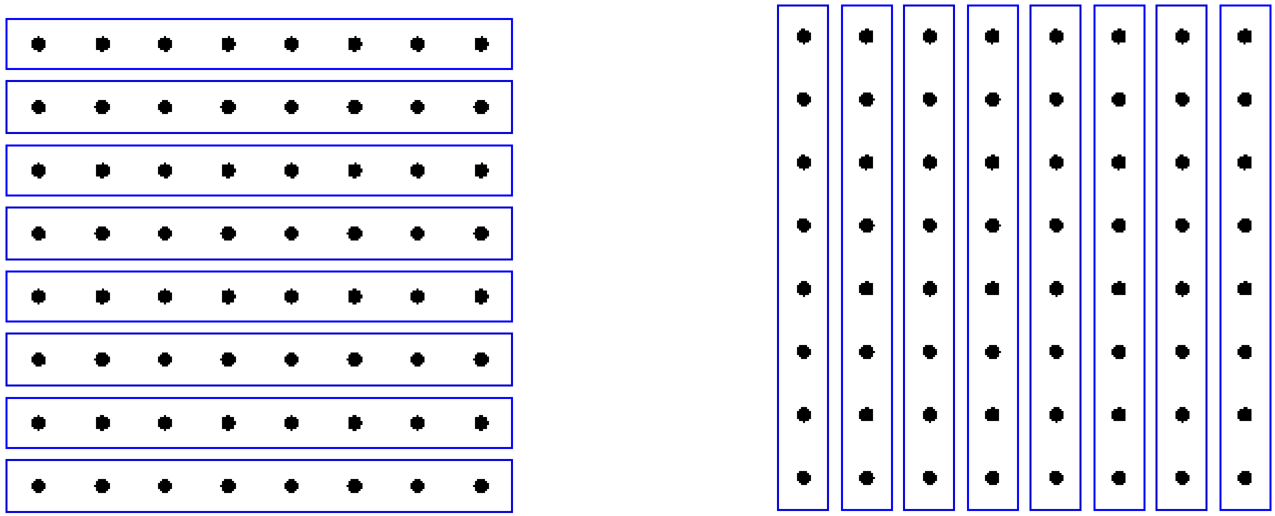
The use of line smoothers

Multigrid for moderately anisotropic problems:

Standard coarsening, e.g.

$$b(x,y)=(1+\cos(x))\cdot(1+\cos(y))$$

and line smoother such as the block-Jacobi or the block-Gauss-Seidel method (all blocks have size n)



Numerical Results

Iteration numbers for T_{nn} corresponding to

$$f(x,y) = (1 - \cos(x)) + 0.005 \cdot (1 - \cos(y))$$

$$b(x,y) = (1 + \cos(x)) \cdot (1 + \cos(y))$$

Smoothing: block-Jacobi with blocks of size n

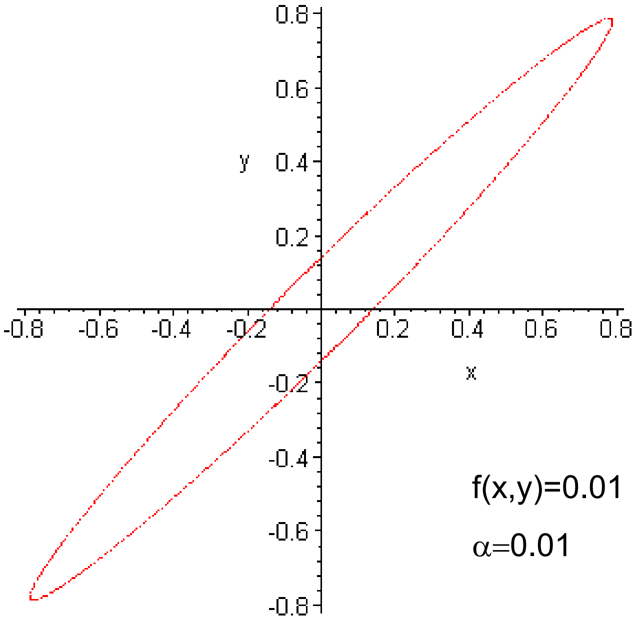
levels	$n{=}2^5{-}1$	$n{=}2^6{-}1$	$n{=}2^7{-}1$	$n{=}2^8{-}1$
2	12	14	14	14
3	15	16	17	17
4	18	19	19	19

