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THE CONVERGENCE OF CIRCLE PACKINGS TO THE RIEMANN MAPPING

BURT RODIN[1] & DENNIS SULLIVAN[2]

1. Introduction. In his address,[3] "The Finite Riemann Mapping Theorem", Bill Thurston discussed his elementary approach to Andreev's theorem (see §2 below) and gave a provocative, constructive, geometric approach to the Riemann mapping theorem (see §3). The method is quite beautiful and easy to implement on a computer (see Appendix 2).

In this paper we prove Thurston's conjecture that his scheme converges to the Riemann mapping. Our proof uses a compactness property of circle packings, a length-area inequality for packings, and an approximate rigidity result about large pieces of the regular hexagonal packing (§3 and Appendix 1).

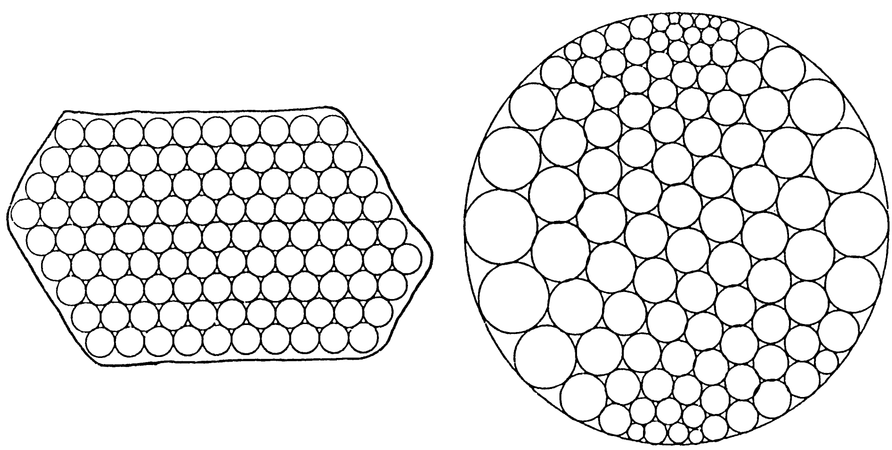

FIGURE 1.1. An approximate conformal mapping

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Our proof is nonconstructive at one point (the Hexagonal Packing Lemma) and does not yield a definite rate of convergence. However, we believe the method will work well in practice. There are two reasons for this belief. First, Thurston's method of finding Andreev packings works very well in practice. Secondly, two circle packings with the same combinatorial triangulation suggest an appropriate model for an approximate conformal mapping (Figure 1.1).

We thank Bob Edwards, Mike Freedman, Troels Jorgensen, and Al Marden for stimulating discussions.

2. Definitions and Preliminaries. Let R be a region in the plane or on the 2-sphere. A *circle packing in* R is a collection of closed disks contained in R and having disjoint interiors. The *nerve* of a circle packing is the embedded 1-complex whose vertices are the centers of the disks and whose edges are the geodesic segments joining the centers of tangent disks and passing through the point of tangency. In this paper we consider only those circle packings whose nerve is the 1-skeleton of a triangulation of some open connected subset in the plane or sphere. A *circle packing **of** the sphere* is one whose associated triangulation is a triangulation of the sphere. Thus the interstices in a circle packing of the sphere are circular cusp triangles. If one of the disks in a circle packing of the sphere is the exterior of the unit disk then the remaining circles are said to be a *circle packing **of** the unit disk*. Note that in the case of a circle packing of the unit disk, the carrier of the associated triangulation is a proper submanifold of the unit disk.

A finite sequence of circles from a circle packing is called a *chain* if each circle except the last is tangent to its successor. The chain is a *cycle* if the first and last circles are tangent.

Let c be a circle in a circle packing. The *flower* centered at c is the closed set consisting of c and its interior, all circles tangent to c and their interiors, and the interiors of all the triangular interstices formed by these circles (Figure 2.1).

We shall refer to the following statement as Andreev's theorem (Andreev [1, 2], Thurston [11; Chapter 13]): *Any triangulation of the sphere is isomorphic to the triangulation associated to some circle packing. The isomorphism can be required to preserve the orientation of the sphere and then this circle packing is unique up to Möbius transformations.*

A topological annulus A in the plane has a modulus mod A that can be defined, without reference to conformal mapping, as the infimum of the L_2-norms of all Borel measurable functions ρ on the plane such that $1 \leqslant \int \rho(z)|dz|$ along all degree one curves in A.

