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Classical Teichmüller theory and (2 + 1) gravity

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Abstract

We consider classical Teichmüller theory and the geodesic flow on the cotangent bundle of the Teichmüller space. We show that the corresponding orbits provide a canonical description of certain $(2+1)$ gravity systems in which a set of point-like particles evolve in universes with topology $\Sigma_g \times \mathbb{R}$ where Σ_g is a Riemann surface of genus $g > 1$. We construct an explicit York's slicing presentation of the associated spacetimes, we give an interpretation of the asymptotic states in terms of measured foliations and discuss the structure of the phase spaces. © 1998 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

Gravity in $(2+1)$ dimensions can be understood as a non trivial toy model of the physically significant $(3+1)$-dimensional case. Einstein equations of $(2+1)$ gravity simplify because the Ricci tensor determines the Riemann tensor: this $(2+1)$-dimensional property allows also for an approach to the problem which is much more geometric than analytic. A matter-free solution of topological type $\Sigma_g \times \mathbb{R}$, where Σ_g is a compact surface of genus g, will be called an empty universe of type g and is a locally Minkowskian 3-manifold fibred by the spatial surfaces $\Sigma(t) = \Sigma_g \times \{t\}$.

Two main approaches have been used to study empty universes of type g: one [1,2] is based on the holonomy representations of $\pi_1(\Sigma_g)$ in the Poincaré group $ISO(2,1)$ and the other [3,4] makes use of the canonical ADM construction. Both of them identify the cotangent bundle T_g^* of the Teichmüller space T_g as the phase space of the empty universes of type g. In fact, the precise correspondence between these two identifications is explicitly known [5] only for genus $g = 1$; for $g > 1$ several open questions remain to be answered.

In this article we shall show that classical Teichmüller theory also allows for an interpretation of T_g^* as a subset of the phase space of certain $(2+1)$ gravity systems with ''matter''. More precisely we will show that, with respect to the dynamics described by the Teichmüller geodesic flow in T_g^* for $g > 1$, each Teichmüller

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line gives a canonical description of the time evolution of a particular set of point-like gravitating particles moving on a 2-dimensional surface of genus g. In the $g=1$ case, our construction reproduces the known [2,4,5] matter-free solutions.

Let us give a short description of the main geometric idea. Our result is based on the properties of the Teichmüller deformations of the conformal structure associated with a Riemann surface. For each Teichmüller line, the corresponding deformations are the result of a stretching of the leaves of a particular measured foliation. This foliation determines a canonical flat structure on the surface with conical singularities which can be interpreted as point-like particles. The effect of these particles is to ‘‘flatten’’ spatial slices. The parameter which determines the strength of the stretching is identified with ‘‘time’’ and, with the spatially flat metric on each time slice, one can construct a spacetime which (in the complement of the particle world-lines) is flat. Moreover, the time slices have constant extrinsic curvature (again, far from the singularities) which is simply related to the time parameter; so, our construction provides a natural York slicing of spacetime.

In the first part of this article, we summarize a few basic results of classical Teichmüller theory and the geometry of quadratic differentials [6–9] for Riemann surfaces of genus $g \geq 1$. Then, we construct the spacetimes of the gravity systems associated with Teichmüller lines and we produce an explicit York slicing description of each of them. Finally, we give an interpretation of the associated asymptotic states in terms of measured foliations on Σ_g and we discuss the structure of the phase space.

2. The geometry of quadratic differentials and Teichmüller theory

Let S be a compact Riemann surface of genus $g \geq 1$. A quadratic differential ω on S is a holomorphic map $\omega{:}TS \to \mathbb{C}$, where TS is the complex tangent bundle on S, such that for any point $p \in S$ the restriction of ω to TS_p is a quadratic form. In a system of local complex coordinates $z = x + iy$, we write $\omega = \varphi(z) dz^2$. Let us denote by $Q(S)$ the set of quadratic differentials on S; $Q(S)$ is a complex vector space and, by Riemann-Roch Theorem, it has complex dimension $3g-3$ when $g \geq 2$ and complex dimension 1 for $g=1$. One can introduce a norm on $Q(S)$ according to

$$\|\omega\| = \int_S |\omega| = \int_S |\varphi| dxdy \, .$$

Any given quadratic differential $\omega \in Q(S) \setminus \{0\}$ selects a particular set of points on S in which ω vanishes. If $p \in S$ is a zero of ω, one can find a local coordinate z at p, with $z(p) = 0$, such that

$$\omega = z^m dz^2 \, .$$

The positive integer m is the order of ω at p and the coordinate z is unique up to rotations of angles $2\pi n/(m+2)$ with integer n.