Anisotropy in other directions

Examples

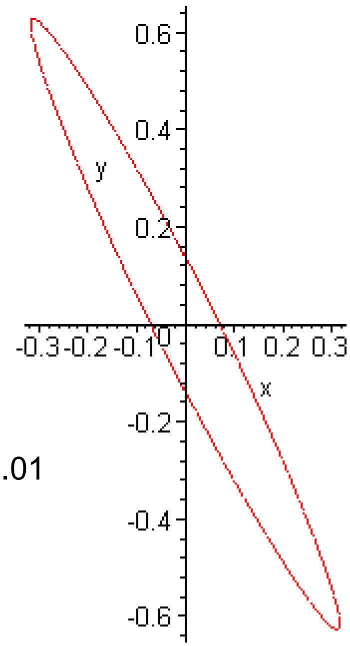
$f(x,y)=\alpha\cdot(1-\cos(x+y)) + (1-\cos(x-y))$

$g(x,y)=(1-\cos(2\cdot x+y)) + \alpha\cdot(1-\cos(x-2\cdot y))$



f(x,y)=0.01

α=0.01



g(x,y)=0.01

α=0.01

Coarsening in other directions

New coordinate system:

$s = x + y$

$t = x - y$

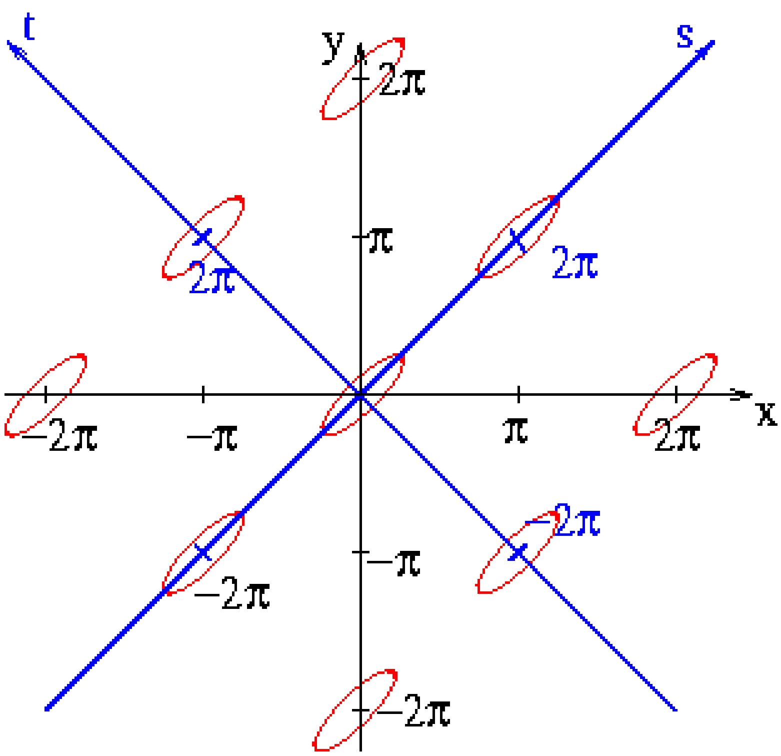
$f(s,t) = \alpha \cdot (1 - \cos(s)) + (1 - \cos(t))$

