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SIMPLICIAL MATTERS

This is a concise summary of some "simplicial" matters.

1. Preliminaries

Given any non empty set X, consider $\mathbb{R}^X$ with the natural real vector space structure. For every $f \in \mathbb{R}^X$, the *support* of f, $S(f) := \{x \in X;\ f(x) \neq 0\}$. Denote by $F(X)$ the subspace of $\mathbb{R}^X$ made by the functions with *finite* support. We fix the inclusion $X \subset F(X)$, by associating to every $x \in X$, the function e_x such that $e_x(x) = 1$, $S(e_x) = \{x\}$. The set of these functions, denoted $\mathcal{B}_X$, is the *standard basis* of $F(X)$ (*ordered* if X is equipped with a total order). For every $Y \subset X$, we fix the inclusion $F(Y) \subset F(X)$, by extending every $f \in F(Y)$ in such a way $S(f)$ is preserved. If Y is finite and ordered then $F(Y)$ is canonically isomorphic to $\mathbb{R}^m$, m being the cardinality of Y. If $\mathcal{F} \subset F(X)$ is finite, it generates the *affine* subspace $A(\mathcal{F})$ formed by the *affine combinations* of the points of $\mathcal{F}$:

$$A(\mathcal{F}) = \{p = \sum_{f \in \mathcal{F}} a_f f;\ \sum_f a_f = 1\} \ .$$

It is well known that for every $f_0 \in \mathcal{F}$,

$$A(\mathcal{F}) = f_0 + TA(\mathcal{F})$$

where the tangent space $TA(\mathcal{F})$ of $A(\mathcal{F})$ does not depend on the choice of f_0 and is the linear subspace generated by the set of vectors $\{f - f_0\}_{f \in \mathcal{F}}$. Set

$$\dim A(\mathcal{F}) = \dim TA(\mathcal{F}) \ .$$

The points of $\mathcal{F}$ are *independent* if $\dim A(\mathcal{F}) = n - 1$, where n is the cardinality of $\mathcal{F}$.
Give $F(X)$ the distance defined by:

$$d(f, g) = \max_{x \in X} |f(x) - g(x)| \ .$$

It induces a topology. For every $F(Y) \sim \mathbb{R}^m$ as above, the subspace topology is the standard euclidean one. Then the topological space $F(X)$ is union of finite dimensional euclidean spaces.

2. Simplices

We develop a parallel traitment either in an "abstract" or a "geometric" setting.
(Abstract simplices) For every $n \geq 0$, the *standard abstract n-simplex* is the *ordered* set:

$$[n] := \{0, 1, \dots, n\} \ .$$

An abstract n-simplex is an *ordered* set S with $n + 1$ elements. For every n simplex S there is the *canonical labelling*

$$l_S : [n] \to S := [s_0, \dots, s_n]$$

where l_S is the unique order preserving bijection. The *support* $|S|$ is the "naked" set obtained by forgetting the order.

(Geometric realization) The *standard geometric realization* of $[n]$ is denoted Δ^n and is obtained as follows: In $\mathbb{R}^{n+1} = \mathbb{R}^{[n]}$, consider the standard ordered basis $\mathcal{B}_{[n]} = \{e_0, \dots, e_n\}$. Then the *support*

$$|\Delta^n| = |\sigma[e_0, \dots, e_n]| := \{(t_0, \dots, t_n) \in \mathbb{R}^{[n]};\ \sum_j t_j = 1,\ t_j \geq 0,\ j = 0, \dots, n\}$$

that is the subset of $A(\mathcal{B}_{[n]})$ made by the *convex combinations* of the points of $\mathcal{B}_{[n]}$. In fact $|\Delta^n|$ is the *convex hull* of the points of $\mathcal{B}_{[n]}$ which are called the *vertices*. Δ^n is obtained by taking into account the given order of the vertices. A geometric realization of an abstract n-simplex $S = [s_0, \dots, s_n]$, denoted $\sigma_S = \sigma[s_0, \dots, s_n]$ is obtained by embedding it as a set of independent points into some $F(X)$, taking as $|\sigma[s_0, \dots, s_n]|$ the convex hull of these points and finally taking into account the order of its vertices. The canonical labelling $l_S : [n] \to S$ "extends" to the *canonical order preserving parametrization*

$$\phi_\sigma : \Delta^n \to \sigma_S$$

given by the restriction of the unique affine map

$$\phi : A(\mathcal{B}_{[n]}) \to A(S) \subset F(X)$$

such that $\phi(e_j) = s_j$, $j = 0, \dots, n$. Δ^n is called the *standard* n-simplex, while any σ_S as before is a n-simplex.

