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On Thurston's Formulation and Proof of Andre Theorem

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In Chapter 13 of his notes [4], W. Thurston states a general result, Theorem (see also Corollary 13.6.2), regarding the existence and uniqueness of circle pack prescribed combinatorial type on closed surfaces. This theorem treats the cases o $g = 1$ and $g \geq 2$; it is pointed out that the case $g = 0$, which is not proved i notes, is a result of E.M. Andreev [1,2].

It is implicit in Thurston's notes that the continuity method used there to Theorem 13.7.1 in the cases of genus $g = 1$ and $g \geq 2$ could be modified to give of the $g = 0$ case. Such a proof would be very different form Andreev's. The pur the present paper is to present such a proof. We separate the statement of the case into two parts: Theorem A below, which deals with standard circle packing Theorem B, which allows the circles to intersect at prescribed angles. These th have applications to conformal mapping [3].

1. A *circle packing* on the Riemann sphere or in the plane is a collection of close or the Riemann sphere on in the Euclidean plane with the property that the in of the disks are disjoint. The *nerve* of such a circle packing is the graph which vertex for each disk and an edge connects two vertices if and only if the corresp closed disks intersect.

Thurston's theorem, in the case of circle packings, is the following:

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Theorem A. *Let T be a triangulation of the Riemann sphere P. There exists a circle packing of P whose nerve is isomorphic to the one dimensional skeleton of T; any two such circle packings are images of each other under some linear fractional transformation or its complex conjugate.*

2. We begin the proof with some formulas for the Euler characteristic. Let V, E, and F denote the number of vertices, edges, and faces in the triangulation T. Then

$$V - E + F = 2. \tag{1}$$

Since $3F = 2E$, we can eliminate E in (1) and obtain

$$2V = F + 4. \tag{2}$$

It will be convenient to label the vertices of the triangulation T by $v_1, v_2, \ldots, v_V$.

Let $r = (r_1, r_2, \ldots, r_V)$ be a vector of V positive numbers. Then r determines a polygonal structure on the topological 2-sphere $|T|$ as follows. Associate to each face of T, with vertices (v_i, v_j, v_k) say, the Euclidean triangle determined by the centers of three mutually (externally) tangent circles of radii r_i, r_j and r_k. Transfer the Euclidean metric on this Euclidean triangle to the associated face of T. Note that the metric is well defined on an edge which is common to two different faces. In this way $|T|$ becomes a locally Euclidean space with cone type singularities at the vertices; we denote this space by T_r.

The *curvature of T_r at the vertex v_i*, denoted by $\kappa_r(v_i)$, is defined as follows. Consider all faces of T_r which have v_i as one of their vertices. Let $\sigma(v_i)$ be the sum of each angle at v_i in each of these triangles. Then

$$\kappa_r(v_i) \equiv 2\pi - \sigma(v_i). \tag{3}$$

Let us note that

$$\sum_{i=1}^{V} \kappa_r(v_i) = 4\pi. \tag{4}$$

Indeed, the left hand side reduces to $2\pi V - \sum \theta$, where $\sum \theta$ denotes the sum of all angles of all triangles of T_r. This sum is equal to πF. An application of Equation (2) now establishes (4).

3. If $\lambda > 0$ then T_r and $T_{\lambda r}$ are similar in the sense that corresponding angles are equal. Therefore $\kappa_r(v_i) = \kappa_{\lambda r}(v_i)$. It turns out to be advantageous to normalize the map

$$r \to f(r) = (\kappa_r(v_1), \kappa_r(v_2), \ldots, \kappa_r(v_V))$$

by restricting its domain to the simplex

$$\Delta = \{(r_1, r_2, \ldots, r_V)) \in R^V : r_1 > 0, r_2 > 0, \ldots r_V > 0 \ \& \ r_1 + r_2 + \ldots + r_V = 1\}. \tag{5}$$

It follows form (4) that the range of f can be taken as the hyperplane

$$Y = \{(y_1, y_2, \ldots, y_V) \in R^V : y_1 + y_2 + \ldots + y_V = 4\pi\}. \tag{6}$$