Let $\omega \in Q(S) \setminus \{0\}$; a vector $X \in TS_p$ is called ω-horizontal if $\omega(X)$ is real and strictly positive. The vector $Y \in TS_p$ is ω-vertical if $\omega(Y)$ is real and strictly negative. Let us now consider the set $\hat{S} = S \setminus \{\text{zeros of } \omega\}$. For every $p \in \hat{S}$, TS_p has one ω-horizontal and one ω-vertical directions which are mutually orthogonal with respect to the Riemannian ω-metric on $\hat{S}$

$$ds^2_\omega = |\varphi| |dz|^2 \, .$$

The integral lines of these two fields of directions are the leaves of the ω-horizontal and ω-vertical foliations on $\hat{S}$. Both the ω-foliations and the ω-metric extend to the whole surface S with singularities at the zeros of the quadratic differential ω. We shall now describe the geometry induced on S by the ω-metric and the associated ω-foliations. We shall firstly concentrate on $\hat{S}$ and then we shall analyze the singular points.

In a neighbourhood of any $p \in \hat{S}$ where the coordinate z satisfies $z(p)=0$, consider the change of coordinates

$$\zeta(z) = \int^{z} \sqrt{\omega} = \int_0^{z} \sqrt{\varphi(z')}\, dz' ,$$

where a fixed choice of the sign for the square root has been made. If $\zeta = \chi + i\eta$, the ω-horizontal lines become $\{\eta = \text{constant}\}$ and the ω-vertical lines correspond to $\{\chi = \text{constant}\}$. Moreover, $\omega = d\zeta^2$ and $ds^2_\omega = d\chi^2 + d\eta^2$. This makes clear that the Riemannian ω-metric on $\hat{S}$ is flat.

Assume now that p is a zero of ω and let $\omega = z^m dz^2$ in a chart such that $z(p)=0$. Let us distinguish two possibilities. When m is even, $\sqrt{\omega}$ has a single valued branch and $\zeta = \zeta(z)$ is a $(m+2)/2$-sheeted branched covering ramified over 0. Consequently, as far as the horizontal lines are concerned, $\zeta^{-1}(\mathbb{R})$ consists of $(m+2)$ analytic rays emanating from 0 which are equally spaced. One finds a similar structure for the vertical lines. When m is odd, one can reduce the discussion to the even case by using the trick of passing to the double covering, in which one puts $z_1^2 = z$, with the effect of getting formally the same result as in the $m =$ even case. For example, the structure of the ω-foliations in a neighborhood of a point in which ω has order 1 is shown in Fig. 1.

The ω-foliations on $\hat{S}$ extend to singular foliations on S which determine $(m+2)$ sectors at a zero of ω of order m; each of these sectors has angles equal to π with respect to ds^2_ω. Consequently, the ω-metric for S is a flat metric with a conical singularity of angle $(m+2)\pi$ at each zero of ω of order m. Finally, the area of S is equal to $\|\omega\|$ and the ω-metric naturally induces a transverse measure on the ω-foliations.

By construction, the flat metric associated to ω admits an atlas for $\hat{S}$ with values in Euclidean $\mathbb{R}^2$ in which the changes of coordinates are compositions of translations and of the rotation of angle π exclusively. In fact, this property characterizes uniquely the flat structures induced by quadratic differentials. More precisely, given a flat metric with conical singularities admitting such a kind of atlas on a topological surface Σ_g, one can reconstruct a conformal structure on Σ_g and a quadratic differential ω inducing the given flat metric. The ω-horizontal and the ω-vertical foliations are just the pull-back on Σ_g of the horizontal and vertical straight lines of $\mathbb{R}^2$.

A version of the Gauss-Bonnet formula gives the relation

$$\sum_i m_i = -2\chi(\Sigma_g) = 4(g-1) ,$$

where $\chi(\Sigma_g)$ is the Euler characteristic and $\{m_i\}$ are the orders of the zeros of ω. In particular, this implies that for $g=1$ a quadratic differential ω has no zeros.