2.1. **Faces.** Let $\sigma = \sigma_S = \sigma[s_0, \dots, s_n]$ be a n-simplex in some $F(X)$. For every $j = 0, \dots, n$, we can define a $(n-1)$-simplex $f_j\sigma$ as follows: remove from S the vertex s_j; by means of the restricted order, we get an ordered set S_j of n independent points in $F(X)$, then we can define $f_j\sigma := \sigma_{S_j}$ by applying the above procedure. Notice that the canonical labelling function $l_{S_j} : [n-1] \to S_j$ verifies

$$l_{S_j}(i) = l_S(i),\ i < j, \quad l_{S_j}(i) = l_S(i+1),\ i \geq j\ .$$

This reflects in the following *compatibility* of the standard parametrizations:

$$\phi_{f_j\sigma} = \phi_\sigma \circ \phi_{f_j\Delta^n}\ .$$

By using the standard relabelling of the vertices, the face operation can be *iterated*. Every $(n-2)$-face can be obtained by removing two elements s_i and s_j, $i < j$, from S. This can be eventually obtained by two different iterations so that we have the relations:

$$f_i(f_j\sigma) = f_{j-1}(f_i\sigma),\ i < j.$$

By fully iterating, we get the faces of all dimensions d, $0 \leq d \leq n$, σ being the unique n-face, the vertices being the 0-faces. By setting

$$f_j|\sigma| = |f_j\sigma|$$

we define a notion of (iterated) faces at the level of supports.
In the abstract setting the most natural way to encode the d-faces of an n-simplex $S = [s_0, \dots, s_n]$ (incorporating the canonical labelling) is in some sense "opposite", that is in terms of *strictly increasing* maps $[d] \to S$, called *d-cofaces*. As above these are generated by iterating the $n+1$ $(n-1)$-cofaces $d^j : [n-1] \to S$ which miss the elements s_j, $j = 0, \dots n$. Similarly as above we have the ("contravariant") relations formally expressed by

$$d^j d^i = d^i d^{j-1}, \quad i < j\ .$$

2.2. **Orientations.** Let S and S' be abs n-simplices with the same support $|S| = |S'|$. We say that S is *co-oriented* with S' if $l_S^{-1} l_{S'}$ is an *even* permutation of the elements of $[n]$. This defines an equivalence relation on the set of n-simplices sharing the same support, there are two equivalence classes, each one is a *combinatorial orientation* of the support.
Let $\sigma = \sigma_S = \sigma[s_0, \dots, s_n]$ be a n-simplex in some $F(X)$. By definition an orientation on $|\sigma|$ is an orientation on the tangent spaces $TA(S)$ of the affine space $A(S)$ which contains $|\sigma|$. This last orientation is defined as follows. Let V be a m-dimensional real vector spaces. If $m = 0$ then an orientation is just a sign ± 1. If $m > 0$, then let us say that an oriented basis $\mathcal{B}$ of V is *co-oriented* with the basis $\mathcal{B}'$ if $\det M^{\mathcal{B}}_{\mathcal{B}'} > 0$. This defines an equivalence relation on the set of oriented bases of V (by Binet) with two equivalence classes, each one being an orientation on V. Then we can define the *combinatorial orientation* ω_σ of $|\sigma|$. If $n = 0$, we stipulate that it is the sign $+1$. If $n > 0$, it is the class of the basis $\{s_1 - s_0, \dots, s_n - s_0\}$ of $TA(S)$. Notice that S and S' induce the same orientation on the common support if and only if for every geometric realizations σ_S and $\sigma_{S'}$ having the same support $|\sigma|$ (in some $F(X)$), they induce the same orientation on $|\sigma|$. The canonical parametrizations preserve the orientations.

2.3. **Boundary orientation.** For every n-simplex $\sigma = \sigma_S$ (in some $F(X)$), the support boundary is defined as

$$\partial|\sigma| = \cup_{j=0}^{n} f_j|\sigma| \ .$$

As a pair of topological subspaces in $F(X)$, $(|\sigma|, \partial|\sigma|) \sim (D^n, S^{n-1})$. The *interior* of $|\sigma| := |\sigma| \setminus \partial|\sigma|$. $|\sigma|$ has its own combinatorial orientation ω_σ, while every $(n-1)$-face $f_j|\sigma|$ is equipped with $\omega_{f_j\sigma}$. We want to define another orientation on each $f_j|\sigma|$ denoted by $\partial_j\omega_\sigma$ and called the *boundary orientation*. It is defined by means of the "*first the outgoing normal convention*": up to a translation, we can assume that $|\sigma| \subset TA(S)$, $f_j|\sigma|$ is contained in a linear hyperplane $L \subset TA(S)$, L divides $TA(S)$ into two half-spaces, one denoted by P does not intersect the interior of $|\sigma|$. Let η be a non zero vector normal to L and pointing towards P. There is only one class $[\mathcal{B}]$ of bases of L such that for every $\mathcal{B}$ in the class, $\eta \oplus \mathcal{B}$ is a basis of $TA(S)$ that represents the orientation ω_σ. Then set