4. For convenience of notation, assume that v_1, v_2, v_3 are the vertices of a single f[illegible] τ_0 of T. We now prove that the existence assertion of Theorem A will follow once it[illegible] shown that the point

$$p_0 = (4\pi/3, 4\pi/3, 4\pi/3, 0, \ldots, 0),$$

for example, lies in the image of the map

$$f : \Delta \to Y.$$

To see this, suppose $f(r_0) = p_0$. If we remove that face τ_0 from T_{r_0} then the remaini[illegible] triangles can be placed isometrically in the plane, one by one, in an orientation p[illegible] serving manner, keeping identified edges coincident. Since the curvature is zero at ea[illegible] interior vertex of this complex it can be shown that we obtain in this way an isomet[illegible] embedding of T_{r_0} less τ_0 onto a triangle in the plane. {To prove that this is so, one c[illegible] first show that the process of placing adjacent faces in the plane yields a well defin[illegible] isometry once the image of an initial face is fixed. For suppose a sequence of adjac[illegible] faces are placed in the plane in this way and suppose the first face in the sequence [illegible] the same as the last. Then the placement of the first and of the last face will agree–t[illegible] is clearly true if the sequence of faces surrounds only one interior vertex of T_{r_0} less [illegible] and can be shown to be true in general by induction on the number of such vertic[illegible] The second step in the proof is to use the fact that this placement process provide[illegible] locally isometric embedding of T_{r_0} less τ_0 into the plane and is an actual embedding [illegible] the boundary of T_{r_0} less τ_0. It is easy to see that a local embedding of a topologi[illegible] disk into the plane which is an actual embedding on the boundary must by a glo[illegible] embedding. One concludes that this placement process is a global isometric embeddi[illegible] of T_{r_0} less τ_0 onto a triangle in the plane}.

We have constructed an isometric embedding, call it ϕ, of T_{r_0} less τ_0 onto a trian[illegible] ABC in the plane. It follows from the definition of T_{r_0} that if we center a circle of radi[illegible] r_i at the point $\phi(v_i)$ we obtain a circle packing in the plane whose nerve is isomorp[illegible] to the one dimensional skeleton of T. Stereographic projection transforms this packi[illegible] to a packing of the Riemann sphere with the same property.

It will be useful when we discuss uniqueness to observe that triangle ABC is nec[illegible] sarily equilateral. To verify this, weld another copy of triangle ABC to this one alo[illegible] corresponding edges. One then obtains an isometric image of all of T_{r_0}. We can calcula[illegible] the curvature at the vertex $\phi(v_1)$ which, we may assume, corresponds to the point [illegible] directly from the definition (3) using this isometric image. We see that the curvture [illegible] A is 2π less the sum $\sigma(A)$ of all angles in this isometric image with this vertex A. Th[illegible] sum $\sigma(A)$ is clearly twice the angular measure $m(A)$ of angle A in triangle ABC. [illegible] the other hand, we know by the definition of p_0 that the curvature must turn out to [illegible] $4\pi/3$. Thus $4\pi/3 = 2\pi - 2m(A)$. Hence $m(A) = \pi/3$. Similarly, $m(B) = m(C) = \pi$[illegible] and so ABC is equilateral.

5. We now show that $f : \Delta \to Y$ is one to one. Let $r' = (r'_1, r'_2, \ldots, r'_V)$ a[illegible] $r'' = (r''_1, r''_2, \ldots, r''_V)$ be distinct points in Δ. Let V_0 be the set of vertices v_i of T f[illegible] which $r'_i < r''_i$. Note that the definition (5) of Δ implies that V_0 is a nonempty prop[illegible] subset of the set of all vertices of T.

Consider a vertex $v \in \mathcal{V}_0$ together with all the faces of T_r' which have v as vertex. In each such face there is an angle at v, and we classify this angle as type α if it is the only angle in this face which has its vertex in $\mathcal{V}_0$, of type β if two vertices in this face are in $\mathcal{V}_0$, and of type γ if all three vertices of this face are in $\mathcal{V}_0$. Now

$$\sum_{v\in\mathcal{V}_0} \kappa_{r''}(v_i) = \sum_{v\in\mathcal{V}_0}(2\pi - \sigma(v)) \tag{7}$$

$$= 2\pi|\mathcal{V}_0| - \sum(\angle s \text{ of type } \alpha) - \sum(\angle s \text{ of type } \beta) - \sum(\angle s \text{ of type } \gamma).$$