Let us now describe the deformations of the conformal structure according to Teichmüller theory. Teichmüller space T_g is the ‘‘orbifold’’ universal covering of the moduli space $\mathscr{M}_g$ of conformal equivalence classes of Riemann surfaces of genus g. The elements of T_g are equivalence classes of ‘‘marked’’ Riemann surfaces, i.e. of homeomorphisms $\phi: \Sigma_g \to S$, where Σ_g is a fixed topological surface of genus g and S is a Riemann surface. Two marked surfaces (S_i, ϕ_i) with $i = 1,2$ are equivalent iff there exists a conformal map $f: S_1 \to S_2$

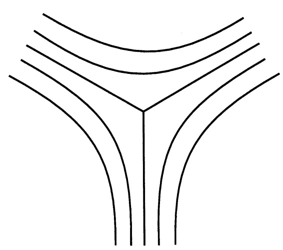

Fig. 1. Horizontal lines at a simple zero of ω.

such that $\phi_1 \circ f \circ \phi_2^{-1}$ is a automorphism of Σ_g which is isotopic to the identity. We will denote by $[\phi:\Sigma_g \to S]$ the element of T_g represented by the marked surface (S,ϕ).

Let $\alpha=[\phi:\Sigma_g \to S] \in T_g$, $\omega \in Q(S)\setminus\{0\}$ and $k\in[0,1)$. One can deform the conformal structure on S as follows: by using the coordinates ζ defined on $\hat{S}$, set

$$\zeta' = \frac{\zeta + k\bar{\zeta}}{1-k} .$$

If $\zeta=\chi+i\eta$, one finds $\zeta'=\tau\chi+i\eta$ where $\tau=(1+k)/(1-k)$. In this way one can define a new flat metric

$$ds_\tau^2 = d\zeta' d\bar{\zeta}' = \tau^2 d\chi^2 + d\eta^2$$

on $\hat{S}$ which coincides with ds_ω^2 for $\tau=1$. Note that any atlas for $(\hat{S},ds_\omega^2)$ with values in $(\mathbb{R}^2, d\chi^2+d\eta^2)$, with changes of coordinates given only by combinations of translations and π-rotations, is also an atlas for $(\hat{S},ds_\tau^2)$ with values in $(\mathbb{R}^2,\tau^2 d\chi^2+d\eta^2)$. Therefore, ds_τ^2 induces a conformal structure S_τ^ω carrying a quadratic differential ω_τ such that

$$ds_\tau^2 = ds_{\omega_\tau}^2 .$$

Moreover, ω_τ and ω have the same zeros with the same orders, the horizontal and vertical foliations are constant in τ apart from the transverse measure: the length of the vertical lines remains unchanged whereas the length of the horizontal lines gets stretched by a factor τ. The line in Teichmüller space

$$\begin{array}{lcl} [1,+\infty) & \to & T_g \\ \tau & \to & \alpha_\tau^\omega = \left[\mathrm{id}\circ\phi : \Sigma_g \to S_\tau^\omega\right] \end{array}$$

is called the Teichmüller ray based on (α,ω). Note that the substitution of ω by $\beta\omega$ with $\beta>0$ does not modifies the Teichmüller ray. Hence, the Teichmüller rays are labeled by the unitary sphere $\partial Q^1(S)$ in $Q(S)$.

Now, fix $\alpha=[\phi:\Sigma_g\to S]\in T_g$ and let $Q^1(S)$ be the unitary open ball in $Q(S)$. Let us define $p_\alpha:Q^1(S)\to T_g$ by

$$p_\alpha(\omega) = \begin{cases} \alpha & \text{if } \omega=0 \\ \alpha_{b(\omega)}^{\tilde{\omega}} & \text{otherwise} \end{cases}$$

where $\tilde{\omega}=\omega/\|\omega\|$ and $b(\omega)=(1+\|\omega\|)/(1-\|\omega\|)$. A fundamental result of Teichmüller theory states that, for every $\alpha\in T_g$, p_α is bijective.