$$\partial_j\omega_\sigma = [\mathcal{B}] \ .$$

Then every $f_j|\sigma|$ has two orientations which might agree or not; we encode this by a sign ±1. This convention is preferable to other possible ones because the following nice formula holds:

$$\omega_{f_j\sigma} = (-1)^j \partial_j\omega_\sigma \ .$$

As it is purely in combinatorial terms, the formula can be used to define a corresponding boundary orientation in the abstract setting.
For every $(n-2)$-face

$$f_i(f_j\sigma) = f_{j-1}(f_i\sigma), \ i < j$$

as above, it is not hard to verify that

$$\partial_i\omega_{f_j\sigma} = -\ \partial_{j-1}\omega_{f_i\sigma} \ .$$

3. Complexes

An *abstract simplicial complex* is a family (of arbitrary cardinality) $\mathcal{S} = \{S\}$ of abs simplices such that whenever $S \in \mathcal{S}$, and S' is an ordered subset of S (with the restricted order), then also $S' \in \mathcal{S}$. In other words $\mathcal{S}$ is closed w.r.t. the (images of) cofaces.
The *standard geometric realization* of an abs simplicial complex $\mathcal{S}$ is obtained as follows: let $V = V_\mathcal{S}$ be the set of vertices (i.e. of 0-simplices) in $\mathcal{S}$. Consider $F(V)$, $V \sim \mathcal{B}_V$ as in the preliminaries. Then every S in $\mathcal{S}$ corresponds to an ordered subset of the standard basis $\mathcal{B}_V$, hence to a geometric simplex σ_S in $F(V)$. Then set

$$K = K_\mathcal{S} = \{\sigma_S\}_{S\in\mathcal{S}} \ .$$

It is easy to verify that this family of simplices in $F(V)$ verifies:

(1) If $\sigma \in K$, and τ is a (iterated) face of σ, then also $\tau \in K$;
(2) If $\sigma, \tau \in K$, then $|\sigma| \cap |\tau| = |\gamma|$, where γ is a (possibly empty) common (iterated) face of both σ and τ.

By definition any family $K = \{\sigma\}$ of simplices (in some $F(X)$) verifying the above properties is a *simplicial complex.* If for every $\sigma = \sigma[s_0, \dots, s_n]$, we consider its set of vertices $v_\sigma = [s_0, \dots, s_n]$ as an ordered set, then we get an abs simplicial complex

$$\mathcal{S}_K = \{v_\sigma\}_{\sigma\in K}$$

called the *vertex scheme* of K. K is *a* geometric realization of $\mathcal{S}_K$. There is a canonical bijection between the set of vertices of K and the set of vertices of the standard geometric realization of $\mathcal{S}_K$, which restricts to a bijection between the set of vertices of every σ and the set of vertices of v_σ. Alike the canonical parametrization, this extends to a unique affine isomorphism between $|\sigma|$ and $|v_\sigma|$. This is an example of a simplicial isomorphism (see next subsection).

3.1. **Simplicial maps.** Let $\mathcal{S}$ and $\mathcal{T}$ be abs simplicial complexes. A simplicial map from $\mathcal{S}$ to $\mathcal{T}$ consists of a map

$$f : V_{\mathcal{S}} \to V_{\mathcal{T}}$$

such that for every S in $\mathcal{S}$, $f(|S|)$ is equal to the support $|T|$ of some $T \in \mathcal{T}$. Notice that we do not require that the restriction of f to $|S|$ is bijective; f is a simplicial isomorphism if it is bijective (with simplicial inverse).
If K and H are simplicial complexes, a simplicial map

$$f : K \to H$$

is the geometric realization of a simplicial map

$$\hat{f} : \mathcal{S}_K \to \mathcal{S}_H \ ;$$

for every $\hat{f}(|S|) = |T|$ as above, we extend $\hat{f}_{|S|} : |S| \to |T|$ to $f : |\sigma_S| \to |\sigma_T|$ by setting

$$f(\sum t_j s_j) = \sum t_j \hat{f}(s_j)$$

for every convex combination of the points of $|S|$; notice in fact that even if there are repetitions in the second term summation, nevertheless it is a convex combination of the points of $|T|$.