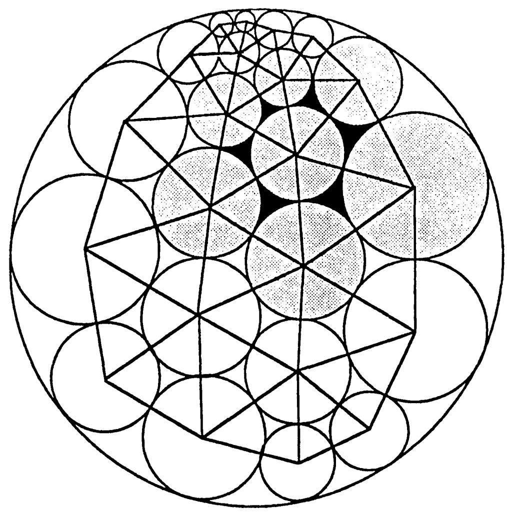

FIGURE 2.1. A packing of the disk, its triangulation, and one of its flowers

An orientation preserving homeomorphism f between two plane domains is called *K-quasiconformal*, $1 \leqslant K < \infty$, if

$$K^{-1}\operatorname{mod} A \leqslant \operatorname{mod} f(A) \leqslant K \operatorname{mod} A$$

for every annulus A in the domain of f. Some basic facts are: (1) K-quasiconformality is a local property (Ahlfors [3, Theorem 1, p. 22], Lehto-Virtanen [7, Theorem 9.1, p. 48]), and (2) a 1-quasiconformal map is conformal and conversely ([3, Theorem 2, p. 23], [7, Theorem 5.1, p. 28]). We shall also make use of the fact that simplicial homeomorphisms are K-quasiconformal for K depending only on the shapes of the triangles involved. To see this note that an affine map is K-quasiconformal for a K depending only on the shapes of one triangle and its image. For a piecewise affine homeomorphism use the fact [7, Theorem 8.3, p. 45] that a homeomorphism of a domain which is K-quasiconformal on the complement of an analytic arc in the domain is actually K-quasiconformal in the entire domain. See [7, Theorem 7.2, p. 39] for the equivalence of definitions using annuli and using quadrilaterals.

3. Thurston's Problem. In Thurston's address (*loc. cit.*), Open Problem No. 1 was the conjecture that the following scheme converges to the Riemann map. As in Figure 1.1, almost fill a simply connected region R with small circles from a regular hexagonal circle packing. Surround these circles by a Jordan curve. Use Andreev's theorem to produce a combinatorially equivalent packing of the unit disk—the unit circle corresponding to the Jordan curve. The correspondence between the circles of the two packings ought to approximate the Riemann mapping.

4. Geometric Lemmas. The following three lemmas will be used in the proof of the main theorem (§5).

Ring Lemma. *There is a constant r depending only on n such that if n circles surround the unit disk* (*i.e., they form a cycle externally tangent to the unit disk*; *see Figure* 4.1) *then each circle has radius at least r.*

