Kreĭn–Langer factorization, J–unitary factorization and generalized Schur functions in the nonstationary setting

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1 Introduction

As is well–known, the classical Schur algorithm associates to a function s analytic and contractive in the open unit disk $\mathbb{D}$ a sequence, finite or infinite of numbers $\rho_n \in \mathbb{D}$ via the recipe

$$\begin{aligned} s_{n+1}(z) &= \frac{s_n(z) - s_n(0)}{z(1 - \overline{s_n(0)}s_n(z))} \\ \rho_n &= s_n(0), \end{aligned}$$

with $s_0(z) = s(z)$. The sequence stops at step n if $|s_n(0)| = 1$.

In the late fifties of the previous century the extension of the Schur algorithm to the case of functions which have a finite number of poles in $\mathbb{D}$, but still have (non–tangential) contractive values on the unit circle was considered by C. Chamfy and J. Dufresnoy; see [?], [?].

The functions considered by Chamfy and Dufresnoy was considered later in a different setting by Kreĭn and Langer, and characterized as functions s analytic in some open subset of $\mathbb{D}$ and for which the kernel $\frac{1-s(z)\overline{s(w)}}{1-z\overline{s(w)}}$ has a finite number of negative squares in the domain of analyticity of s. Such functions are called generalized Schur functions. Kreĭn and Langer proved that these are exactly functions of the form

$$s(z) = \frac{s_0(z)}{b(z)} \tag{1.1}$$

where s_0 is a Schur function and b is a finite Blaschke product. When assuming that b and s_0 have no common zeros, the degree of b is equal to the number of negative squares of the kernel k_s.

Problem 1.1 *If one start from a finite polynomial $s_0 + \cdots + z^N s_N$ and ask for all its extensions as Taylor series of generalized Schur functions at the origin, one has the Schur–Takagi problem.*

The solution of this problem is given by the generalized Schur algorithm. The strategy is as follows: one considers the space $\mathcal{M}_N(s)$ spanned by the polynomials $f_0, f_1, \ldots$ defined by

$$\begin{aligned}
f_0(z) &= \begin{pmatrix} 1 \\ -s_0^* \end{pmatrix} \\
f_1(z) &= z\begin{pmatrix} 1 \\ -s_0^* \end{pmatrix} + \begin{pmatrix} 0 \\ -s_1^* \end{pmatrix} \\
f_2(z) &= z^2\begin{pmatrix} 1 \\ -s_0^* \end{pmatrix} + z\begin{pmatrix} 0 \\ -s_1^* \end{pmatrix} + \begin{pmatrix} 0 \\ -s_2^* \end{pmatrix} \\
&\vdots \\
f_N(z) &= z^N\begin{pmatrix} 1 \\ -s_0^* \end{pmatrix} + z^{N-1}\begin{pmatrix} 0 \\ -s_1^* \end{pmatrix} + \begin{pmatrix} 0 \\ -s_N^* \end{pmatrix}
\end{aligned} \tag{1.2}$$

When this subspace is non degeenrate in $\mathbf{H}_{2,J}$ it is a finite dimensional backward shift invariant reproducing kenrel space, with reproducing kernel of the form $\frac{J-\Theta(z)J\Theta(w)^*}{1-zw^*}$. The set of all solutions to the Schur–Takagi problem is given by the corresponding linear

$$s(z) = \frac{a(z)\sigma(z)+b(z)}{c(z)\sigma(z)+d(z)} \stackrel{\text{def.}}{=} T_{\Theta(z)}(\sigma(z)) \tag{1.3}$$

where

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

is a polynomial matrix–valued function which is moreover J–unitary on the real line, with $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

In fact every generalized Schur functions is of this form when Θ is obtained from the Schur–Takagi problem for some N. form.