(Subcomplex-skeletons) In both settings, a subcomplex is a subfamily which is a simplicial complex by itself. For example a simplex of the given complex with all its faces form a subcomplex. There are some distinguished subcomplexes. Given K, for every $n \geq 0$, K_n denotes the set of n-simplices belonging to K. Then set

$$K^n = \cup_{j=0}^n K_j \ .$$

It is a subcomplex of K called its *n-skeleton*.
The restriction of a simplicial map to a subcomplex is a simplicial map.

(Simplicial complex categories) In this way we have defined two categories either abstract or geometric, having as **Objects** the (abs) simplicial complexes and as **Arrows** the (abs) simplicial maps; the equivalence are the simplicial isomorphisms. The verification that the composition is well defined is easy. The standard geometric realization provides a covariant **functor** $\Rightarrow_{AG}$ from the abs to the geometric category. By associating to every geometric complex its vertex scheme, we define a functor $\Rightarrow_{GA}$. Clearly $\Rightarrow_{GA}\Rightarrow_{AG}$ equals the "identity abstract functor". This holds for $\Rightarrow_{AG}\Rightarrow_{GA}$ "up to simplicial isomorphisms". These categories mainly are of combinatorial nature. We have not yet considered the topological aspects of the story.

As we have also a notion of face for the supports, we can define similarly a notion of *complex of simplicial supports*. Clearly every simplicial complex $K = \{\sigma\}$, incorporates the complex of supports $||K|| = \{|\sigma|\}$. On the other hand every complex of supports is induced by some simplicial complex; for example we can give the set of vertices a total order and restrict it to the vertex set of every simplex. We have also the intermediate notion of complex of *oriented* simplicial supports $\{|\sigma|, \omega_{|\sigma|}\}$, where every $|\sigma|$ is equipped with an orientation. Every K as above incorporates also the complex of supports endowed with the combinatorial orientations.

4. Topology

Let K be a simplicial complex in some $F(X)$. We define the support

$$|K| = \cup_{\sigma \in K} |\sigma| \subset F(X) \ .$$

As we have given $F(X)$ a topology, the first naif idea would be to consider $|K|$ with the subspace topology. But this is not a good choice mainly because simplicial maps (even isomorphisms) are in general not continuous. We have the family of inclusions

$$\{i_\sigma : |\sigma| \to |K|\} \ .$$

Every $|\sigma|$ is a compact subset of some standard finite dimensional euclidean space, moreover two different geometric realizations of some S are related by a very tame homeomorphism built by means of two standard parametrizations. Then we give $|K|$ the *final topology* w.r.t. such a family of maps, the finer topology such that every inclusion is continuous. Equivalenty, $A \subset |K|$ is closed (open) if and only if for every $\sigma \in K$, $|\sigma| \cap A$ is closed (open) in $|\sigma|$. We have also the family of maps

$$\{i_\sigma \circ \phi_\sigma : \Delta^{\dim \sigma} \to |K|\}$$

equivalently we take the final topology w.r.t. this family of maps.
We give a sparse list of properties of this topology.

- If K is finite then $|K| \subset \mathbb{R}^m$ for some m and in this case we have the usual subspace topology, so that $|K|$ is compact.
- In general the topology on $|K|$ is finer than the subspace topology in $F(X)$. For example consider the simplicial complex in $\mathbb{R}^2$, made by the 1-simplices of the form $\sigma[0, x]$ where x belongs to the unitary circle S^1. As a set $|K| = D^2$, but $0 \in |K|$ has no any countable basis of neighbourhoods.
- If L is a subcomplex of K, then $|L|$ is a subspace of $|K|$ and is closed.
- A map $f : |K| \to T$, T being any topological space, is continuous if and only if every map $f \circ i_\sigma : |\sigma| \to X$ is continuous.
- Simplicial maps $f : K \to H$ induce continuous maps $|f| : |K| \to |H|$. A simplicial isomorphism induces a homeomorphism.
- Points are closed in $|K|$ and $|K|$ is normal, hence Haussdorff.
- If L is a subcomplex of K then there is a open neighbourhood of $|L|$ which retracts by deformation to $|L|$. $|K|$ is locally contractible (hence it has a nice universal covering which is itself the support of a simplicial complex and the deck trasformations are simplicial).

  Note. *The last 3 items hold also in the more general setting of CW complexes and a proof can be obtained by using so called ϵ-neighbourhoods of closed sets in $|K|$.*
- A closed subset of $|K|$ is compact if and only if it is contained in the support of a *finite* subcomplex. One implication is obvious. As for the other, assume that A is compact and denote by K_A the subset of $\sigma \in K$ such that the intersection between A and the interior of σ is not empty. It is enough to show that K_A is finite. In fact by the very definition of the topology of $|K|$ every subset of K_A is closed; then K_A itself is closed and inherits the discrete topology. Hence it is compact as a closed subset of the compact set A, and it is finite as it is discrete.