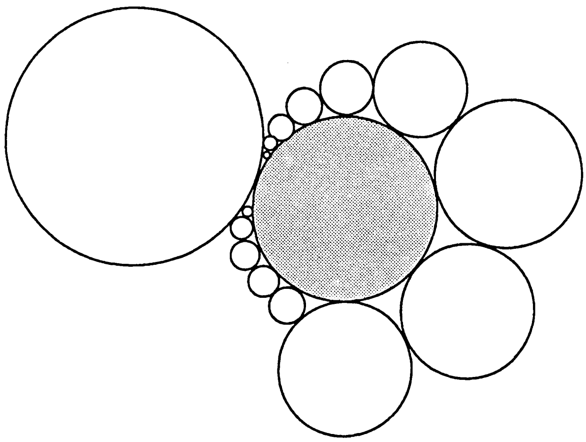

FIGURE 4.1. Fifteen circles surround the unit circle

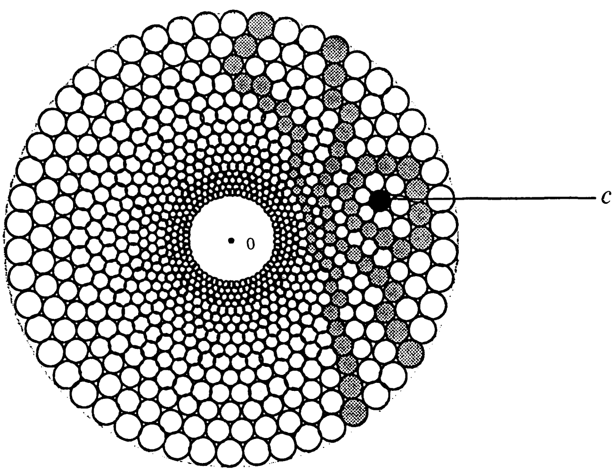

FIGURE 4.2. Chains of combinatorial lengths 13, 21, 31 separate c from 0

Proof. Fix n. There is a uniform lower bound for the radius of the largest outer circle c_1 (namely, that which occurs when all n outer circles are equal). A circle c_2 adjacent to c_1 also has a uniform lower bound for its radius because if c_2 were extremely small then a chain of $n - 1$ circles starting from c_2 could not escape from the crevasse between c_1 and the unit circle. Repeat this reasoning for the circle c_3 adjacent to c_2, and so on.

Length-Area Lemma. *Let c be a circle in a circle packing in the unit disk. Let $S_1, S_2, \cdots, S_k$ be k disjoint chains which separate c from the origin and from a point of the boundary of the disk. Denote the combinatorial lengths of these chains by $n_1, n_2, \cdots, n_k$ (see Figure 4.2). Then*

$$\text{radius}(c) \leqslant \left(n_1^{-1} + n_2^{-1} + \cdots + n_k^{-1}\right)^{-1/2}.$$

Proof. Suppose the chain S_j consists of circles of radius r_{ji}. Then by the Schwarz inequality

$$\left(\sum_i r_{ji}\right)^2 \leqslant n_j \sum_i r_{ji}^2.$$

Let $s_j = 2\sum_i r_{ji}$ be the geometric length of S_j. We obtain

$$s_j^2 n_j^{-1} \leqslant 4 \sum_i r_{ji}^2$$

$$\sum_j s_j^2 n_j^{-1} \leqslant 4 \sum_{ji} r_{ji}^2 \leqslant 4.$$

Thus $s = \min\{s_1, s_2, \cdots, s_k\}$ satisfies

$$s^2 \leqslant 4\left(n_1^{-1} + n_2^{-1} + \cdots n_k^{-1}\right)^{-1}.$$

Since s is greater than the diameter of c, the above inequality proves the Lemma.

Hexagonal Packing Lemma. *There is a sequence s_n, decreasing to zero, with the following property. Let c_0 be a circle in a finite packing P of circles in the plane and suppose the packing P around c_0 is combinatorially equivalent to n generations of the regular hexagonal circle packing about one of its circles. Then the ratio of radii of any two circles in the flower around c differs from unity by less than s_n (see Figure 4.3).*

Corollary. *A circle packing in the plane with the hexagonal pattern is the regular hexagonal packing.*

Remark. The above corollary is actually used in the proof of the Hexagonal Packing Lemma and arose in the Colorado Geometry Seminar with Thurston in 1980. A proof is outlined in Appendix 1 below.

Proof of the Hexagonal Packing Lemma. Suppose that for each $n = 1, 2, \cdots$ we have a circle packing P_n which is combinatorially equivalent to n generations of the regular hexagonal circle packing centered around the unit circle c_0.

