Esercizio 2

$$\Phi_\lambda(x,y) = (xy - \arctan(xy),\ x^2y + 4\lambda y)$$

(i) $D_1 \Phi_{\lambda,1}(x,y) = \dfrac{x^2y^3}{1+(xy)^2}$, $D_1\Phi_{\lambda,2}(x,y) = 2xy$

$D_2 \Phi_{\lambda,1}(x,y) = \dfrac{x^3y^2}{1+(xy)^2}$, $D_2\Phi_{\lambda,2}(x,y) = x^2+4\lambda$

da cui

$$\det J\Phi_\lambda(x,y) = \frac{(4\lambda - x^2)x^2y^3}{1+(xy)^2} = 0 \iff$$

(*) $x = 0$ o $y = 0$ o $x = \pm 2\sqrt{\lambda}$.

Affinché il cerchio aperto B_λ di raggio 1 e centro in $(1,\lambda)$ non intersechi una delle rette (*), occorre e basta che

$$\begin{cases} 2 \le 2\sqrt{\lambda} \\ \lambda \ge 1 \end{cases} \iff \lambda \ge 1$$

(ii) Si ha $\det J\Phi_1(x,y) = \dfrac{(4-x^2)x^2y^3}{1+(xy)^2} > 0$

se $0 < x < 2$ e $y > 0$. Dunque

$$\iint_{\Phi_1(T)} du\,dv = \iint_{T} \det J\Phi_1(x,y)\,dx\,dy =$$

$$= \int_1^2 dx \int_0^{x^{-1}} \frac{(4-x^2)x^2y^3}{1+(xy)^2}\,dy = \frac{1}{2}\int_1^2 (4-x^2)\,dx.$$

$$\int_0^{x^{-2}} \frac{x^2 t}{1+x^2 t}\,dt = \frac{1}{2}\int_1^2 (4-x^2)\Big[\, t -\frac{1}{x^2}\log(1+x^2 t)\Big]_0^{x^{-2}} dx = \frac{1-\log 2}{2}\int_1^2 \frac{4-x^2}{x^2}\,dx =$$

$$= \frac{1-\log 2}{2}.$$

Esercizio 3

(i) La superficie sferica di equazione

$$x^2+(y-1)^2+(z-1)^2=4$$

si parametrizza in modo standard

$$\begin{cases} x = 2\cos\theta\sin\varphi \\ y = 1+2\sin\theta\sin\varphi \\ z = 1+2\cos\varphi \end{cases}$$

con $0\le\theta\le 2\pi$, $0\le\varphi\le\pi$.

Se $z\geq 2$, si ha $1+2\cos\varphi\geq 2 \iff$

$\cos\varphi \geq 1/2 \iff 0\le\varphi\le\pi/3$

da cui

$$\Sigma = \begin{cases} x = 2\cos\theta\sin\varphi \\ y = 1+2\sin\theta\sin\varphi \\ z = 1+2\cos\varphi \end{cases} \quad (\theta,\varphi)\in[0,2\pi]\times[0,\pi/3].$$

Ne segue

$$m(\Sigma) = \iint_{[0,2\pi]\times[0,\pi/3]} 4\sin\varphi\, d\theta\, d\varphi = 8\pi\int_0^{\pi/3}\sin\varphi\, d\varphi =$$

$$= 4\pi.$$

(ii) Per il teorema del rotore applicato a $\underline{F}$ e Σ, si ha

$$\Phi(\operatorname{rot} \underline{F}, \Sigma) = \oint_{\partial^+\Sigma} \underline{F} \, ds$$

dove $\partial^+\Sigma$ è dato da

$$\begin{cases} x^2 + (y-1)^2 + (z-1)^2 = 4 \\ z = 2 \end{cases} \Longleftrightarrow$$

$$x^2 + (y-1)^2 = 3 \quad , \quad z = 2$$

circonferenza da percorrere nel verso positivo, parametrizzabile da

$$\begin{cases} x(t) = \sqrt{3} \cos t \\ y(t) = \sqrt{3} \sin t + 1 \\ z(t) = 2 \end{cases} \qquad t \in [0, 2\pi]$$

Dunque, poiché

$$\underline{F}(x(t), y(t), z(t)) = \Big(3\sin^2 t + 1 + 2\sqrt{3} \sin t , \; 3\cos^2 t + 1 - 2\sqrt{3} \cos t , \; 2\sqrt{3} \cos t (\sqrt{3} \sin t + 1)\Big)$$

si ha

$$\oint_{\partial^+\Sigma} \underline{F} \, ds = \int_0^{2\pi} \Big(-(3 \sin^2 t + 1 + 2\sqrt{3} \sin t) \sqrt{3} \sin t +$$

$$+ (3\cos^2 t + 1 - 2\sqrt{3} \cos t) \sqrt{3} \cos t \Big) \, dt =$$

$$= \int_0^{2\pi} (-6 \sin^2 t - 6 \cos^2 t) \, dt = -12\pi .$$

$\mathcal{E}$

Es. 2

$$\Phi_\lambda(x,y) = (xy - \arctan(xy),\ xy^2 + 16\lambda x)$$

(i) $D_1 \Phi_{\lambda,1}(x,y) = \dfrac{x^2 y^3}{1+(xy)^2}$, $D_1 \Phi_{\lambda,2}(x,y) = y^2 + 16\lambda$