Consider three mutually tangent circles in the plane and their triangle of centers. If one of the circles shrinks and the other two either expand or stay the same size, and if the three circles always remain mutually tangent, then the angle in the triangle of centers with vertex at the center of the shrinking circle will (strictly) increase. If two of the circles shrink and the other either expands or stays the same size, then in the triangle of centers the sum of the two angles which have their vertices at the centers of the shrinking circles will increase. These observations show that if r'' is replaced by r' in equations (7) then

$$\sum_{v\in\mathcal{V}_0} \kappa_{r''}(v_i) > \sum_{v\in\mathcal{V}_0} \kappa_{r'}(v_i). \tag{8}$$

Indeed, in passing from r'' to r' the radii at $v \in \mathcal{V}_0$ shrink and so the first two quantities in

$$\sum(\angle s \text{ of type } \alpha),\ \sum(\angle s \text{ of type } \beta),\ \sum(\angle s \text{ of type } \gamma)$$

will each increase, and the third will remain constant. Since $\mathcal{V}_0$ is a nonempty proper subset of vertices, not all angles are of type γ. It follows that the inequality in (8) is strict and that $f : \Delta \to Y$ must be one-to-one.

6. We now examine the behaviour of $f(r)$ as r tends to a boundary point $s = (s_1, s_2, \ldots, s_v)$ of Δ. It will turn out–and this seems very remarkable–that f cannot be extended continously to the boundary of Δ, yet the set of accumulation points of $f(r)$ as r tends to the boundary of Δ form the boundary of a polyhedron. Let $\mathcal{V}_0$ be the set of vertices v_i in T for which $s_i = 0$; $\mathcal{V}_0$ is a nonempty proper subset of $\mathcal{V}$. We classify the angles of T_r into types α, β, γ as above. Then as $r \to s$ we have

$$\begin{aligned} &\sum(\angle s \text{ of type } \alpha) \ \to\ \pi|\alpha|,\\ &\sum(\angle s \text{ of type } \beta) \ \to\ \pi|\beta|/2, \\ &\sum(\angle s \text{ of type } \gamma) \ \to\ \pi|\gamma|/3, \end{aligned} \tag{9}$$

where $|x|$ denotes the number of angles of type x. Therefore equation (7) yields

$$\lim_{r\to s} \sum_{v\in\mathcal{V}_0} \kappa_r(v) = 2\pi|\mathcal{V}_0| - \pi|\alpha| - \frac{\pi|\beta|}{2} - \frac{\pi|\gamma|}{3} \tag{10}$$

$$= 2\pi|\mathcal{V}_0| - \pi\cdot(\text{no. of faces with a vertex in } \mathcal{V}_0).$$

From (8) and (10) we see that the image $f(\Delta)$ of $f : \Delta \to T$ lies in the boun convex polyhedron Y_0 formed by intersecting Y with the half spaces

$$\sum_{i\in I} y_i > 2\pi|I| - \pi\cdot(\text{no. of faces with a vertex in } \mathcal{V}_I \equiv \{v_i : i \in I\}) \tag{11}$$

as I varies over all nonempty proper subsets of $\{1, 2, \ldots, V\}$. We have also seen t the accumulation points of $f(r)$ as $r \to \partial\Delta$ lie on the hyperplanes

$$\sum_{i\in I} y_i = 2\pi|I| - \pi\cdot(\text{no. of faces with a vertex in } \mathcal{V}_I \equiv \{v_i : i \in I\}). \tag{12}$$

which form the boundary of Y_0.

7. We know that $f : \Delta \to Y_0$ is a continuous 1-1 mapping. Hence, by Invaria of the Domain, f is a homeomorphism. We also know that $f(r) \to \partial Y_0$ as $r \to \partial\Delta$ follows by elementary topology that $f : \Delta \to Y_0$ is surjective. Indeed, merely pass to one point compactifications and apply the simple fact that if X, Y are Hausdorff sp with X compact and connected and Y connected, and if $\phi : X \to Y$ is continuous open, then ϕ is surjective.

8. We complete the proof of the existence part of Theorem A by showing t $p_0 = (4\pi/3, 4\pi/3, 4\pi/3, 0, \ldots, 0)$ is in the image Y_0 of $f : \Delta \to Y$ (see Section According to (11), this can be done by showing that for every nonempty proper sub I of $\{1, 2, \ldots, V\}$,

$$\sum_{i\in I} p_i > 2\pi|I| - \pi\cdot(\text{no. of faces with a vertex in } \mathcal{V}_I \equiv \{v_i : i \in I\}), \tag{13}$$

where $p_0 = (p_1, p_2, \ldots, p_V) = (4\pi/3, 4\pi/3, 4\pi/3, 0, \ldots, 0)$.