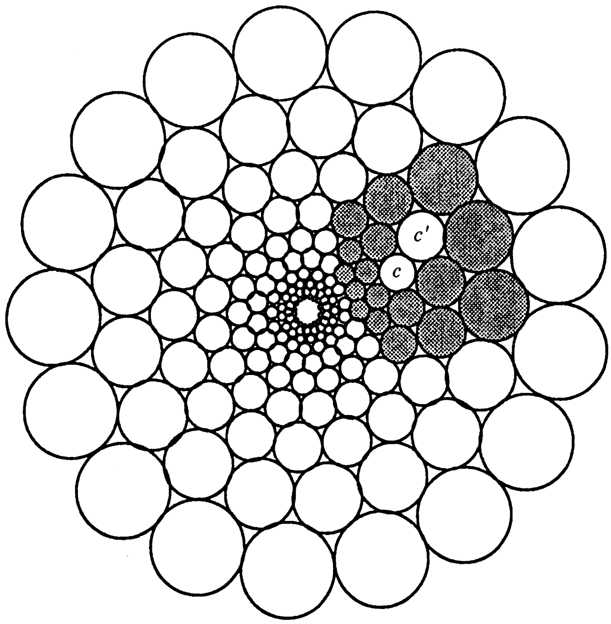

FIGURE 4.3. Two generations: $1 - s_2 < (\text{radius } c_1)/(\text{radius } c_2) < 1 + s_2$

Let us apply the Ring Lemma to the P_n's. It shows that the radii of circles of generation k in P_k, $P_{k+1}, \cdots$ are uniformly bounded away from zero and infinity. Therefore we can select a subsequence of the P_n's so that the generation one circles converge geometrically. A further subsequence can be selected so that the generation two circles converge geometrically, and so on.

In this way a limit infinite packing of a planar region is obtained. This packing has the combinatorics of the regular hexagonal packing and the region is an increasing sequence of disks and so is connected and simply connected.

If the lemma were false we could select the subsequences so that in the limit packing one of the six circles around c_0 would have radius different from one. This contradicts the uniqueness of packings in the plane with the pattern of the hexagonal packing (see Appendix 1).

5. Convergence of circle packings to the Riemann mapping. Let R be a simply connected bounded region in the plane with two distinguished points z_0 and z_1. For a sufficiently small $\varepsilon > 0$ consider the regular hexagonal packing H_ε of the plane by circles of radius ε. Let c_0 be a circle whose flower contains z_0. Form all chains $c_0, c_1, \cdots, c_k$ of circles from H_ε, starting with c_0, such that the flowers of the circles in the chain are contained in R. The circles which appear in such chains are called *inner circles*. The set of inner circles is denoted I_ε.

The circles in H_ε which are not inner circles but which are tangent to inner circles will be called *border circles*. The set B_ε of border circles can be cyclically ordered to form a cycle called the *border*. It has the property that the linear polygon obtained by joining the centers in order is a Jordan curve surrounding the inner circles. The inner circles I_ε and border circles B_ε form a packing C_ε whose nerve is the 1-skeleton of a triangulation T_ε.

Complete T_ε to a topological triangulation T_ε^* of the sphere by adding a vertex at ∞ plus disjoint (except for ∞) Jordan arcs from ∞ to the centers of the border circles.

By Andreev's theorem there is a circle packing of the sphere whose associated triangulation is isomorphic to T_ε^* by an isomorphism that preserves the orientation of the sphere. This circle packing is unique up to Möbius transformations. We partially normalize this packing so that the exterior of the unit disk is the disk whose center corresponds to the vertex ∞ of T_ε^*. We then have a correspondence $c \to c'$ of circles c in $C_\varepsilon \equiv I_\varepsilon \cup B_\varepsilon$ with circles c' in a circle packing C_ε' of the unit disk. We further normalize the situation by a Möbius transformation fixing the unit disk so that c_0' is centered at the origin and c_1', where c_1 is a circle whose flower contains z_1, is centered on the positive real axis.

The correspondence $c \to c'$ of $C_\varepsilon \to C_\varepsilon'$ defines an approximate mapping of R into the unit disk D in the following sense. Let z be any point in R. For ε sufficiently small, z will be in the flower of some c in I_ε. As $\varepsilon \to 0$ such a flower will be surrounded by more and more generations of cycles in I_ε. Therefore, by the Length-Area Lemma and the divergence of the harmonic series, the radius of c' shrinks to zero. Thus c' determines an approximate position for the image of z.