The strategy is as follows: assuming the function analytic in a neighborhood of the origin we consider its Taylor expansion at the origin

$$s(z) = s_0 + zs_1 + z^2s_2 + \cdots \tag{1.4}$$

The main idea is to consider the space $\mathcal{M}(s)$ spanned by the polynomials $f_0, f_1, \ldots$ defined by

$$\begin{aligned}
f_0(z) &= \begin{pmatrix} 1 \\ -s_0^* \end{pmatrix} \\
f_1(z) &= z\begin{pmatrix} 1 \\ -s_0^* \end{pmatrix} + \begin{pmatrix} 0 \\ -s_1^* \end{pmatrix} \\
f_2(z) &= z^2\begin{pmatrix} 1 \\ -s_0^* \end{pmatrix} + z\begin{pmatrix} 0 \\ -s_1^* \end{pmatrix} + \begin{pmatrix} 0 \\ -s_2^* \end{pmatrix} \\
&\vdots
\end{aligned} \tag{1.5}$$

$\mathcal{M}(s)$ is an indefinite subsace of $\mathbf{H}_{2,J}$ (it will be positive when s is a Schur function).

The map τ

$$f \mapsto \begin{pmatrix} 1 & -s \end{pmatrix} f \tag{1.6}$$

is an isometry from $\mathcal{M}(s)$ into the Pontryagin space $\mathcal{P}(s)$; the range of τ is dense in $\mathcal{P}(s)$. It follows (see [**?**, Proposition 6.3]) that $\mathcal{M}(s)$ contains a maximum finite dimensional negative subspace and that there is an N such that the span of $f_0, \ldots, f_N$ is nondegenerate. It is then a $\mathcal{P}(\Theta_N)$ space where Θ_N is a J–unitary polynomial. It is elementary when N is taken minimal. The procedure may be
The case of J-inner polynomials correspond to the classical Schur algorithm. A fundamental result of Chamfy and Dufresnoy is that there are three kinds of elementary J–unitary polynomials, from which the Θ are built; see [**?**] for a recent account using the theory of reproducing kernel spaces. iterated.

The purpose of this paper is to extend this analysis to time–varying setting, that is when one replaces analytic functions by upper triangular operators and the complex numbers by diagonal operators. In particular we characterize the counterparts of the generalized Schur functions in the time–varying case and develop a Schur–type algorithm for such functions. A number of difficulties arise. In particular:

- The notion of finite number of negative squares does not exist (it could be extended, but does not seem helpful). We are in the Kreĭn space case.
- What is the time–varying analogue of generalized Schur functions? By analogy with the Kreĭn–Langer representation we will consider operators of the form TV^*, where T an upper triangular contraction and V is inner. We will consider the Schur–Takagi problem in this class.
- What is the analogue of the expansion (**??**) for such operators?

Our main goal is to solve the Schur–Takagi Problem;

Problem 1.2 *Given $N+1$ diagonal operators $S_{[0]}, \ldots, S_{[N]}$ find all operators of the form TV^* where T is an upper triangular contraction and V is inner of finite degree such that*

$$T - \left(\sum_0^N Z^n S_{[n]} \right) V \in Z^{N+1}\mathcal{U},$$

(meaning that the first $N+1$ diagonals of $T - \left(\sum_0^N Z^n S_{[n]}\right) V$ vanish).

We solve this problem by reducing to the case of operator–valued functions of a complex variable by using the Zadeh transform; see Problem **??**. A similar strategy was used in [**?**] to study Brune sections in the time–varying setting.

2 Preliminaries: the time–varying case

In this section we review the non–stationary setting. We follow the analysis and notations of [?] and [?]. Let $\mathcal{M}$ be a separable Hilbert space, "the coefficient space". As in [?, Section 1], the set of bounded linear operators from the space $\ell^2_{\mathcal{M}}$ of square summable sequences with components in $\mathcal{M}$ into itself is denoted by $\mathcal{X}\left(\ell^2_{\mathcal{M}}\right)$, or $\mathcal{X}$. The space $\ell^2_{\mathcal{M}}$ is taken with the standard inner product. Let Z be the bilateral backward shift operator

$$(Zf)_i = f_{i+1}, \qquad i = \ldots, -1, 0, 1, \ldots$$

where $f = (\ldots, f_{-1}, \boxed{f_0}, f_1, \ldots) \in \ell^2_{\mathcal{M}}$. The operator Z is unitary on $\ell^2_{\mathcal{M}}$ i.e. $ZZ^* = Z^*Z = I$, and

$$\pi^* Z^j \pi = \begin{cases} I_{\mathcal{M}} & \text{if} \quad j = 0 \\ 0_{\mathcal{M}} & \text{if} \quad j \neq 0. \end{cases}$$

where π denote the injection map

$$\pi : u \in \mathcal{M} \to f \in \ell^2_{\mathcal{M}} \quad \text{where} \quad \begin{cases} f_0 = u \\ f_i = 0, \quad i \neq 0 \end{cases} .$$