So far we have constructed a functor $\Rightarrow_{GT}$ from the simplicial category of (geometric) simplicial complexes and simplicial maps to the category **TOP** by associating to every K the topological space $|K|$ and to every simplicial map f, the corresponding $|f|$ (also called simplicial) between the respective supports. We note that simplicial homeomorphisms are very rigid (being governed by the combinatorics at the level of the vertex shemes) while arbitrary homeomorphisms (even between topological supports of simplicial complexes) are very flexible. For example the boundary of a triangle and the boundary of a square cannot be simplicially homeomorphic in spite of the fact that they are apparently homeomorphic. This suggests to look for an intermediate category which keeps some features of the simplicial one but with more flexible morphisms.

5. The PL category

The **Objects** are the *polytopes* that is by definition topological spaces P which can be realized as topological support of some simplicial complex K; in such a case $P = |K|$ and sometimes one says that $|K|$ is a *triangulation* of P. The **Arrows** are the PL maps: a map

$$f : P \to Q$$

between polytopes is PL if there exist triangulations $P = |K|$, $Q = |H|$ such that $f : |K| \to |H|$ is simplicial.

At present it is not so evident that this is a category that is *it is not evident that the composition of two PL maps is PL* because at the intermediate polytope in a composition we might deal with different triangulations. The key to solve this difficulty is the notion of *subdivision*.
A subdivision of a simplicial complex K is a complex H such that

(1) For every $\sigma \in H$ there is $\tau \in K$ such that $|\sigma| \subset |\tau|$;
(2) For every $\tau \in K$, $|\tau|$ is the union of a *finite* number of supports of simplices of H.

A consequence of these properties (in particular of the finiteness in (2)) is that $|K| = |H|$ *as topological spaces*. Here is the key fact:

If $P = |K| = |K'|$ are two triangulations of the polytope P, then there exists K" such that $P = |K"|$ and K" is a common subdivision of both K and K'.

Let us sketch a proof. The family $K \cap K'$ of intersections between supports of simplices of K and K' respectively is a "complex of convex cells" (in the sense that a complex of simplicial supports is a complex of convex cells of a special type). In the realm of these more general complexes $K \cap K'$ is a common subdivision of both K and K'. The idea is that we can triangulate $K \cap K'$ to get a desidered K", moreover *without adding new vertices*. Let us give the set of vertices (i.e. of 0-cells) of $K \cap K'$ a total order. We construct K" by induction on the dimension of the cells. The set of vertices is untouched. Assume that we have triangulated the union of cells of dimension less than n. For every n-cell C consider the *smallest* vertex v_C of C. Then the simplices that triangulate C are of the form $\sigma[v_C, v_1, \dots, v_s]$ where $\sigma[v_1, \dots, v_s]$ is anyone of the already constructed simplices of dimension less than n that are contained in the boundary of C and do not contain v_C. The global order ensures the coherence of the construction.

□

By using this key fact we see that the above definition of PL map is equivalent to

$f : P \to Q$ is PL if for every triangulations $P = |K|$, $Q = |H|$, there are subdivisions K' of K, H' of H respectively such that $f : |K'| \to |H'|$ is simplicial.

Eventually one realizes that the composition of PL maps is PL and that *being PL*-homeomorphic *is a transitive relation on the class of polytopes.*

5.1. **On the product of simplicial complexes.** The product $K \times H$ of two simplicial complexes is not a symplicial complex. Assuming for simplicity that the ordering of the simplex vertices is the restriction of global vertex orders, then $K \times H$ is a complex of convex cells with (lexicographically) totally ordered vertices. Then we can adopt the above triangulation method *without introducing new vertices* depicted above and get a simplicial complex denoted $K \boxtimes H$. Note that $|K \boxtimes H|$ is in general different from the topological product $|K| \times |H|$, it has a finer topology; nevertheless these two topologies share the same family of compact sets. We get the product topology if at least one among K and H is finite.

5.2. **Barycentric subdivision.** This is a standard fundamental way to subdivide. For every simplex $\sigma = \sigma[s_0, \dots, s_n]$ its *barycenter* is the point

$$\hat{\sigma} := \sum_j \frac{1}{n+1} s_j \ .$$

Let K be a simplicial complex. The barycentric subdivision $K^{(1)}$ of K is obtained (as in the previous sketch of proof) by induction on the dimension of the simplices. The 0-simplices of K are not subdivided. Assume that we have subdivided the $(n-1)$-skeleton K^{n-1} of K. For every n-simplex σ of K the simplices that subdivide it are of the form $\sigma[\hat{\sigma}, v_1, \dots, v_s]$ where $\sigma[v_1, \dots v_s]$ is any simplex of $(K^{n-1})^{(1)}$ contained in $\partial|\sigma|$.