Theorem. *The isomorphism $C_\varepsilon \to C_\varepsilon'$ of circle packings determines an approximate mapping which, as $\varepsilon \to 0$, converges to a conformal homeomorphism of R with the unit disk.*

Proof. As described above, we have a circle packing C_ε in R and an isomorphic circle packing C_ε' of the unit disk D. The associated isomorphic triangulations are denoted T_ε and T_ε'. Let R_ε, D_ε be the carriers of T_ε, T_ε' and let f_ε: $R_\varepsilon \to D_\varepsilon$ be the simplicial map determined by the correspondence of vertices of T_ε and T_ε'. We may assume that f_ε is orientation preserving.

It is clear from the construction that R_ε converges to R in the sense that R is the union of the R_ε and any compact subset of R is contained in all R_ε with sufficiently small positive ε (this is a special case of Carathéodory domain convergence). It is also true that D_ε converges to D in the same sense; this follows from the Length-Area Lemma which shows that the radii of the border circles of C_ε' tend uniformly to zero as $\varepsilon \to 0$ (each border circle is separated from the origin of D by many disjoint chains of combinatorial length $\leqslant 6, 12, 18, \cdots$).

The Ring Lemma shows that the angles of the triangles in T_ε' are bounded away from zero independently of ε. The Ring Lemma shows this directly for the inner triangles of T_ε'. The proof of the Ring Lemma can also be applied to border circles to show that the ratio of radii of a border circle to any adjacent circle in C_ε' is bounded above. It is also bounded away from zero. Thus the maps f_ε: $R_\varepsilon \to D_\varepsilon$ are uniformly K-quasiconformal since they map equilateral triangles to triangles of uniformly bounded distortion.

Since the f_ε are K-quasiconformal they are equicontinuous on compact subsets of R (the same is true for the family f_ε^{-1} on compact subsets of D). This standard fact can be seen, for example, by observing that if z, z' vary in a compact subset of R and $|z - z'|$ becomes arbitrarily small then $|f_\varepsilon(z) - f_\varepsilon(z')|$ cannot remain bounded away from zero for a sequence of $\varepsilon \to 0$. Indeed, one can surround z, z' with annuli A of arbitrarily large modulus; hence $f_\varepsilon(A)$ has arbitrarily large modulus and surrounds $f_\varepsilon(z)$, $f_\varepsilon(z')$. This is impossible if $f_\varepsilon(z)$, $f_\varepsilon(z')$ lie in a bounded region and their distance apart is bounded away from zero.

From the equicontinuity on compacta it follows that the f_ε form a normal family. From the Caratheodory domain convergence of R_ε to R and of D_ε to D it follows that every limit function f is a K-quasiconformal mapping of R onto D with $f(z_0) = 0$ and $f(z_1) > 0$. That $D \supset f(R)$ follows from $f(z_0) = 0$. To see that $D = f(R)$ pick w_0 in D. Let G be a subdomain of D with $0 \in G$, $w_0 \in G$, and $D_\varepsilon \supset G$ for all sufficiently small $\varepsilon > 0$. Consider the restrictions of f_ε^{-1} to G; denote them by g_ε. Choose $\varepsilon(n) \to 0+$ such that $f_{\varepsilon(n)} \to f$ and $g_{\varepsilon(n)} \to g$ uniformly on compacta. Now $R \supset g(G)$ because $g(0) = z_0$. It follows from $f_{\varepsilon(n)}(g_{\varepsilon(n)}(w_0)) = w_0$, using the uniform convergence of $f_{\varepsilon(n)}$ near $g(w_0)$, that $f(g(w_0)) = w_0$. Thus $D = f(R)$. f is one-to-one since the roles of R and D can be reversed.

The Hexagonal Packing Lemma shows that the simplicial mapping f_ε restricted to a fixed compact subset of R maps equilateral triangles to triangles of T_ε' that become arbitrarily close to equilateral as $\varepsilon \to 0$. Therefore any limit function f of the f_ε's will be 1-conformal and therefore conformal. Since $f(z_0) = 0$ and $f(z_1) > 0$ we see that all limit functions are equal to the unique Riemann mapping with this normalization.

