QR factorization and QR iteration methods for quasiseparable matrices

Yuli Eidelman and Israel Gohberg
Tel Aviv University, Israel
Vadim Olshevsky
University of Connecticut, USA

We study a class of block structured matrices $A = \{A_{ij}\}_{i,j=1}^N$ whose entries are specified as follows:

$$A_{ij} = \begin{cases} p_i a_{i-1} \cdots a_{j+1} q_j, & 1 \le j < i \le N, \\ d_i, & 1 \le i = j \le N, \\ g_i b_{i+1} \cdots b_{j-1} h_j, & 1 \le i < j \le N. \end{cases}$$

Here $p_i,\ q_j,\ a_k$ are matrices of sizes $m_i \times r'_{i-1},\ r'_j \times m_j,\ r'_k \times r'_{k-1}$ respectively; these elements are said to be lower generators of the matrix R with orders r'_k. The elements $g_i,\ h_j,\ b_k$ are matrices of sizes $m_i \times r''_i,\ r''_{j-1} \times m_j,\ r''_{k-1} \times r''_k$ respectively; these elements are said to be upper generators of the matrix R with orders r''_k. The diagonal entries d_k are matrices of sizes $m_k \times m_k$. Set $n_L = \max_{1 \le k \le N-1} r'_k,\ n_U = \max_{1 \le k \le N-1} r''_k$, then the matrix A is said to be *quasiseparable of order* (n_L, n_U).
For quasiseparable matrices using a modification of the Dewilde-Van der Veen method we obtain the QR-type factorization $A = VUR$, where V, U are unitary matrices, V is a block lower triangular, U is a block upper triangular, S is a block upper triangular with square invertible blocks on the main diagonal. V, U, S are quasiseparable of the orders $(n_L, 0), (0, n_L), (0, n_L + n_U)$. Generators of V, U, S are computed in $O(N)$ operations using QR factorizations for the matrices of small sizes obtained via generators of original matrices.
Next for a quasiseparable matrix A with scalar entries we consider the QR iteration procedure

$$\begin{cases} A - \sigma I = QR, \\ A_1 = \sigma I + RQ, \end{cases}$$

where Q is a unitary matrix and R is an upper triangular matrix. We obtain that the iterant A_1 is a lower quasiseparable of the order n_L matrix and develop a $O(N)$ algorithm to compute it lower generators. Thus for a Hermitian quasiseparable of any order matrix we obtain a fast QR iteration method with $O(N)$ operations per step.