For every $\alpha,\beta\in T_g$ such that $\beta=p_\alpha(\omega)$, define

$$d_T(\alpha,\beta) = \frac{1}{2}\log\left(\frac{1+\|\omega\|}{1-\|\omega\|}\right).$$

d_T is a well defined distance on T_g called the Teichmüller distance; in fact, d_T is induced by a Finslerian metric on T_g with respect to a compatible differential structure on T_g, the Teichmüller rays are geodesic rays and each p_α is a diffeomorphism. The cotangent bundle T_g^* is identified with

$$T_g^* \approx \{(\alpha,\omega)|\alpha=[(S,\phi)]\in T_g,\ \omega\in Q(S)\}.$$

$(\alpha_\tau^\omega,\omega_\tau/\tau)$ with $\tau\in(0,\infty)$ is the Teichmüller geodetic flow on T_g^* governed by the Lagrangian $\|\omega\|^2/2$. Note that we have extended each Teichmüller ray to a complete oriented Teichmüller line by setting $\tau\in(0,\infty)$ in the previous formulae. According to this definition, a Teichmüller line based on (α,ω) is just the union of two Teichmüller rays: one of them is based on (α,ω) and the other is based on $(\alpha,-\omega)$.

3. Spacetime for a gravity system

The key observation, which allows us to associate a $(2+1)$ gravity system to each Teichmüller line, is contained in the following change of coordinates. In $\mathbb{R}^3$ with coordinates (u,y,τ), consider the upper half-plane $\Pi = \{\tau > 0\}$ with the metric of signature $(++-)$ given by

$$\tau^2 du^2 + dy^2 - d\tau^2$$

This metric is flat; indeed, under the change of coordinates

$$x = \tau \operatorname{sh} u, \quad y = y, \quad t = \tau \operatorname{ch} u,$$

the set Π goes onto the open domain $\Delta = \{t > 0, x^2 - t^2 < 0\}$ of the standard Minkowski space with coordinates (x,y,t) and metric $dx^2 + dy^2 - dt^2$. The constant-time hyperplanes $\{\tau = \tau_0\}_{\tau_0 > 0}$ have constant extrinsic mean curvature equal to $1/(2\tau_0)$, so we say that they realize a natural York slicing of Π. Note that the isometry group of Π is isomorphic with the subgroup of the Poincaré group $ISO(2,1)$ having Δ as invariant subset and consists of combinations of translations parallel to the ($\tau =$ constant) planes and the π-rotation around the τ axis. Then, as τ varies we see a one-parameter family of flat metrics on $\mathbb{R}^2$ which is formally the same occurring in the Teichmüller deformation.

Consider $g > 1$ and the Teichmüller line based on $\xi = (\alpha, \omega) \in T_g^*$, with $\omega \neq 0$, and the homeomorphism

$$\psi : \Sigma_g \times (0,\infty) \to M$$

where M is a 3-manifold fibred by the surfaces $S_\tau^\omega \equiv \psi(\Sigma_g \times \{\tau\})$ with $\tau \in (0,\infty)$. Let S_τ^ω be endowed with the flat metric $ds^2_{\omega_\tau}$ with conical singularities. Let us define $\hat{S}_\tau^\omega = S_\tau^\omega \setminus \{\text{zeros of } \omega_\tau\}$ and $M' = \cup_\tau \hat{S}_\tau^\omega$. One can give M' the metric

$$ds^2_\xi = ds^2_{\omega_\tau} - d\tau^2 , \tag{1}$$

which is flat and locally Minkowskian on M'; in fact, it is immediate to produce an atlas of M' modeled on the natural York slice of Π so that the surfaces $\hat{S}_\tau^\omega$ correspond to the τ-constant hyperplanes.

Let us consider the extension of the 3-dimensional metric (1) to the entire manifold M. Since the metric (1) has vanishing shift-vectors, the conical singularities of $ds^2_{\omega_\tau}$ on S_τ^ω survive in the three-dimensional context and contribute to the three-dimensional curvature for any τ. We shall now prove that the 3-manifold M is flat with the exception of the world-lines (that we call the singular lines) associated with the zeros of the quadratic differential ω; at each zero of ω of order m, one has a spatial conical singularity of angle $(m+2)\pi$.