Semicoarsening with

$b(s,t) = 1 + \cos(t)$

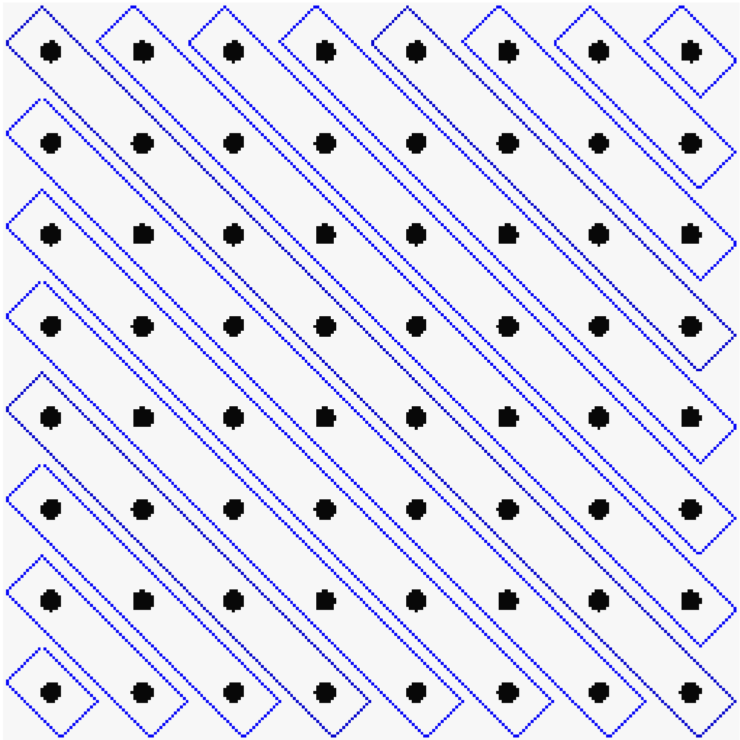
Full coarsening with

$b(s,t) = (1 + \cos(s)) \cdot (1 + \cos(t))$



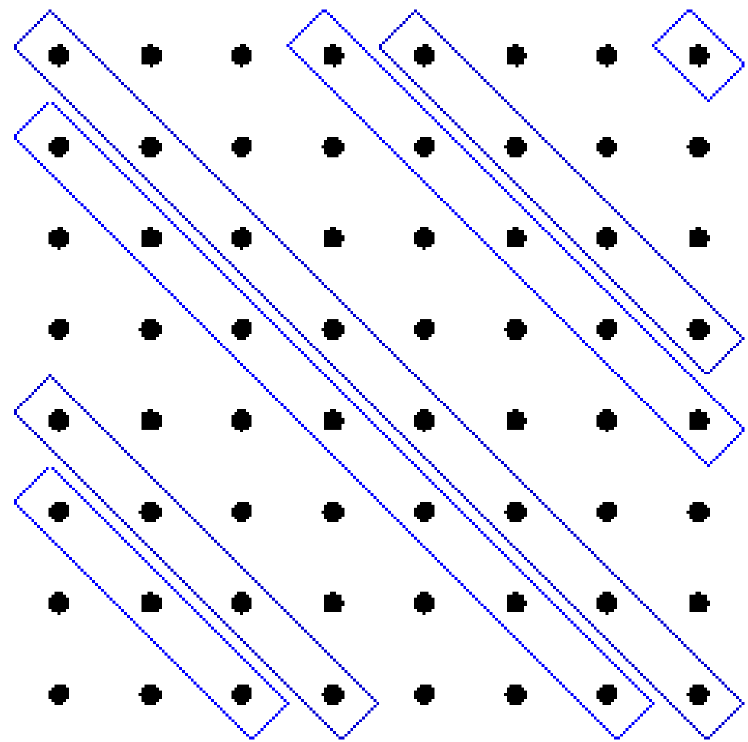
Semicoarsening in matrix notation

- Permutation of T_{nn} and partitioning into blocks, one block for each diagonal
- B_S=diag(B_1,…,B_n,…,B_1) with B_1=2 , B_k=tridiag(1,2,1)
- Coarse grid matrix computed by $B_S \cdot T_{nn} \cdot B_S$ and then by eliminating every second row/column within each block



Full coarsening in matrix notation

- $B_F=B_S+B_T$, where B_T has blocks $BI_k=tridiag_k(0.5,1,0.5)$ in the second lower block diagonal
- Coarse grid matrix computed by $B_F \cdot T_{nn} \cdot B_F$ and then by eliminating every second row/column within each block and every second block row/column
- Multilevel method:
  Apply semicoarsening until the level curves are circles, then proceed with full coarsening



Numerical results

Iteration numbers for the matrices T_{nn} corresponding to the generating functions

$$f(x,y) = 0.01 \cdot (1 - \cos(x+y)) + (1 - \cos(x-y))$$

coarsening	$n{=}2^6{-}1$	$n{=}2^7{-}1$	$n{=}2^8{-}1$
t,st,st	43	45	45
t,t,t,st	17	18	18
t,t,t,t,t	21	21	21

$$f(x,y) = 0.001 \cdot (1 - \cos(x+y)) + (1 - \cos(x-y))$$

coarsening	$n{=}2^6{-}1$	$n{=}2^7{-}1$	$n{=}2^8{-}1$
t,st,st	82	97	104
t,t,t,st	28	32	34
t,t,t,t,t	17	17	17

Standard coarsening and line smoothing

Coarsening: b(x,y)=(1+cos(x))(1+cos(y))

Smoothing: Block-Jacobi or block-GS with blocks of variable size (one for each diagonal)

Problem: f(x,y) has zeros at (0,0) and (π,π)

Solution: For computation of the coarse grid matrix consider T_{nn} as a block-BTTB matrix with blocks of size 2

Example: $f(x,y)=0.05\cdot(1-\cos(x))+(1-\cos(y))$

levels	$n=2^5-1$	$n=2^6-1$	$n=2^7-1$	$n=2^8-1$
2	13	14	14	13
3	21	25	26	25