$D_2 \Phi_{\lambda,1}(x,y) = \dfrac{x^3 y^2}{1+x^2y^2}$, $D_2 \Phi_{\lambda,2}(x,y) = 2xy$

da cui $\det J\Phi_\lambda(x,y) = \dfrac{x^3 y^2 (y^2 - 16\lambda)}{1+x^2y^2} = 0 \Leftrightarrow$

(**) $x = 0$ o $y = 0$ o $y = \pm 4\sqrt{\lambda}$.

Affinché B_λ non intersechi una delle rette (*), ~~occorre~~ basta che

$$\begin{cases} 4 \le 4\sqrt{\lambda} \\ 2\lambda \geq 2 \end{cases} \Longleftrightarrow \lambda \geq 1$$

(ii) Si ha

$$|\det J\Phi_1(x,y)| = \frac{(16-y^2)\,x^3y^2}{1+x^2y^2}$$

se $2 \le y \le 4$, $x \geq 0$. Pertanto

$$\iint_{\Phi_1(T)} du\,dv = \iint_T \frac{(16-y^2)\,x^3y^2}{1+x^2y^2}\,dx\,dy =$$

$$= \int_2^4 (16-y^2)\,y^2\,dy \int_0^{y^{-1}} \frac{x^3}{1+x^2y^2}\,dx =$$

$$= \frac{1}{2}\int_2^4 (16-y^2)\,dy \int_0^{y^{-2}} \frac{y^2 t}{1+y^2 t}\,dt =$$

$$= \frac{1}{2}\int_2^4 (16-y^2)\left[t - \frac{1}{y^2}\log(1+y^2 t)\right]_0^{y^{-2}} dy =$$

$$= \frac{1-\log 2}{2}\int_2^4 \frac{16-y^2}{y^2}\,dy = 1-\log 2 .$$

Es 3.

(i) La superficie sferica di equazione

$$(x-1)^2+y^2+(z-1)^2=4$$

si parametrizza

$$\begin{cases} x = 1+2\cos\theta\sin\varphi \\ y = 2\sin\theta\sin\varphi \\ z = 1+2\cos\varphi \end{cases}$$

con $0\le\theta\le 2\pi$, $0\le\varphi\le\pi$.

Se $z\le 0$ si ha $1+2\cos\varphi\le 0 \Leftrightarrow$

$$\cos\varphi\le -1/2 \Leftrightarrow \frac{2\pi}{3}\le\varphi\le\pi$$

da cui

$$\Sigma = \begin{cases} x = 1+2\cos\theta\sin\varphi \\ y = 2\sin\theta\sin\varphi \\ z = 1+2\cos\varphi \end{cases}$$

con $(\theta,\varphi)\in[0,2\pi]\times\left[\frac{2\pi}{3},\pi\right]$.

Quindi

$$m(\Sigma_1) = \iint_{[0,2\pi]\times[\frac{2\pi}{3},\pi]} 4\sin\varphi \, d\theta\, d\varphi = 8\pi \int_{2\pi/3}^{\pi} \sin\varphi \, d\varphi =$$

$$= 4\pi .$$

(ii) Applichiamo il teorema del rotore a $\underline{F}$ e Σ_1, osservando che $\partial^+\Sigma_1$ è dato da

$$\begin{cases} y^2 + (x-1)^2 = 3 \\ z = 0 \end{cases}$$ che si parametrizza

nel seguente modo

$$\begin{cases} x(t) = \sqrt{3}\cos t + 1 \\ y(t) = \sqrt{3}\sin t \\ z(t) = 0 \end{cases} \qquad t \in [0, 2\pi]$$

da cui

$$F(x(t), y(t), z(t)) = (3\sin^2 t + 1 - 2\sqrt{3}\sin t, \; 3\cos^2 t + 1 + 2\sqrt{3}\cos t, \; 0).$$

Pertanto

$$\oint_{\partial^+\Sigma} \underline{F}\, ds = \int_0^{2\pi} \Big(-(3\sin^2 t + 1 - 2\sqrt{3}\sin t)\sqrt{3}\sin t + (3\cos^2 t + 1 + 2\sqrt{3}\cos t)\sqrt{3}\cos t \Big)\, dt =$$

$$= \int_0^{2\pi} (6\sin^2 t + 6\cos^2 t)\, dt = 12\pi .$$