Remarks. There are a number of questions about packings and conformal mapping that are still open. Thurston's Open Problem No. 2 (*loc. cit.*) concerned existence and uniqueness questions for infinite packings. The Hexagonal Packing Lemma is a uniqueness result for infinite packings of given combinatorial type (it can be generalized to give uniqueness for any pattern with a uniformly bounded number of edges at each vertex). It follows from the proof of the Length-Area Lemma, using the spherical metric, that: *If a simply connected plane region is the carrier of an infinite circle packing, and if* $\sum n_k^{-1}$ *diverges, where* n_k *is the number of circles in generation* k *from a fixed circle, then the region must be the whole plane.*

Let $c \to c'$ be an approximate Riemann mapping function as in the Theorem. Is it true that the function $c \to \text{radius}(c')/\text{radius}(c)$ is an approximation to the modulus of the derivative of the Riemann mapping function? A related question is to estimate the s_n of the Hexagonal Packing Lemma.

Appendix 1

The Uniqueness of the Regular Hexagonal Packing

We sketch the proof that the regular hexagonal circle packing of the plane is the only packing of a simply connected planar region which has the hexagonal pattern.

The simplicial map between the triangulation associated to the regular hexagonal packing H and any other one H' with the same combinatorics is a homeomorphism of the plane with the carrier of H', and we may assume it is orientation preserving. By the Ring Lemma this homeomorphism is a quasiconformal mapping. Therefore the carrier of H' is also the plane.

This simplicial map can be modified to become a quasiconformal mapping of the plane which sends the circles of H to the corresponding circles of H'. Extend it to ∞ to become a quasiconformal map of the sphere and denote it by g.

For each n consider the finite packings H_n and H_n' consisting of the first n generations around fixed base circles. Let G_n and G_n' be the Schottky groups generated by inversions in the circles of H_n and H_n' respectively.

A fundamental domain for G_n' on the sphere consists of the triangular interstices of H_n' together with the unbounded component of the complement of the union of the circles in H_n'. There is a similar picture for G_n. Now G_n and G_n' are geometrically finite groups which are quasiconformally conjugate (Marden [8]). The map g provides a K-quasiconformal mapping between the Riemann surfaces of G_n and G_n'. Thus by Maskit [9] there is a K-quasiconformal conjugacy g_n between G_n and G_n' agreeing with g in the unbounded component and conformal on the triangular interstices.

We take a limit of these g_n to obtain a K-quasiconformal conjugacy h between $G_\infty = \cup G_n$ and $G_\infty' = \cup G_n'$ which is conformal on each triangle. Thus we have a new quasiconformal homeomorphism h which sends circles in

H_n to corresponding circles in H_n', which is conformal in the triangular interstices, and which conjugates G_n to G_n'. Thus h is conformal on all of the sphere except the limit sets of these groups. If we knew these had measure zero we could conclude that h was conformal everywhere and hence a Euclidean similarity.

Without knowing this we can reach the same conclusion by appealing to two theorems from Sullivan [10; Theorems I and IV]. These theorems together give: *If the 3-dimensional fundamental domain of a discrete group G of conformal transformations of the 2-sphere intersects the limit set of G in a set of measure zero, then any quasiconformal homeomorphism conjugating G to another discrete group is already conformal on the entire sphere if it is conformal off the limit set.* In our case the fundamental domain in the 3-ball is obtained by removing the half spaces determined by the disks of the packing. This convex set intersects the limit set in a countable set.

Appendix 2

Thurston's Algorithm for calculating the Andreev Realization

The theorem in §5 shows that the problem of approximating the Riemann map of a region reduces to constructing a circle packing with a prescribed pattern. Thurston has presented the following elegant algorithm for accomplishing this (the higher genus case is treated in [11] pages 13.44 ff.).

Let T be an imbedded triangulation of the extended plane. We may assume that infinity is an interior point of one of the faces. We shall construct a circle packing in the nonextended plane whose nerve is isomorphic to the 1-skeleton of T.

Let v_1, v_2, v_3 denote the vertices of the face which contains infinity. Denote the remaining vertices of T by v_4, $v_5, \cdots, v_n$. In the realization of T by means of a circle packing, the radius of the circle whose center corresponds to v_i is denoted r_i; the circles whose centers correspond to v_1, v_2, v_3 are normalized to have radius 1. The values of r_4, $r_5, \cdots, r_n$ will be found by successive approximation (see Figure A2.1).