Consider a tubular neighbourhood V of a singular line, $V \subset M$; we shall denote by V' the complement of the singular line in V. For $\tau = 1$ suppose that, in local coordinates, one has $\omega = z^m dz^2$. In a neighbourhood of $z = 0$, the two-dimensional metric induced by the Teichmüller deformation is given by

$$ds^2_{\omega_\tau} = \frac{4}{(m+2)^2}\, d\left(\frac{z^{(m+2)/2} + k\bar{z}^{(m+2)/2}}{1-k}\right) d\left(\frac{\bar{z}^{(m+2)/2} + kz^{(m+2)/2}}{1-k}\right).$$

By using polar coordinates $z = r\cos\theta + ir\sin\theta$, one has

$$z^{(m+2)/2} = r^{(m+2)/2}\cos\left(\frac{m+2}{2}\theta\right) + ir^{(m+2)/2}\sin\left(\frac{m+2}{2}\theta\right) = A(r,\theta) + iB(r,\theta),$$

and, since $\tau = (1+k)/(1-k)$, one finds

$$ds^2_{\omega_\tau} = \frac{4}{(m+2)^2}\left\{\tau^2(dA(r,\theta))^2 + (dB(r,\theta))^2\right\}.$$

Let us assume that the order m of the zero of ω is even. We shall now prove that V' is isometric with a $(m+2)/2$-branched covering of a locally Minkowskian manifold. Indeed, consider $\mathbb{R}^3$ with coordinates (x,y,t) endowed with the usual Minkowski metric $ds^2 = dx^2 + dy^2 - dt^2$ and the map given by

$$t = \tau \, \mathrm{ch}\left[\frac{2}{m+2} A(r,\theta)\right], \quad x = \tau \, \mathrm{sh}\left[\frac{2}{m+2} A(r,\theta)\right], \quad y = \frac{2}{m+2} B(r,\theta) . \tag{2}$$

The pull-back of the metric ds^2 in the coordinates (r,θ,τ) is

$$\frac{4}{(m+2)^2} \tau^2 (dA(r,\theta))^2 + \frac{4}{(m+2)^2} (dB(r,\theta))^2 - d\tau^2 = ds^2_{\omega_\tau} - d\tau^2$$

which coincides precisely with the three-dimensional metric ds^2_ξ on V'. The singular line in V is mapped into the straight line $(x=0, y=0)$ in $\mathbb{R}^3$ which can be interpreted as the world-line of a ‘‘static particle’’. The map (2) corresponds to a $(m+2)/2$-fold cyclic covering of the complement of the $(x=0, y=0)$ straight line in $\mathbb{R}^3$; therefore, the extension of the metric (1) in V has a conical spatial singularity of angle $(m+2)\pi$. When m is odd, one can apply the same argument to the double covering of V' and one obtains the same final result; namely, the conical singularity is of angle $(m+2)\pi$.

To sum up, each Teichmüller line is canonically associated with a 3-manifold M equipped with the metric (1) of Lorentz signature which is locally flat and has conical spatial singularities along the world-lines associated with the zeros of the corresponding quadratic differential. Thus, M can be interpreted [8–12] as the spacetime of a certain $(2+1)$ gravity system containing point-like gravitating particles moving on a two-dimensional surface of genus $g>1$. In fact, for a localized particle of mass $\mu>0$, the associated conical singularity in its rest frame has angle [11]

$$2\pi(1-4G\mu) ,$$

where G is the three-dimensional gravitational constant. As noted by ’t Hooft [13], in $(2+1)$ dimensions the sign of G is not fixed a priori; this property is also connected with the existence of a Chern-Simons interpretation [1] of $(2+1)$ gravity. With positive G, the conical singularity associated with the world-line of a particle of small mass μ has angle less than 2π; whereas for negative G the conical singularity has angle greater than 2π. As far as our particular gravity systems are concerned, the conical singularities of angles $(m+2)\pi$ in M admit a physical interpretation in terms, for instance, of particles of mass $\mu = -m/8G$ with negative G. The number of particles associated with the zeros of $\omega \in Q(S) \setminus \{0\}$ and their masses are constrained by the Gauss-Bonnet formula and the total mass is equal to $\chi(\Sigma_g)/4G$.

When $g=1$, the same construction reproduces the nonstatic matter-free solutions studied in [2–5]. Our interpretation based on the Teichmüller flow gives also a clear explanation of the already noted [4] fact that the orbit in T_1, which is identified with the Poincaré disc, is in fact geodesic.