This subdivision has many interesting properties:

- $K^{(1)}$ only depends on the complex of symplicial supports associated to K.
- The vertex order of every simplex of $K^{(1)}$ is the restriction of a global order on its set of vertices.
- Simplicial maps preserve the barycentric subdivision; precisely every simplicial map

$$f : K \to H$$

induces in a canonical way a simplicial map

$$f^{(1)} : K^{(1)} \to H^{(1)} \ .$$

- The barycentric subdivision can be iterated: $K^{(n)} := (K^{(n-1)})^{(1)}$.
- If K is finite, so that $|K|$ is contained in some finite dimensional euclidean space $\mathbb{R}^m$ and is endowed with the restriction of the euclidean distance, then

$$\lim_{n\to\infty} \max_{\sigma\in K^{(n)}} \delta(|\sigma|) = 0$$

where δ denotes the diameter. This last properties is intuitive, however an actual proof needs some (elementary) extimations where it is important that we are using the barycenters.

Another nice application of the barycentric subdivision is the following important *simplicial approximation theorem* in finite dimensional PL geometry

Let $f : |K| \to |H|$ be a continuous map between triangulated compact polytopes. Then there are an iterated barycentric subdivision $K^{(n)}$ of K and a simplicial map $g : |K^{(n)}| \to |H|$ which is homotopic to f.

Here is a sketch of proof. For every simplex σ in a finite simplicial simplex say T, its *open star* $St(\sigma)$ is the union of the interior of the simplices of T which contains σ. Hence it is an open set. The open stars of the vertices of T form an open covering of $|T|$. If the intersection of open stars of some set of vertices of T is non empty, then those vertices are vertices of a simplex τ of T and the intersection of the vertex stars is the open star of τ. Now, let $\mathcal{U}$ be the open covering of $|H|$ made by the open stars of its vertices. As f is continuous, then $\mathcal{U}' = f^{-1}(\mathcal{U})$ is a open covering of $|K|$. $|K|$ is a compact metric space as it is endowed with the restriction of the euclidean metric on some $\mathbb{R}^m$. Let $\epsilon > 0$ be a Lebesgue number for $\mathcal{U}'$. Let n big enough in such a way that every simplex of $K^{(n)}$ has diameter less than $\epsilon/2$. Then the open star of every vertex v of $K^{(n)}$ has diameter less than ϵ. It follows that there is a vertex $g(v)$ of H such that $f(St(v)) \subset St(g(v))$. We want to show that $g : V_{K^{(n)}} \to V_H$ defined so far is simplicial. If σ is a simplex of $K^{(n)}$, and if $x \in St(\sigma)$ then $f(x)$ belongs to the intersection of the images $f(St(v))$ where v varies among the vertices of σ; hence the intersection of the open stars $St(g(v))$'s is non empty and finally the $g(v)$'s are vertices of a simplex H. So we can define the continuous simplicial map $g : |K^{(n)}| \to |H|$ which extends the above abs simplicial map. Finally we note that for every $x \in |K^{(n)}|$ both $f(x)$ and $g(x)$ belongs to a same simplex of H and we define a homotopy by the convex combination $tg(x) + (1-t)f(x)$, $t \in [0,1]$.

□

This has the following corollary:

Let $f : X \to Y$ be a continuous map and assume that X (Y) is homeomorphic to a compact polytope P (Q). Assume also that $\dim P < \dim Q$. Then f is homotopic to a continuous map $g : X \to Y$ which is not surjective (in fact the image complement is open and dense in Y).

By elementary linear algebra if $g : P \to Q$ is PL then it has the desidered properties. We reduce to this case by applying the above approximation theorem to $\hat{f} : P \to Q$, $\hat{f} = p \circ f \circ h$ where $h : P \to X$ and $p : Y \to Q$ are homeomorphisms.

□

By applying this corollary to continuous maps $S^m \to S^n$, $1 \leq m < n$, we derive that

$$\pi_m(S^n) = 0\ .$$

6. Δ-complexes

We want to extend the notion of simplicial complex. Here is a few motivating examples: two simple compact arcs with common pair of endpoints can be naturally 'triangulated' by means of two 1-simplices, but this is not a simplicial complex; if we glue together two edges of a triangle we get a sort of 'triangulation' of the cone over S^1 but again it is not a simplicial complex; a similar generalized triangulation of the torus $S^1 \times S^1$ is obtained from the ordinary triangulation of a square by two 2-simplices and one diagonal 1-simplices , by gluing together the pair of opposite sides. It is natural to look for a more general notion of complex which would cover such a kind of examples. The way we will do it would sound a bit esoteric but it opens the door towards a wide new territory. We will limit to a quick glance to this matter.