If $|I| = V - 1$ then every face has a vertex in $\mathcal{V}_I \equiv \{v_i : i \in I\}$. Therefore the ri hand side of (13) is, by (2),

$$2\pi(V - 1) - \pi\cdot F = 2\pi. \tag{14}$$

For subsets I of this cardinality the left hand side of (13) becomes

$$\sum_{i\in I} p_i = 8\pi/3 \text{ or } 12\pi/3. \tag{15}$$

Thus p_0 satisfies (13) when $|I| = V - 1$.

If $|I| = V - 2$ similar reasoning shows that the right hand side of (13) is zero wl the left hand side is at least $4\pi/3$. Thus p_0 satisfies (13) in this case also.

We shall show that p_0 satisfies (13) when $1 \le |I| \le V - 3$ by proving that the ri hand side of (13) will be negative in these cases. First we rewrite the right hand side i more invariant form. Let F_1, F_2, F_3 denote, respectively, the number of faces of T wh have exactly 1,2,3 vertices in $\mathcal{V}_I$. Then the right side of (13) is $\pi(2|I| - F_1 - F_2 - F_3)$. I

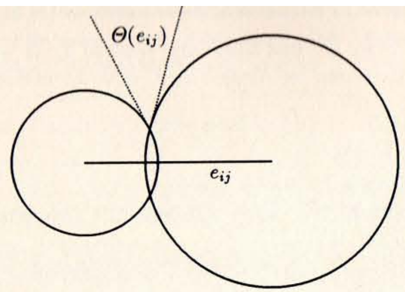

Figure 1. The angle of intersection of two disks

E_2 denote the number of edges of $\mathcal{T}$ which have both of their boundary vertices in $\mathcal{V}_I$. Then the simplicial complex $\mathcal{T}_I$ spanned by the vertices of $\mathcal{V}_I$ has Euler characteristic $\chi_0 = |I| - E_2 + F_3$. Since $3F_3 + F_2 = 2E_2$, we can eliminate E_2 from the expression for χ_0 and obtain $2\chi_0 = 2|I| - F_3 - F_2$. Therefore the condition (13) that $(p_1, p_2, \ldots, p_V)$ lies in the image of f can be rewritten as

$$\sum_{i \in I} p_i > \pi(2\chi_0 - F_1) \qquad (17)$$

for every nonempty proper subset I of $\{1, 2, \ldots, V\}$.

We wish to show that the right hand side of (17) is negative for $1 \leq |I| \leq V - 3$. For that purpose we may assume that $\mathcal{T}_I$ is connected. The Euler characteristic of a connected simplicial 2-complex can be interpreted as $2 - 2g - n$ where g is the genus and $n = 1, 2, \ldots$ is the connectivity. In our case $g = 0$ and $\chi_0 = 2 - n$. If $n \geq 3$ there is nothing to prove. If $n = 1$ or 2, one of the components of the complement of $\mathcal{T}_I$ contains at least two vertices in $\mathcal{V} - \mathcal{V}_I$. Therefore we can find an edge (x', y') where the vertices x' and y' are in $\mathcal{V} - \mathcal{V}_I$. We can even choose x' and y' so that there is an edge (y', a') with a' in $\mathcal{V}_I$. If we examine the star of y' we can find a triangle face (x, y, a) of $\mathcal{T}$ with $a \in \mathcal{V}_I$ and $x, y \in \mathcal{V} - \mathcal{V}_I$.

Now look at the union of the star of x and the star of y. If all the vertices adjacent to x and y belong to $\mathcal{V}_I$ the $\{x, y\}$ is a component of the complement of $\mathcal{T}_I$. Since there are at least three vertices in $\mathcal{V} - \mathcal{V}_I$ we must be in the case $n = 2$. Therefore the right hand side of (17) is negative in this case since (x, y, a) is an F_1 type triangle. In the remaining case the set of vertices adjacent to x and to y contains $a \in \mathcal{V}_I$ and some vertex z $(\neq x, y) \in \mathcal{V} - \mathcal{V}_I$. It follows that there are three F_1 type triangles in the union of the stars of x and y.