We define the space of upper triangular operators by

$$\mathcal{U}\left(\ell^2_{\mathcal{M}}\right) = \left\{ A \in \mathcal{X}\left(\ell^2_{\mathcal{M}}\right) \mid \pi^* Z^i A Z^{*j} \pi = 0 \quad \text{for} \quad i > j \ \right\},$$

and the space of lower triangular operators by

$$\mathcal{L}\left(\ell^2_{\mathcal{M}}\right) = \left\{ A \in \mathcal{X}\left(\ell^2_{\mathcal{M}}\right) \mid \pi^* Z^i A Z^{*j} \pi = 0 \quad \text{for} \quad i < j \ \right\}.$$

The space of diagonal operators $\mathcal{D}\left(\ell^2_{\mathcal{M}}\right)$ consists of the operators which are both upper and lower triangular. As for the space $\mathcal{X}$, we usually denote these spaces by $\mathcal{U}$, $\mathcal{L}$ and $\mathcal{D}$.

Let $A^{(j)} = Z^{*j} A Z^j$ for $A \in \mathcal{X}$ and $j = \ldots, -1, 0, 1, \ldots$; note that $\left(A^{(j)}\right)_{st} = A_{s-j,t-j}$ and that the maps $A \mapsto A^{(j)}$ take the spaces $\mathcal{L}$, $\mathcal{D}$, $\mathcal{U}$ into themselves. Clearly, for A and B in $\mathcal{X}$ we have that $(AB)^{(j)} = A^{(j)} B^{(j)}$ and $A^{(j+k)} = \left(A^{(j)}\right)^{(k)}$.

In [?] it is shown that for every $F \in \mathcal{U}$, there exists a unique sequence of operators $F_{[j]} \in \mathcal{D}$, $j = 0, 1, \ldots$ such that

$$F - \sum_{j=0}^{n-1} Z^j F_{[j]} \in Z^n_{\mathcal{M}} \mathcal{U}.$$

In fact, $\left(F_{[j]}\right)_{ii} = F_{i-j,i}$ and we can formally represent $F \in \mathcal{U}$ as the sum of its diagonals

$$F = \sum_{n=0}^{\infty} Z^n F_{[n]}. \tag{2.1}$$

More generally one can associate to an element $F \in \mathcal{X}$ a sequence of diagonal operators such that, formally $F = \sum_{\mathbb{Z}} Z^n F_{[n]}$. Recall the well known fact that even when F is a bounded operator the formal sums $\sum_{n=0}^{\infty} Z^n F_{[n]}$ and $\sum_{-\infty}^{0} Z^n F_{[n]}$ need not define bounded operators. See e.g. [**?**, p. 29] for a counterexample.

When the operator F is in the Hilbert–Schmidt class (we will use the notation $F \in \mathcal{X}_2$) the above representation is not formal but converges both in operator and Hilbert–Schmidt norm. Indeed, each of the diagonal operator $F_{[n]}$ is itself a Hilbert–Schmidt operator and we have:

$$\|F\|_{\mathcal{X}_2}^2 = \sum_{n=0}^{\infty} \|F_{[n]}\|_{\mathcal{X}_2}^2 < \infty \tag{2.2}$$

and

$$\begin{aligned} \|F - \sum_{-M}^{N} Z^n F_{[n]}\|^2 &\leq \|F - \sum_{-M}^{N} Z^n F_{[n]}\|_{\mathcal{X}_2}^2 \\ &= \sum_{-\infty}^{-M-1} \|F_{[n]}\|_{\mathcal{X}_2}^2 + \sum_{N+1}^{\infty} \|F_{[n]}\|_{\mathcal{X}_2}^2 \\ &\to 0 \quad \text{as} \quad N, M \to \infty. \end{aligned}$$

Here we used the fact that the operator norm is less that the Hilbert–Schmidt norm:

$$\|F\| \leq \|F\|_{\mathcal{X}_2}. \tag{2.3}$$

See e.g. [**?**, EVT V.52].