In the remaining part of this article, we will call these universes associated with Teichmüller lines (for $g \geq 1$) the Teichmüller universes.

4. Asymptotic states

For each Teichmüller universe M, which is associated to a Teichmüller line based on (α,ω), we have selected a canonical time τ on M realizing a York slicing of its matter-free part M'; the constant-time spatial surfaces have mean extrinsic curvature $1/(2\tau)$ and area $\tau\|\omega\|$. Let us consider the asymptotic behaviour of the universe in the ‘‘initial time’’ $\tau \to 0$ limit and in the ‘‘final time’’ $\tau \to \infty$ limit. It seems physically interesting to note that, for asymptotic times where time and metric do not exist, one still has something more than just a ‘‘topological shadow’’ (the genus): in fact, metrics degenerate to measured foliations. We shall now elaborate on this point.

As we have seen in first section, T_g is star-shaped by the geodesic Teichmüller rays emanating from any fixed base point $\alpha \in T_g$. By adding the end-point of each ray, one gets the so-called Teichmüller compactification $\overline{T}_g(\alpha)$. When $g > 1$, $\overline{T}_g(\alpha) \approx \overline{B}^{6g-6}$ and this compactification actually depends on α; whereas $\overline{T}_1(\alpha) \approx \overline{B}^2$.

It is natural to identify the end-point of each ray by the associated ω-vertical measured foliation $\mathscr{F}_v = \mathscr{F}_v(\omega)$. The physical interpretation of the asymptotic states, which are defined by measured foliations, is based on a set of observables connected to the length of simple curves. Indeed, let us denote by $\mathscr{S}$ the set of isotopy classes of essential simple curves on Σ_g. For each $\tau \in [1,\infty)$ and for each $\gamma \in \mathscr{S}$, set

$$\ell_\tau(\gamma) = \inf_{C \in \gamma} \ell_\tau(C) ,$$

where $\ell_\tau(C)$ is the length of the curve C with respect to the metric ds_τ^2 on Σ_g. Let us denote by ρ_v the transverse measure of the foliation $\mathscr{F}_v$ and define

$$i(\mathscr{F}_v,\gamma) = \inf_{C \in \gamma} \rho_v(C) .$$

Then, it is not hard to show that for every $\gamma \in \mathscr{S}$ one has (see [9])

$$\lim_{\tau \to \infty} \frac{\ell_\tau(\gamma)}{\tau} = i(\mathscr{F}_v,\gamma) .$$

Each Teichmüller line based on (α,ω) determines two end-points on the boundary of $\overline{T}_g(\alpha)$. The measured foliation associated with the asymptotic $\tau \to \infty$ final configuration is given by the ω-vertical foliation $\mathscr{F}_v(\omega)$. Whereas the ω-horizontal foliation $\mathscr{F}_h(\omega) = \mathscr{F}_v(-\omega)$ describes the asymptotic $\tau \to 0$ initial configuration. Therefore, $\mathscr{F}_h(\omega)$ and $\mathscr{F}_v(\omega)$ are interpreted as the asymptotic states of the Teichmüller universe.

When $g=1$, the Teichmüller compactification $\overline{T}_1(\alpha)$ does not depend on α and coincides with the $g=1$ version of Thurston compactification. In this case, the Teichmüller universes are completely determined by the asymptotic states up to an overall rescaling.

For $g>1$, the compactification $\overline{T}_g(\alpha)$ nontrivially depends [9] on the choice of the base point α. Consequently, in order to reconstruct the spacetime of the universe, the knowledge of the asymptotic states must be supplemented by a specific choice of the base point $\alpha \in T_g$.

5. On the phase spaces

Several mathematical results in Teichmüller theory, see [9,14–16], can be reinterpreted in the present $(2+1)$ gravity context. In this section we present a few examples. Assume $g>1$ and $G<0$ as before. Given a set of N gravitating point-like particles moving on a surface of genus g, the type of this gravity system is, by definition,

$$(g,\{\alpha_i\})$$

where

$$\alpha_i = 2\pi(1-4G\mu_i)$$

and $\mu_i > 0$ is a mass for each $i = 1,2,\dots,N$. Two basic general questions arise:

(1) Determine all the couples $(g,\{\alpha_i\})$ which are the type of any gravity system.