9. The proof of the existence part of Theorem A is now complete. We have seen that a given triangulation $\mathcal{T}$ of the Riemann sphere P can be realized as the nerve of a circle packing of P. By means of a linear fractional transformation of P we can always arrange the realization so that any three preassigned mutually tangent circles will have

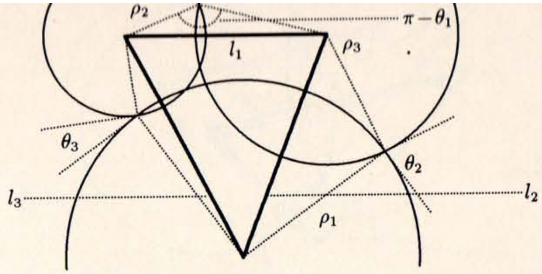

Figure 2. Prescribed intersection angles

equal radii; ∞ will be an interior point of the face whose vertices are their centers. T nerve of this packing on the finite complex plane forms a triangulation of an equilate triangle by straight line segments.

Let the circles which correspond to the vertices $v_1, v_2, \ldots, v_V$ have radii $r_1, r_2, \ldots,$ respectively. We may assume that $r_1 + r_2 + \ldots + r_V = 1$. For $r_0 = (r_1, r_2, \ldots, r_V)$ we calculate the curvatures $f(r_0)$ of $\mathcal{T}_{r_0}$ by means of this triangulated equilateral trian as was done in the last paragraph of Section 4. If v_1, v_2, v_3 correspond to the vertices the equilateral triangle we find that

$$f(r_0) = (4\pi/3, 4\pi/3, 4\pi/3, 0, \ldots, 0).$$

By the injectivity of f, r_0 is uniquely determined. Therefore the radii of the cir in this normalized circle packing are uniquely determined. It is clear that two ci packings with the same abstract nerve and with corresponding radii equal are (pro of improper) rigid motions of each other. This proves that all circle packings wh realize the same triangulation of a 2-sphere are linear fractional transformations of e other followed possibly by a reflection. This completes the proof of Theorem A.

10. We now consider a generalization of Theorem A in the spirit of Theorem 13 in Thurston's notes *(loc. cit)* for the case $g = 0$. Let $\mathcal{T}$ be a triangulation of the 2-sph let $\mathcal{E}$ be the set of edges in $\mathcal{T}$, and let $\Theta : \mathcal{E} \to [0, \pi/2]$ be any function. A family closed disks on the Riemann sphere P or in the plane will be said to *realize* the d $\mathcal{T}, \Theta$ provided the following conditions are satisfied: (a) the nerve of $\mathcal{C}$ is isomorphi $\mathcal{T}$, and (b) two disks C_i and C_j in $\mathcal{C}$ intersect if and only if their angle of intersection radian measure $\Theta(e_{ij})$, where e_{ij} is the edge of $\mathcal{T}$ which spans the vertices correspon to C_i and C_j. (The *angle of intersection* of two disks is the one in the exterior of two disks; see Figure 1.) We shall prove the following result.

Theorem B. *Let $\mathcal{T}$ be a triangulation of the 2-sphere. Let $\Theta : \mathcal{E} \to \mathbf{R}$ be a func defined on the edges of $\mathcal{T}$ with the property that $0 \leq \Theta(e) \leq \pi/2$ for all $e \in \mathcal{E}$. Ass that Θ has the following two properties: (i) If $e_1 + e_2 + e_3$ is a cycle of edges in $\mathcal{T}$ t $\Theta(e_1) + \Theta(e_2) + \Theta(e_3) < \pi$, and (ii) if $e_1 + e_2 + e_3 + e_4$ is a cycle of distinct edges i*

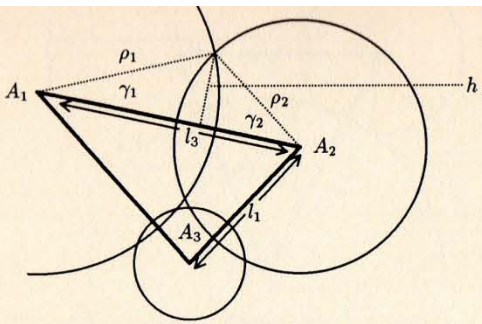

Figure 3

ten $\Theta(e_1)+\Theta(e_2)+\Theta(e_3)+\Theta(e_4) < 2\pi$. Then there exists a family C of round disks on he Riemann sphere which realizes the data T, Θ. This family C is uniquely determined p to a linear fractional transformation or its conjugate.