Definition 2.1 *The Hilbert space of upper triangular (resp. diagonal) Hilbert–Schmidt operators will be denoted by* $\mathcal{U}_2$ *(resp. by* $\mathcal{D}_2$*).*

Definition 2.2 *Let* $\Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{U}^{2\times 2}$ *and let* $W \in \mathcal{D}$*. We define*

$$\Theta^\wedge(W) = \begin{pmatrix} A^\wedge(W) & B^\wedge(W) \\ C^\wedge(W) & D^\wedge(W) \end{pmatrix}. \tag{2.4}$$

An important result is the inner–outer factorization theorem; see [**?**, Theorem 7.1 p. 150].

Theorem 2.3 *Every* $T \in \mathcal{U}$ *admits a factorization* $T = T_\ell V$ *where* T_ℓ *is left–outer and* V *is a causal isometry. The operator* V *is inner if and ony if the left kernel of* T^* *reduces to zero.*

This allows us to define the counterparts of the generalized Schur functions in terms of Kreĭn–Langer type factorization

3 The Zadeh transform

Definition 3.1 *Let* $U \in \mathcal{U}$ *with formal representation* $U = \sum_{n=0}^{\infty} Z^n U_{[n]}$*. The Zadeh transform of* U *is the* $\mathcal{X}$*–valued function defined by*

$$U(z) = \sum_{n=0}^{\infty} z^n Z^n U_{[n]}, \qquad z \in \mathbb{D}. \tag{3.1}$$

We note that $\|U_{[n]}\| \le \|U\|$ and hence the series (**??**) converges in the operator norm for every $z \in \mathbb{D}$ and that $U(z)$ is called in [**?**] the symbol of U; see [**?**, p. 135].

Theorem 3.2 *Let U, U_1 and U_2 be upper-triangular operators. Then,*

$$\|U(z)\| \le \|U\|, \qquad z \in \mathbb{D} \tag{3.2}$$

and

$$(U_1U_2)(z) = U_1(z)U_2(z). \tag{3.3}$$

Proof: A proof of the first claim can be found in [**?**, Theorem 5.5 p 136]. The key ingredient in the proof is that every upper triangular contraction is the characteristic function of a unitary colligation; see [**?**, Theorem 5.3 p. 135]. To prove the second claim we remark, as in [**?**, p. 136] that

$$U(z) = \Lambda(z)U\Lambda(z)^{-1} \tag{3.4}$$

where $z \neq 0$ and where $\Lambda(z)$ denotes the *unbounded* diagonal operator defined by

$$\Lambda(z) = \text{diag } \begin{pmatrix} \cdots & z^2 I_{\mathcal{M}} & zI_{\mathcal{M}} & I_{\mathcal{M}} & z^{-1}I_{\mathcal{M}} & z^{-2}I_{\mathcal{M}} & \cdots \end{pmatrix}. \tag{3.5}$$

Of course some care is needed with (**??**). What is really meant is that the *a priori* unbounded operator on the right coincides with the bounded operator on the left on a dense set (for instance on the set of sequences with finite support):

$$\Lambda(z)U\Lambda(z)^{-1}u = U(z)u \tag{3.6}$$

where $u \in \ell^2_{\mathcal{M}}$ is a sequence with finite support.

We now proceed as follows to prove (**??**). We start with a sequence u as above. Then:

$$\begin{aligned} \Lambda(z)^{-1}u \in \ell^2_{\mathcal{M}} & \quad \text{(since } u \text{ has finite support)} \\ U_2\Lambda(z)^{-1}u \in \ell^2_{\mathcal{M}} & \quad \text{(since dom } U_2 = \ell^2_{\mathcal{M}}) \\ \Lambda(z)U_2\Lambda(z)^{-1}u \in \ell^2_{\mathcal{M}} & \quad \text{(by (\textbf{??}))} \\ \Lambda(z)^{-1}\Lambda(z)U_2\Lambda(z)^{-1}u \in \ell^2_{\mathcal{M}} & \quad \text{since it is equal to } U_2\Lambda(z)^{-1}u \end{aligned}$$

and so (still for sequences with finite support)

$$U_1\Lambda(z)^{-1}\Lambda(z)U_2\Lambda(z)^{-1}u = U_1U_2\Lambda(z)^{-1}u$$

and applying (**??**) we conclude that

$$\Lambda(z)U_1\Lambda(z)^{-1}\Lambda(z)U_2\Lambda(z)^{-1}u = \Lambda(z)U_1U_2\Lambda(z)^{-1}u = (U_1U_2)(z)u \in \ell^2_{\mathcal{M}}.$$