(2) For each type, describe the phase space of the universes realizing it.

To our knowledge, a complete answer to question (1) is not known. As far as question (2) is concerned, it seems to be generally accepted that the dimension of the phase space is given by

$$12g - 12 + 4N = \dim_{\mathbb{R}} T^*_{g,N}$$

where $T^*_{g,N}$ denotes the cotangent bundle of the Teichmüller space for N-punctured surfaces of genus g.

These questions can be specialized in the framework of Teichmüller universes; the answers in this case suggest a few general speculations. For Teichmüller universes question (1) has a complete answer:

$(g,\{\alpha_i\})$ is the type of a Teichmüller universe if and only if each α_i is of the form $\alpha_i = (m_i+2)\pi$

with $m_i \in \mathbb{N}$, $m_i \geq 1$, the Gauss-Bonnet relation is satisfied and $(g,\{\alpha_i\}) \neq (2,\{1,3\})$.

Apart from the exceptional case, we already mentioned the ''only if'' part of this statement; the ''if'' part is nontrivial, see [14]. On the other hand, not every universe of this type is a Teichmüller universe. For example, for any $\omega \in Q(S) \setminus \{0\}$ consider the static universe $S \times \mathbb{R}$ with the product metric $ds_\omega^2 - dt^2$.

Let us now consider question (2); first of all there is a subtle problem concerning which kind of ''isomorphism relation'' one stipulates to work with. This problem includes the assumption on whether particles with equal masses may be or may not be distinguished from each other. These two possibilities reflect on the choice of the mapping class group for the N-punctured surfaces of genus g which must be used in passing from the ''Teichmüller space'' level to the ''moduli space'' level. Apart from this subtle point, the phase space of Teichmüller universes has a rather complicated structure, see [15,16]. Each admissible type $(g,\{\alpha_i=(m_i+2)\pi\})$ splits into two augmented types $(g,\{\alpha_i\},\epsilon)$ where $\epsilon = \pm 1$. T_g^* is stratified according to the augmented types as follows: $(\alpha,\omega) \in T_g^*$ belongs to the stratum of type $(g,\{\alpha_i\},\epsilon)$ if and only if it determines a universe of type $(g,\{\alpha_i\})$ and $\epsilon = 1$ iff ω is the square of a holomorphic Abelian differential. The stratum corresponding to the type $(g,\{\alpha_i\},\epsilon)$, if nonempty, is a submanifold of T_g^* of real dimension

$$4g + 2\sum_j \nu(j) + \epsilon - 3,$$

where, for integer $j \geq 1$, $\nu(j)$ is the cardinality of $\{m_i | m_i = j\}$. This stratification is invariant for the Teichmüller flow. When at least one m_i is odd, the stratum of type $(g,\{\alpha_i\},1)$ is empty. Whereas when all the m_i are even, the type $(g,\{\alpha_i\})$ correspond to two nonempty strata of different dimensions, with the exception of $(2,\{4\})$ as the stratum $(2,\{4\},-1)$ is empty.

The maximal dimension is realized onlyin the stable case in which all $m_i = 1$. It should be noted that this top dimensional stratum is a topological nontrivial open dense subset of T_g^* (in particular, its fundamental group is nontrivial).

These facts suggest the following speculations/conjectures.

C1. For a given type $(g,\{\alpha_i\})$ which does not correspond necessarily to a Teichmüller universe, the expected dimension $\dim_{\mathbb{R}} T_{g,N}^*$ is realized only for a stable situation (in a sense to be specified). For gravity systems admitting Teichmüller universes, we conjecture that $\dim_{\mathbb{R}} T_{g,N}^*$ is realized only for the type

$$(g,\{1,1,\dots,1\}), \quad N = 4g-4.$$

C2. Even at the ''Teichmüller space'' level, the phase space of a given type is topologically not trivial (in contrast with the case of empty universes).

C3. The fact that T_g^* contains all the strata corresponding to different Teichmüller universes for given g, seems to indicate that it would be natural to look for a ''global phase space'' in which any significant change of configurations takes place; for instance, non connected surfaces and different genera should also be admitted. In fact, already in T_g^* one could describe decays of particles in the framework of Teichmüller universes.

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