11. The proof proceeds as before except for several additional complications. The rst complication arises in constructing T_r. The metric on a face with vertices v_i, v_j, v_k hould be the Euclidean metric of the triangle of centers of three disks which inters-ct pairwise at the nonobtuse angles $\Theta(e_{ij})$, $\Theta(e_{jk})$, and $\Theta(e_{ki})$ and which have the adii prescribed by r. The following lemma from [4] guarantees the existence of such a onfiguration.

Lemma 2. *For any three nonobtuse angles $\theta_1, \theta_2, \theta_3$ and any three positive numbers $_1, \rho_2, \rho_3$, there is a unique configuration in the plane consisting of three disks having hese radii and intersecting in these angles.*

The proof of Lemma 2 refers to Figure 2. Determine the sides l_1, l_2, l_3 of the desired riangle of centers as follows. Side l_1 is the length of the third side of a triangle which as sides ρ_2 and ρ_3 and included angle $\pi - \theta_1$. Sides l_1 and l_2 are determined similarly. 'o see that l_1, l_2, l_3 satisfy the triangle inequality note that property (i) of Theorem B nplies $l_1 \le \rho_2+\rho_3 \le l_2+l_3$, and similarly for l_2 and l_3, because the angles of intersection re nonobtuse. Thus the configuration shown in Figure 2 can always be constructed and uniquely determined.

12. Having constructed T_r we proceed as before to define the curvatures at the ertices $v_1, v_2, \ldots, v_V$ and thereby obtain the map

18) $$r \to f(r) = (\kappa_r(v_1), \kappa_r(v_2), \ldots, \kappa_r(v_V)) : \Delta \to \Sigma.$$

'he previous proof (Section 5) that f is one to one can be imitated with the help of the ollowing lemma from [4].

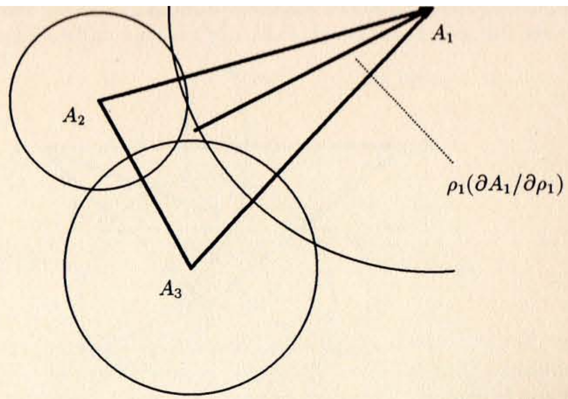

Figure 4

Lemma 3. *Consider three circles in the plane which intersect pairwise in nonobtu angles. If one radius decreases and the other two remain the same, and if the circl continue to intersect each other at the original angles then, in the triangle of center the angle at the vertex of the shrinking circle will increase and the other two angles w decrease.*

Let the radii be ρ_1, ρ_2, ρ_3 and let the triangle of centers have sides of lengths $l_1, l_2,$ and vertices at A_1, A_2, A_3. If we fix A_2 as the origin, so $\|A_1\| = l_3$, then we can differe tiate $A_1 = l_3 u$, where u is a unit vector, to obtain

(19) $$\frac{\partial A_1}{\partial \rho_1} = \frac{\partial l_3}{\partial \rho_1}\frac{A_1}{\|a_1\|} + l_3 B,$$

where B is orthogonal to A_1. Note that (Figure 3) $l_3 = \rho_1\cos\gamma_1 + \rho_2\cos\gamma_2$ and so

(20) $$\frac{\partial l_3}{\partial \rho_1} = -\rho_1\frac{\gamma_1}{\partial \rho_1}\sin\gamma_1 - \rho_2\frac{\gamma_2}{\partial \rho_1}\sin\gamma_2 + \cos\gamma_1.$$

Since $\rho_2\sin\gamma_2 = \rho_1\sin\gamma_1 \equiv h$, (20) becomes

(21) $$\frac{\partial l_3}{\partial \rho_1} = -h\left(\frac{\partial \gamma_1}{\partial \rho_1} + \frac{\partial \gamma_2}{\partial \rho_1}\right) + \cos\gamma_1.$$