These same equalities prove (**??**). $\square$

Theorem 3.3 *Assume that $V \in \mathcal{U}$ is unitary. Then the Zadeh transform $z \mapsto V(z)$ is an inner $\mathcal{U}$–valued function*

Proof: First, we note that the operator–valued function $V(z)$ is analytic in the open unit disk $\mathbb{D}$ and satisfy by Theorem **??**

$$\|V(z)\| \leq \|V\|$$

Therefore, by [**?**, Theorem A p. 84] the limit $\lim_{r\to 1} V(re^{it})$ in the strong operator topology exists for almost every $t \in [0, 2\pi]$ and it is easily checked that

$$\lim_{r\to 1} V(re^{it}) = V(e^{it}) = \sum_{n=0}^{\infty} e^{int} Z^n V_{[n]}.$$

But

$$V(e^{it}) = \Lambda(e^{it}) V \Lambda(e^{it})^*,$$

where $\Lambda(e^{it}) = \mathrm{diag}(e^{int})$ is unitary. Hence also $V(e^{it})$ is unitary. $\square$

Proposition 3.4 *Let V be an upper triangular inner function. Then the reproducing kernel Hilbert space with reproducing kernel*

$$\frac{I - V(z)V(w)^*}{1 - zw^*}$$

is infinite dimensional. In particular, $z \mapsto V(z)$ is not a finite (operator–valued) Blaschke product.

Proof:

$\square$

The Zadeh transform also makes sense for sequences $S_{[n]}$ of diagonal operators which do not define bounded operators.

The Zadeh transform allows us to define the analogue of the Taylor expansion at the origin for certain non upper triangular operators.

Proposition 3.5 *Let T and V be upper triangular operators and assume that the main diagonal of V is invertible. Then there exists a sequence of diagonal operators such that*

$$T(z)V(z)^{-1} = \sum_{0}^{\infty} z^n Z^n S_{[n]} \tag{3.7}$$

for z in a neighbourhood of the origin.

Proof:

□

In the case when the operator V in Proposition **??** has an upper triangular inverse, the series (**??**) is just the Zadeh transform of the operator TV^{-1}. In general, however, the radius of convergence of this series may be strictly less than one.
Of particular interest to us is the class of operators S of the form

$$S = TV^* \tag{3.8}$$

where T is an upper triangular contraction and V is inner of finite degree. The representation (**??**) can be viewed as the non-stationary counterpart of the representation (**??**). In this class we consider the *Schur–Tagaki interpolation problem*:

Problem 3.6 *Given $N+1$ diagonal operators $S_{[0]}, \ldots, S_{[N]}$ find all operators of the form TV^* where T is an upper triangular contraction and V is inner of finite degree such that*

$$T(z)V(z)^{-1} = \sum_0^N z^n Z^n S_{[n]} + o(z^n) \text{ as } z \to 0.$$

Proposition 3.7 *The set of power series of the form*

$$\sum_0^\infty z^n Z^n D_{[n]}$$

converging in a neighborhood of the origin form an algebra. An element is invertible in this algebra if and only if $D_{[0]}$ is invertible.

Definition 3.8 *We denote by $\mathcal{Z}$ the above algebra.*

4 The solution

In a way similar to the classical case we build from the interpolation data (or from the whole sequence) a backward–shift invariant subspace of $\mathcal{U}_{2,J}$ in the following way.

We reduce the study to the case of functions of the complex variable z. Let

$$\mathcal{M} = \text{l.s. } \{f_0, \ldots .f_N\} \subset \mathbf{H}_2(\mathcal{L}_2).$$

where

$$
\begin{aligned}
f_0(z) &= \begin{pmatrix} I \\ S_{[0]}^* \end{pmatrix}\\
f_1(z) &= z\begin{pmatrix} I \\ S_{[0]}^* \end{pmatrix} + \begin{pmatrix} 0 \\ S_{[1]}^* Z^* \end{pmatrix}\\
&\vdots\\
f_N(z) &= z^N\begin{pmatrix} I \\ S_{[0]}^* \end{pmatrix} + z^{N-1}\begin{pmatrix} 0 \\ S_{[1]}^* Z^* \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ S_{[N]}^{(N-1)*} Z^{(N-1)}* \end{pmatrix},
\end{aligned}
\tag{4.1}
$$

