Multigrid preconditioning for anisotropic positive semidefinite Block-Toeplitz systems

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Multigrid methods are highly efficient solution techniques for positive semidefinite block-Toeplitz-Toeplitz-block (BTTB) systems. They can be used as standalone solvers or as preconditioners for Krylov subspace methods such as the conjugate gradient algorithm. However, many applications lead to anisotropic BTTB systems, i.e. BTTB matrices corresponding to a generating function of the form

$$f(x,y) = 1 - \cos(x) + \alpha \cdot (1 - \cos(y)) \;\; (\alpha \ll 1)\ .$$

Although such a system is positive semidefinite, standard multigrid fails in this case due to weak coupling in the y-direction. Convergence can be retained by a suitable combination of semicoarsening steps and full coarsening steps, or alternatively by the use of line smoothers. If the direction of weak coupling is either along x- or y-axis, these solution techniques for anisotropic BTTB systems can be described by the corresponding generating functions. However, in many examples such as

$$f(x,y) = 1 - \cos(x + k * y) + \alpha \cdot (1 - \cos(k * x - y)) \;\; (\alpha \ll 1\ ,\ k \in \mathbb{N})$$

weak coupling occurs in other directions. This leads to problems which are much harder to solve. Therefore, we shall mainly present two approaches of extending the above techniques to these problems. BTTB matrices with blocks of variable size are used to develop multigrid methods based on semicoarsening, and interpretation of BTTB matrices as block-BTTB matrices guarantees convergence by shifting zeros of the generating functions.