The term in parentheses is zero since $\gamma_1 + \gamma_2$ is a constant equal to the fixed ang of intersection of circles 1 and 2. Therefore $\frac{\partial l_3}{\partial \rho_1} = \cos\gamma_1$. Thus

(22) $$\rho_1\frac{\partial l_3}{\partial \rho_1} = \rho_1\cos\gamma_1,$$

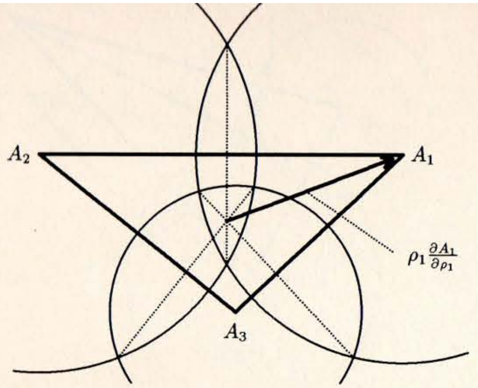

Figure 5

ich is the distance form A_1 to the point of intersection of the radical axis of circles 1 l 2 (that is, the line through their points of intersection) with the line joining their ters. It follows that $\rho_1 \dfrac{\partial A_1}{\partial \rho_1}$ is the vector with its tail at A_1 and its tip on the radical s of circles 1 and 2. By symmetry, its tip is also on the radical axis of circles 1 and 3. ıce $\rho_1 \dfrac{\partial A_1}{\partial \rho_1}$ is the vector from the common point of intersection of the three radical s to the point A_1 (Figures 4 and 6; in the context of Theorem B the configuration igure 6 will not occur because condition (ii) in that theorem insures that the three ‹s have empty intersection).

Therefore, if ρ_1 decreases and ρ_1 and ρ_2 remain unchanged, the vertex A_1 will move ard the intersection point Q of the radical axes. If we show that Q lies in the triangle enters it will follow that the angle at A_1 decreases and the proof of Lemma 3 will complete.

If Q did not lie in the triangle of centers then one of the sides of that triangle would arate Q from the vertex not on that side. Suppose side A_2A_3 separates Q from A_1 n Figure 6. First note that the circle centered at A_1 does not intersect side A_2A_3 ause if it did then the distance A_2A_3 would be greater than the sum of lengths of the gents from A_2 and A_3 to circle 1 and the lengths of these tangents are upper bounds r_2 and r_3 since circles 2 and 3 must intersect circle 1 in nonobtuse angles (see Figure Thus in Figure 7 the circle centered at A_1 in the upper half plane determined by A_3 cannot enter the lower plane. It follows that the radical axis of circles 1 and 2 not enter the shaded region of Figure 7. This contradicts the fact that Q lies on that ical axis. Therefore Q must lie in the triangle of centers.

13. We now consider the modifications to Section 6 that are needed for the present case. Consider three circles which intersect pairwise in nonobtuse angles. Let the radius

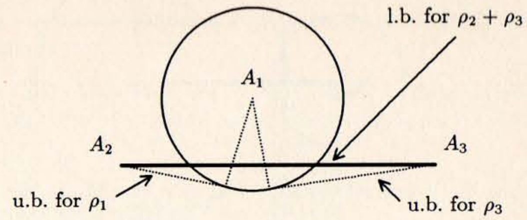

Figure 6

of circle 1 shrink to zero while the intersection angles and the other two radii remain constant. Then in the triangle of centers the angle at the center of circle 1 will increase to the limiting value $\pi - (\theta(e))$, where e is the opposite edge. If the radii of two circles shrink to zero then in the triangle of centers the sum of the two angles at the centers of these two circles will tend to the limiting value π. When all three radii shrink to zero we use the fact that the sum of the three angles is constantly equal to π. Thus the three equations (9) are to be replaced by

$$\begin{aligned} &\textstyle\sum(\angle s \text{ of type } \alpha) \;\to\; \sum(\pi - \Theta(e(\alpha)), \\ (23)\qquad &\textstyle\sum(\angle s \text{ of type } \beta) \;\to\; \pi|\beta|/2, \\ &\textstyle\sum(\angle s \text{ of type } \gamma) \;\to\; \pi|\gamma|/3. \end{aligned}$$