that is,

$$
\begin{pmatrix} f_0(z) & f_1(z) & \cdots & f_N(z) \end{pmatrix} = C(I-zA)^{-1} \tag{4.2}
$$

where

$$
C = \begin{pmatrix} I & 0 & \cdots & 0 \\ S_{[0]}^* & S_{[1]}^* Z^* & \cdots & S_{[N]}^* Z^{(N-1)*} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & I & \cdot & & 0 & \\ 0 & 0 & I & 0 & 0 & \\ & & & & & \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.3}
$$

and consider the operator Stein equation

$$
\mathbb{Q} - A^*\mathbb{Q}A = C^*JC. \tag{4.4}
$$

Theorem 4.1 *Assume that the solution to* (**??**) *is invertible. Then $\mathcal{M}$ is a reproducing kernel Kreĭn space and there exists a J–unitary operator polynomial such that its reproducing kernel is of the form*

$$
\frac{J - \Theta(z)J\Theta(w)^*}{1-zw^*}. \tag{4.5}
$$

Finally it holds that

$$
\mathcal{M} = \mathbf{H}_2(\mathcal{L}_2) \ominus \Theta\mathbf{H}_2(\mathcal{L}_2). \tag{4.6}
$$

and

$$
\begin{pmatrix} I & -(S_{[0]} + zZS_{[1]} + \cdots + z^N Z^N S_{[N]}) \end{pmatrix} \Theta(z) = z^{N+1}Z^{N+1}\begin{pmatrix} \alpha(z) & \beta(z) \end{pmatrix} \tag{4.7}
$$

where $\alpha, \beta \in \mathcal{Z}$.

Proof: The proof follows the case of finite dimensional case, and we only sketch it. Set

$$
\Theta(z) = I - (1-z)C(I-zA)^{-1}\mathbb{Q}^{-1}(I-A)^{-1}C^*J. \tag{4.8}
$$

$$
\frac{J-\Theta(z)J\Theta(w)^*}{1-z\overline{w}} = \frac{J-(I-(1-z)F(z)\mathbb{P}^{-1}F(1)^*J)J(I-(1-w)F(w)\mathbb{P}^{-1}F(1)^*J)^*}{1-z\overline{w}},
$$

then, after some developments we get:

$$
C(I-zA)^{-1}\mathbb{P}^{-1}(I-wA)^{-*}C^* =
$$

$$= C\Big(\frac{(1-\overline{w})(I-A)^{-1}\mathbb{P}^{-*}(I-wA)^{-*}+(1-z)(I-zA)^{-1}\mathbb{P}^{-1}(I-A)^{-*}-}{1-z\overline{w}}$$

$$\frac{(1-z)\overline{(1-w)}(I-zA)^{-1}\mathbb{P}^{-1}(I-A)^{-*}C^*JC(I-A)^{-1}\mathbb{P}^{-*}(I-wA)^{-*}}{1-z\overline{w}}\Big)C^*$$

or,

$$(I-zA)^{-1}\mathbb{P}^{-1}(I-A)^{-*}[(1-z\overline{w})(I-A)^*\mathbb{P}^*(I-A)](I-A)^{-1}\mathbb{P}^{-*}(I-wA)^{-*} =$$

$$= (I-zA)^{-1}\mathbb{P}^{-1}(I-A)^{-*}[(1-\overline{w})(I-A)^*\mathbb{P}(I-zA)+(1-z)(I-wA)^*\mathbb{P}^*(I-A)-$$

$$-(1-z)\overline{1-w}C^*JC](I-A)^{-1}\mathbb{P}^{-*}(I-wA)^{-*}$$

Therefore we get:

$$C^*JC =$$

$$= \frac{(1-\overline{w})(I-A)^*\mathbb{P}(I-zA)+(1-z)(I-wA)^*\mathbb{P}^*(I-A)-(1-z\overline{w})(I-A)^*\mathbb{P}^*(I-A)}{(1-z)(1-\overline{w})}$$

So, after some calculations under the assumption that $\mathbb{P}^* = \mathbb{P}$ we get:

$$C^*JC = \mathbb{P} - A^*\mathbb{P}A.$$

□

Proposition 4.2 *The function $\Theta(z)$ belongs to $\mathcal{Z}^{2\times 2}$.*

Theorem 4.3 *Let W be an upper triangular contraction with Zadeh transform $W(z)$. Then, the operator valued function $T_{\Theta(z)}(W(z)) \in \mathcal{Z}$ and is a solution to Problem* **??**.