Let $(r_{k1}, r_{k2}, \cdots, r_{kn})$, a vector of positive real numbers with $r_{k1} = r_{k2} = r_{k3} = 1$, be the kth approximation to $(r_1, r_2, \cdots, r_n)$. To each triangle in T there is associated, by means of $(r_{k1}, r_{k2}, \cdots, r_{kn})$, a Euclidean triangle as follows: if the vertices of the triangle are $v_{m(1)}, v_{m(2)}, v_{m(3)}$ then the associated Euclidean triangle has sides of lengths $r_{km(1)} + r_{km(2)}$, $r_{km(1)} + r_{km(3)}$, $r_{km(2)} + r_{km(3)}$. In order to obtain the $(k + 1)$-st approximation $(r_{(k+1)1}, r_{(k+1)2}, \cdots, r_{(k+1)n})$

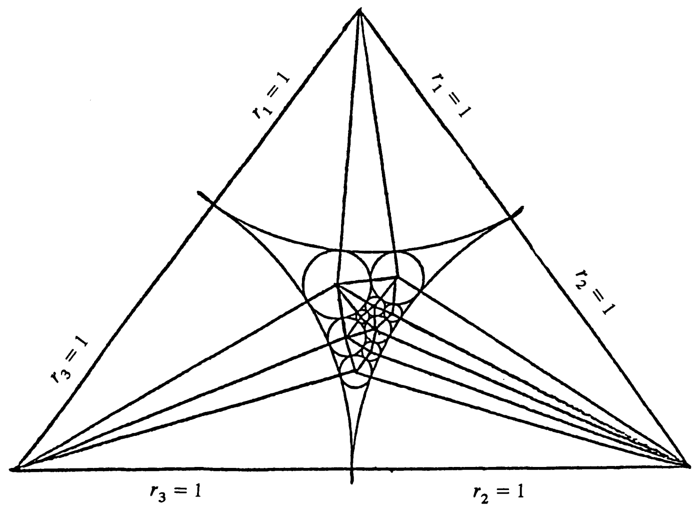

FIGURE A2.1

from the kth approximation one first defines the curvature of $(r_{k1}, r_{k2}, \cdots, r_{kn})$ at the vertex $v_i (i \geqslant 4)$ to be $2\pi - \sum_{j=1}^{6} \theta(j)$, where the $\theta(j)$'s are the angles at v_i in the Euclidean triangles which $(r_{k1}, \cdots, r_{kn})$ associates to those triangles of T belonging to the star of v_i. The curvature of $(r_{k1}, \cdots, r_{kn})$ at $v_i (i \geqslant 4)$ is a monotonically increasing function of r_{ki}. For $i \geqslant 4$ define $r_{(k+1)i}$ to be the unique positive number which makes the curvature of $(r'_{k1}, \cdots, r'_{kn})$ at v_i vanish, where $r'_{kj} = r_{kj}$ for $j \neq i$ and $r'_{ki} = r_{(k+1)i}$. Define $r_{(k+1)i} = 1$ for $i = 1, 2, 3$. This determines the $(k+1)$-st approximation. Since the change in r_{ki} primarily affects the curvature at v_i and has relatively little effect on the curvatures at other vertices, this process would be expected to converge fairly rapidly.

Suppose the process converges to $(r_1, \cdots, r_n)$, a vector of positive numbers with $r_1 = r_2 = r_3 = 1$ and such that the curvature of $(r_1, \cdots, r_n)$ at each v_i $(i \geqslant 4)$ is zero. It is then fairly clear that the Euclidean triangles associated to the triangles in T other than (v_1, v_2, v_3) by $(r_1, \cdots, r_n)$ can be laid flat in the plane, with their interiors disjoint, so that edges identified in T actually coincide in the plane. They induce an embedded triangulation of a triangle of sides $2, 2, 2$. The 1-skeleton of this triangulation is isomorphic to T and it is the nerve of the packing of circles of radii $1, 1, 1, r_4, \cdots, r_n$ with centers where the points corresponding to the vertices $v_1, \cdots, v_n$ of T have been placed.

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UNIVERSITY OF CALIFORNIA AT SAN DIEGO
CITY UNIVERSITY OF NEW YORK