Equation (10) is replaced by

$$(24)\qquad \lim_{r\to s} \sum_{v\in\mathcal{V}_0} \kappa_r(v) = 2\pi|\mathcal{V}_0| - \sum(\pi - \Theta(e(\alpha)) - \frac{\pi|\beta|}{2} - \frac{\pi|\gamma|}{3}$$

$$= 2\pi|\mathcal{V}_0| - \pi\cdot \text{ (no. of faces with a vertex in } \mathcal{V}_0) + \sum_\alpha \Theta(e(\alpha)).$$

We conclude that the image Y_0 of $f : \Delta \to Y$ is the hyperplane formed by intersecting Y with the half spaces

$$(25)\qquad \sum_{i\in I} y_i > 2\pi|I| - \pi\cdot \text{ (no. of faces with a vertex in } \mathcal{V}_I) + \sum_\alpha \Theta(e(\alpha))$$

for each nonempty proper subset I of $\{1, 2, \ldots, V\}$.

14. As in Section 8, the existence assertion of Theorem B will follow from showing that $p_0 = (4\pi/3, 4\pi/3, 4\pi/3, 0, \ldots, 0)$ is in the image Y_0 of $f : \Delta \to Y$. According to (25) this is equivalent to showing that for each nonempty proper subset I of $\{1, 2, 3, \ldots, V\}$,

$$(26)\qquad \sum_{i\in I} p_i > 2\pi|I| - \pi\cdot \text{ (no. of faces with a vertex in } \mathcal{V}_I) + \sum_\alpha \Theta(e(\alpha))$$

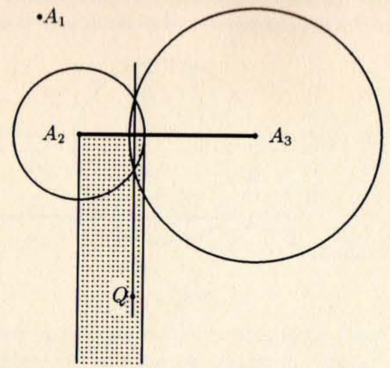

Figure 7

'here $p_0 = (p_1, p_2, \ldots, p_V) = (4\pi/3, 4\pi/3, 4\pi/3, 0, \ldots, 0)$ and where the summation enotes the sum over angles of type α of the value of Θ at the edge opposite the angle f type α.

If $|I| = V - 1$, (26) holds. Indeed, there are no angles of type α in this case, so the ırlier calculation (Equations (14) and (15)) applies.

If $|I| = V - 2$ the left hand side of (26) is at least $4\pi/3$. The first two terms on the ght hand side cancel by (2) and the summation term is at most π since there can be ; most two terms in the sum. Thus (26) holds in this case as well.

For the cases $1 \leq |I| \leq V - 3$, rewrite (26) in the form

$$\sum_{i \in I} p_i > \pi(2\chi_0 - F_1) + \sum_{\alpha} \Theta(e(\alpha)) = 2\pi\chi_0 - \sum_{e}(\pi - \Theta(e)) \quad \text{!7)}$$

. the same way that (13) was rewritten as (17); here the sum of F_1 terms (each term ıtisfies $\pi/2 \leq \pi - \Theta(e) \leq \pi$) is taken over the edges in the F_1 type triangles opposite ıe angles of type α. These edges form a 1-cycle in the mod 2 homology of $\mathcal{T}$.

As before, we have to show that the right hand side of (27) is negative and we may ısume that the complex spanned by $\mathcal{V}_I$ is connected ($\chi_0 \leq 1$). The negativity is clear $\chi_0 = 0$. The reasoning in Section 8 following (17) showed that if $\chi_0 = 0$ then $F_1 \geq 1$, ıd if $\chi_0 = 1$ then $F_1 \geq 3$. Therefore the right hand side is negative if $\chi_0 = 0$; it will so be negative if $\chi_0 = 1$ and $F_1 \geq 5$. The remaining cases $\chi_0 = 1$ and $F_1 = 3$ or 4 e covered by properties (i) and (ii) of Theorem B. In case $F_1 = 4$ one can eliminate e case that the four edges are not distinct because in that case $\mathcal{V} - \mathcal{V}_I$ will consist of e vertices on these edges. Since they do not span a triangle face, one of the vertices , v_2, v_3 is in $\mathcal{V}_I$. Therefore the left hand side of (27) is at least $4\pi/3$ and the desired equality (27) holds, even though the right hand side may be zero.

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