Proof: We proceed in a number of steps:

STEP 1: *It holds that:*

$$\Theta(z)^*J\Theta(z) = J, \quad z \in \mathbb{T}.$$

STEP 2: *Let $\Theta(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$. Then, $D(z)$ is invertible for all $z \in \mathbb{T}$.*

Indeed,

STEP 3: *Assume that $D(0)$ is invertible. The function $\Sigma_{12}(z) = B(z)D(z)^{-1}$ belongs to $\mathcal{Z}$ and is a solution to Problem* **??**.

It follows from (**??**) that

$$B(z) - (S_{[0]} + zZS_{[1]} + \cdots + z^NZ^NS_{[N]})D(z) = z^NZ^N\beta(z)$$

Since $D(0)$ is invertible, $D(z)$ is invertible in $\mathcal{Z}$ and the result follows.

□

5 $\mathcal{P}(\Theta)$ spaces and J–unitary factorizations

An operator $\Theta \in \mathcal{U}^{2\times 2}$ such thay

$$\Theta J \Theta^* = \Theta^* J \Theta = J$$

is called J–unitary. Writing $\Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we see that D is an invertible operator. The operator Θ is called J–inner if D^{-1} is upper triangular. In this paper we are in particular interested in the case where D^{-1} *is not* an upper triangular operator.

If Θ is J–unitary, it is clear that the operator of multiplication by Θ on the left is an isometry from the Kreĭn space $\mathcal{U}_J$ into itself. We set

$$\mathcal{P}(\Theta) = \mathcal{U}_J \ominus \Theta \mathcal{U}_J. \tag{5.1}$$

We note that in general $\mathcal{P}(\Theta)$ will be a Kreĭn space. It will be a Hilbert space if and only if Θ is J–inner.

We are interested in factorisation of an element $\Theta \in \mathcal{U}_J$ into factors which are themselves in $\mathcal{U}_J$. Because of the polynomial hypothesis such a factorization is always minimal as illustrated in the following proposition.

Proposition 5.1 *Let Θ_1 and Θ_2 be in $\mathcal{U}_J$. Then $\Theta_1\Theta_2 \in \mathcal{U}_J$ and*

$$\mathcal{P}(\Theta_1\Theta_2) = \mathcal{P}(\Theta_1) + \Theta_1\mathcal{P}(\Theta_2). \tag{5.2}$$

For $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \in \mathcal{U}_{2,J}$ we define

$$R_0 F = \begin{pmatrix} R_0 F_1 \\ R_0 F_2 \end{pmatrix}. \tag{5.3}$$

Proposition 5.2 *Let $\Theta \in \mathcal{U}_J$. Then the space $\mathcal{P}(\Theta)$ is R_0–invariant.*

Proof:

$\square$

Proposition 5.3 *There is a one–to-one correspondence between J–unitary factorizations of Θ and non-degenerate R_0–invariant subspaces of $\mathcal{P}(\Theta)$.*

Proposition 5.4 *There exists an observable pair (C, A) such that*

$$\mathcal{P}(\Theta) = \text{ran } C(I - ZA)^{-1} \tag{5.4}$$

Proposition 5.5 *Let $\mathcal{M}$ a non–degenerate R_0–invariant subspace of $\mathcal{U}_J$. Then $\mathcal{M} = \mathcal{P}(\Theta)$ for some J–unitary Θ.*

6 The case of polynomial Θ's

Proposition 6.1 *There exists an observable pair (C, A) such that $(ZA)^N = 0$ for some $N \in \mathbb{N}$ and*

$$\begin{aligned} \mathcal{P}(\Theta) &= \text{ran } C(I - ZA)^{-1} \\ &= \text{ran } C \dot{+} \text{ran } CZA \dot{+} \cdots \dot{+} \text{ran } C(ZA)^{N-1}. \end{aligned} \tag{6.1}$$

7 Elementary factors

8 Generalized Schur functions

Proposition 8.1 *Let $\Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{U}_J$ and let S_0 be an upper tiangular contraction, and assume that the main diagonal of $CS_0 + D$ is invertible. Then, the operator $S = (AS_0 + B)(CS_0 + D)^{-1}$ has a Taylor series at the origin*

Proof: We note that $D^{-1}C$ is a strict contraction and so the operator $CS_0 + D$ is invertible and the linear fractional transformation makes sense.