6.1. **The category Δ.** It has as **Objects** the standard abstract simplices $[n]$ for every $n \geq 0$. The **Arrows** are the coface maps, that is the *strictly increasing* maps $[s] \to [n]$. By varying n they are generated by the maps $d^j : [n-1] \to [n]$ which miss the point $j \in [n]$, $j = 0, \dots, n$. As already remarked they verify the relations

$$d^j d^i = d^i d^{j-1}, \quad i < j \ .$$

6.2. **A categorial reformulation of symplicial complexes.** Let K be a symplicial complex. Let us recall that K_n denotes the *set* of n-simplices of K. By means of the $(n-1)$-faces of the simplices in K_n, we define maps

$$f_j : K_n \to K_{n-1}$$

verifying the relations

$$f_i f_j = f_{j-1} f_i, \ i < j.$$

Then by associating to every $[n]$ the set K_n, and to every d^j the map f_j, we actually define a **contravariant functor $\mathfrak{f}_K$ from the category Δ to the category SET**.
The *topological realization* $|\mathfrak{f}_K|$ of this functor, homeomorphic to $|K|$, is defined as follows:

Take the topological space

$$\tilde{K} := \amalg_n (K_n \times |\Delta^n|)$$

where "$\amalg$" denotes the disjoint union, every $|\Delta^n|$ has the usual topology and every set K_n is endowed with the discrete topology. Consider the equivalence relations $\sim$ on this space generated by the identifications

$$(\sigma, q) \sim (f_j \sigma, p)$$

where $p \in |\Delta^n|$, $q \in |\Delta^{n+1}|$, $\sigma \in K_{n+1}$, $q = \phi_{f_j \Delta^{n+1}}(p)$. Finally

$$|\mathfrak{f}_K| := \tilde{K} / \sim \ .$$

6.3. **Δ-complex.** By definition, a Δ-complex is any contravariant functor $\mathfrak{f}$ from the category Δ and the category **SET**. Its topological realization $|\mathfrak{f}|$ is formally defined as above. This covers a wider range of complexes than the simplicial ones. The simple motivating examples at the beginning of this section are Δ-complexes.

After all, the topological space $X = |\mathfrak{f}|$ can be equivalently described in the following way which could be taken as a *definition* of a *Δ-complex over X* and which is suited for practical topological use. There is a natural family of maps

$$\{\psi_\sigma : |\Delta^{d(\sigma)}| \to X\}$$

such that whenever ψ_σ belongs to the family, then also $\psi_\sigma \circ \phi_{f_j \Delta^{d(\sigma)}}$ does it. This family can be interpreted as a unique map defined on $\tilde{K}$ with values in X. This verifies the following properties:
(1) The restriction to the interior of every simplex is injective;
(2) Every $x \in X$ is contained in the image of the interior of *one* simplex.

Then we give X the final topology with respect to this family of maps, that is the quotient topology if we consider them as a unique map. In this way the restriction to the interior of any simplex is a homeomorphism onto its image.
Simplicial complexes are characterized among general Δ-complexes by the properties that the restriction to every entire $|\sigma|$ is a homeomorphism onto its image and that they are completely determined by the combinatorial behaviour of the vertex shemes.

(Δ-barycentric subdivision) By using the last description of a Δ-complex, we see that the barycentric subdivision makes sense also for Δ-complexes. One can see that

The second barycentric subdivision of a Δ-complex is a simplicial complex.

Then Δ-complexes can be considered also as a more efficient way (that is by possibly using less simplices) than simplicial complexes to 'triangulate' the same class of topological spaces.

6.4. **On Δ-maps.** Once we have defined a Δ-simplex as a functor, then a 'map' from $\mathfrak{f}$ to $\mathfrak{f}'$ 'must' be a *natural transformations* $\mathcal{T}$ of functors, that is a rule which assigns to every object $[n]$ of the category Δ a map $T_{[n]} : \mathfrak{f}([n]) \to \mathfrak{f}'([n])$ such that for every arrow $f : [m] \to [n]$ in Δ, $T_{[m]}$ and $T_{[n]}$ form a suitable commutative square with $\mathfrak{f}(f)$ and $\mathfrak{f}'(f)$. However this is quite restrictive:

A simplicial map which includes some dimension decreasing collapse cannot be recovered as a natural transformations of the corresponding functors.

On the oder hand we see that

- Simplicial isomorphisms or more generally simplicial covering maps can be recovered as natural transformations of the corresponding functors
- every Δ-maps beetween Δ-complexes induces a simplicial map between the respective second barycentric subdivisions.