□

Write

$$(AS_0 + B)(z)(CS_0 + D)(z)^{-1} = \sum_0^\infty z^n Z^n X_{[n]}.$$

Proposition 8.2 *Let $\Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{U}_J$ and let S_0 be an upper tiangular contraction. Then, the operator*

$$S = (AS_0 + B)(CS_0 + D)^{-1} = TV^{-1}$$

where T is an upper triangular contraction and where V is a finite unitary upper triangular operator.

Proof:

□

$$\Omega = \{V \in \mathcal{D} \ ; \ \ell_V < 1\}$$

First recall:

Proposition 8.3 *Let $S \in \mathcal{U}$ be such that $\|S\| \le 1$. Then the function*

$$\sum_{n=1}^\infty V^{[n]} \left(I - S^\wedge(V) S^\wedge(W)^*\right)^{(n)} W^{[n]*} \tag{8.1}$$

is positive for $V, W \in \Omega$.

Definition 8.4 *A sequence $D_{[0]}, D_{[1]}, \cdots$ of diagonal operators defines a generalized Schur function if the kernel (??) has a finite number of negative squares in a neighborhood of the origin.*

More generally one will allow kernels of the form:

$$\sum_{n=1}^{\infty} V^{[n]} \left(J - \Theta^\wedge(V) J \Theta^\wedge(W)^*\right)^{(n)} W^{[n]*} \tag{8.2}$$

where J is the signature operator

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{8.3}$$

Theorem 8.5 *An element $S \in \mathcal{X}$ is of the form $S = T_\Theta(D)$ where $\Theta \in \mathcal{U}_J$ and D a unitary diagonal operator if and only if*

Proof:

□

9 The Schur algorithm: time–varying approach

Rather that the space $\mathcal{M}$ introduced in (??) one can introduced the space $\widetilde{\mathcal{M}}$ spanned (with coefficients in $\mathcal{L}_2$) by the operators:

$$\begin{aligned} F_0 &= \begin{pmatrix} I \\ S_{[0]}^* \end{pmatrix} \\ F_1 &= \begin{pmatrix} I \\ S_{[0]}^* \end{pmatrix} Z + \begin{pmatrix} 0 \\ S_{[1]}^* \end{pmatrix} \\ &\vdots \\ F_N &= \begin{pmatrix} I \\ S_{[0]}^* \end{pmatrix} Z^N + \begin{pmatrix} 0 \\ S_{[1]} \end{pmatrix} Z^{N-1} + \cdots + \begin{pmatrix} 0 \\ S_{[N]}^* \end{pmatrix}, \end{aligned} \tag{9.1}$$

that is,

$$\begin{pmatrix} F_0 & F_1 & \cdots & F_N \end{pmatrix} = \widetilde{C}(I - Z\widetilde{A})^{-1} \tag{9.2}$$

where

$$\widetilde{C} = \begin{pmatrix} I & 0 & \cdots & 0 \\ S_{[0]}^* & S_{[1]}^* & \cdots & S_{[N]}^* \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} 0 & I & \cdot & & 0 & \\ 0 & 0 & I & 0 & 0 & \\ & & & & & \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.3}$$

Theorem 9.1 *Assume that the (unique) solution to the nonstationary Stein equation*

$$\mathbb{P} - A^*\mathbb{P}^{(1)}A = \widetilde{C}^* J \widetilde{C} \tag{9.4}$$

is invertible. Then there exists a J–unitary $\Theta \in \mathcal{U}_J$ such that

$$\widetilde{\mathcal{M}} = \mathcal{U}_{2,J} \ominus \Theta\mathcal{U}_{2,J}. \tag{9.5}$$

Problem 9.2 *Given a pair $(U, V) \in \mathrm{Proj}$ find all pairs $N, \Theta \in \mathbb{N} \times \mathcal{U}_J$ such that*

$$(U, V)\Theta = Z^N(U_N, V_N) \tag{9.6}$$

for some pair $(U_N, V_N) \in \mathrm{Proj}$.