The possible presence of collapses is the key difficulty. A way to overcome it is to enlarge the category Δ to the category $\hat{\Delta}$ which has the same objects but allows as arrows increasing but not necessarily strictly increasing maps $[m] \to [n]$. Thus we have to add to the coface generators d^j, also 'codegeneracy' maps $s_j : [n+1] \to [n]$ which sends both j and $j+1$ to j. By definition a *simplicial set* is a contravariant functor from $\hat{\Delta}$ to **SET**. Every Δ-complex can be canonically completed to a simplicial sets (by adding to the maps f_j all possible 'degeneracy' maps); arbitrary simplicial maps can be incorporated into natural transformations of simplicial sets. This is the beginning of an extremely rich and complex theory firstly aimed to build a performant combinatorial homotopy theory and which has wide ramifications in a lot of fields including algebraic geometry. All this is beyond the aims and the possibilities of the present note. To our aims ordinary simplicial maps or possibly some Δ-maps will suffice. To uniformise the terminology, sometimes a Δ-complex is called a Δ-*set*.

7. The Δ-K(G,1)

Given any group G, a $K(G,1)$ is a path-connected topological space X such that the fundamental group $\pi_1(X) \sim G$, while $\pi_n(X) = 0$ for every $n > 0$. For every G, we are going to construct in a canonical way a Δ-set whose topological realization is a $K(G,1)$.

If in the above functorial definition we replace the category **SET** by any subcategory like **GROUP**, **RING**, etc., we get the notion of Δ-*group* (*simplicial group*) etc.

For every group G consider the following Δ-group: associate to the object $[n]$ the product group $G^{[n]}$; every element of this product can be encoded as $\Delta[g_0,\dots,g_n]$ i.e. by labelling the ordered vertices of Δ^n by the $(n+1)$-uples $(g_0,\dots,g_n) \in G^{[n]}$. Every face $f_j\Delta^n$ determines in a natural way a group homorphism

$$f_j : G^{[n]}) \to G^{[n-1]} \ ;$$

then we complete the functor by associating to every elementary coface d^j the map f_j.

There is an underlying Δ-set, and we denote by $E(G)$ its topological realization. We claim that

(1) $E(G)$ is contractible to the point $\Delta[u]$ corresponding to $u = 1_G$;
(2) The natural action of G on every $G^{[n]}$ by left multiplication:

$$(g,(g_0,\dots,g_n)) \to (gg_0,\dots,gg_n)$$

determines a natural self-transformation of the fuctor, hence and action on $E(G)$ by a Δ-isomorphism; this action is free and properly discontinuous.

As for (1), every $\Delta(g_0,\dots,g_n)$ is a face of $\Delta(u,g_0,\dots,g_n)$; this last retracts to its distinguished vertex u by the canonical cone structure based on the opposite face. These local retractions are compatible because they restrict to the faces; then it gives us a global retraction of $E(G)$. Notice that this retraction moves $[u]$ around the loop $\Delta[u,u]$, hence $E(G)$ is contractible but it does not retract by deformation onto $\Delta[u]$.

As for (2) it is clear that the action is free. Recall that "properly discontinuous" means that for every compact subset A of $E(G)$, the set of $g \in G$ such that $gA \cap A \neq \emptyset$ is *finite*. In our situation this holds because G acts by Δ-maps and every compact set is contained in the support of a finite subcomplex.

If $B(G)$ denotes the quotient space of the above action, by generals fact about covering maps we have that the projection

$$p_G : E(G) \to B(G)$$

is a universal covering map (with contractible universal covering spaces) and $\pi_1(B(G)) \sim G$. Moreover it follows from the construction that:

- $B(G)$ itself is a Δ-set, p_G is a Δ-map;
- the Δ-trasformations of $E(G)$ given by the action coincide with the covering trasformations;
- as $E(G)$ is contractible, then $\pi_n(B(G)) = 0$ for every $n > 0$.

Thus we have constructed the *canonical* Δ-**K(G,1)**.

The construction is functorial: every group homomorphism

$$\phi : G \to H$$

induces in a canonical way a covariant natural transformation $\mathcal{T}_\phi$ of the corresponding Δ-groups and eventually a Δ-map

$$\delta_\phi : B(G) \to B(H) \ .$$

Thus we have constructed a covariant functor from the category **GROUP** to the category of Δ-sets and Δ-maps. By composing with the functor from the second category to **TOP** we have also a functor **GROUP**$\Rightarrow$**TOP**. Every topological invariant can be lifted to an invariant for groups. This is a way to start a geometric/topological group theory.