Introduction to Teichmüller Theory

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1 Introduction

This set of notes contains basic material on Riemann surfaces, Teichmüller spaces and Kleinian groups. It is based on a course I taught at University of Utah in 1992–1993. This course was a prequel to the 1993-1994 course on Thurston's Hyperbolization Theorem which later became a book [K].

Contents

2 Conformal geometry on surfaces.

Conformal maps are smooth maps in domains in $\overline{\mathbb{C}}$ with derivatives in

$$CSO(2) = \mathbb{R}_+ \times SO(2)$$

The special feature of the complex dimension 1 is that the classes of (locally) biholomorphic and (locally) conformal maps in subdomains of $\bar{\mathbb{C}}$ coincide.

Definition 2.1. *Riemann surface is a (connected) 1-dimensional complex manifold.*

Classes of (locally) biholomorphic and (locally) conformal maps in $\overline{\mathbb{C}}$ coincide. Therefore, each complex curve 1-1 corresponds to a conformal structure on the 2-dimensional surface S (maximal atlas with conformal transition maps).

Riemann surface with punctures X is obtained from a Riemann surface $\bar{X}$ by removing some discrete set of points. We mainly will be interested in Riemann surface X of "finite type" (g, p) which have $g =$ genus of compact surface $\bar{X}$; $p =$ number of punctures.

Infinitesimally, conformal structure is a reduction of the principal $GL(2, R)$ (frame) bundle $F(S)$ over S to a principal bundle with the structure group $CSO(2)$. The Riemannian structure on S is a reduction of $F(S)$ to a principal subbundle with the structure group $SO(2)$. Thus, each conformal structure can be obtained from a Riemannian metric (this is true in arbitrary dimension). This is a general fact of the reduction theory: the quotient $CSO(n)/SO(n) \cong \mathbb{R}_+$ is contractible. Therefore we can use:

Theorem 2.2. *Suppose that G is a Lie group and H is its Lie subgroup so that G/H is contractible. Then for each manifold M any principal G-bundle can be reduced to a principal H-subbundle.*

As we shall see, for the surfaces the converse is true as well:

Theorem 2.3. *(Gauss' theorem on isothermal coordinates). For each Riemannian surface (S, ds^2) there exists a local system of coordinates such that $ds^2 = \rho(z)|dz|^2$. I.e. any Riemannian metric in dimension 2 is locally conformally-Euclidean.*

Notice that the system of coordinates on S where ds^2 has the type $ds^2 = \rho(z)|dz|^2$ is a conformal structure on S. Really, the transition maps are isometries between metrics $\lambda_1|dz|, \lambda_2|dz|$ on domains in $\overline{\mathbb{C}}$, thus they are conformal maps with respect to the Euclidean metric. Two metrics define one and the same conformal structure if they are "proportional".

Theorem 2.4. *(Uniformization theorem). For any Riemann surface S the universal cover of S is conformally- equivalent either to the (a) unit disc Δ or to (b) $\mathbb{C}$ or to (c) $\overline{\mathbb{C}}$.*

These classes of Riemann surface correspond to the following types:
(a) $(0,0)$ (rational type),
(b) $(1,0),(0,1),(0,2)$ (elliptic type: torus, complex plane, $\mathbb{C}^* = \mathbb{C} - \{0\}$),
(c) other (hyperbolic type).

The proof of these theorems will be given later as a corollary from some existence theorem in PDEs in the case of surfaces of finite type.

The groups of conformal automorphisms of Δ, $\mathbb{C}$, $\bar{\mathbb{C}}$ consist of linear-fractional transformations. In the case (a) our surface is simply connected.

The fundamental group Γ of X acts properly on $\tilde{X}$. Thus, in the case (b) the group Γ consists only of Euclidean isometries. As we shall see later in the case (c) all conformal automorphisms preserve the hyperbolic metric.

1) Torus. Metric can be obtained by identification of sides of Euclidean rectangle.

2) $exp(\mathbb{C}) = \mathbb{C}^*$ - the universal covering.

We will be mainly interested in surfaces of "hyperbolic type".

Our strategy in proving U.T.: (1) use some geometry to construct a complete hyperbolic metric on S. Then (2) use some analytic technique to prove that each metric is conformally hyperbolic.

Hyperbolic plane: $\mathbb{H}^2 = \{z : Im(z) > 0\}$ with the hyperbolic metric $ds = |dz|/Im(z)$ (that has curvature -1). Recall that the group of biholomorphic automorphisms of the upper half-plane consists of linear-fractional transformations, i.e. equals $PSL(2,\mathbb{R})$. Suppose that $f \in PSL(2,\mathbb{R})$; then $Im(fz) = Im(z)|f'(z)|$; thus, f is an isometry of $\mathbb{H}^2$.

Definition 2.5. *A hyperbolic surface X is a complete connected 2-dimensional Riemannian surface of the constant curvature -1.*

The universal cover $\tilde{X}$ of X is again complete, hence it is isometric to $\mathbb{H}^2$. Therefore we get an equivalent definition of a hyperbolic surface:

Definition 2.6. *Let G be a properly discontinuous group of isometries of $\mathbb{H}^2$ which acts freely. Then $X = \mathbb{H}^2/G$ is a hyperbolic surface.*

We will use two models of the hyperbolic plane $\mathbb{H}^2$: the upper half-plane and the unit disk. Geodesics in the hyperbolic plane are the arcs of Euclidean circles orthogonal to $\partial\mathbb{H}^2$. Proof: use the inversion and the property that between each 2 points the geodesic is unique.

Horoballs and hypercycles. Horoballs in the unit disc model (Δ) of the hyperbolic plane are Euclidean discs in Δ which are tangent to the boundary of Δ. If h is a geodesic in $\mathbb{H}^2$ then the boundary of its r-neighborhood is called a "hypercycle".

Definition 2.7. *(***Types of isometries***.) Consider a space X of negatively pinched sectional curvature $-b < K_X < a < 0$. Then an isometry g of X is called* **elliptic** *if it has a fixed point in X. An isometry is called* **parabolic** *if it has a single fixed point in $\bar{X} = X \cup \partial X$. An isometry is called* **hyperbolic** *(or* **loxodromic***) if it has exactly two fixed points in $\bar{X} = X \cup \partial X$.*

Examples: $z \to 2z$ (hyperbolic) ; $z \to z + a$ (parabolic); Euclidean rotation of Δ around the center (elliptic).

Remark 2.8. *Suppose that p in a puncture on X. Then p has a neighborhood U which is conformally equivalent to a puncture on $\mathbb{C}/<\gamma>$ where γ is a translation. Really, U can be realized as a neighborhood of $0 \in \mathbb{C}$. Then the universal covering of $\mathbb{C}^*$ is*

$$\exp : \mathbb{C} \to \mathbb{C}^*$$

and the deck-transformation group is $< z \mapsto 2i\pi + z >$.

Non-example. Now suppose that the boundary of the strip

$$1 \leq Re(z) \leq 2\ ; Im(z) > 0$$

is identified by the homothety (hyperbolic transformation):

$$h : z \mapsto 2z$$

denote the result by A. Let's prove that the neighborhood $\partial_1 A$ of ∞ at A isn't conformally equivalent to a neighborhood of ∞ in $\mathbb{C}$.

Notice that A is conformally isomorphic to $\{z \in \mathbb{C} : Re(z) > 0, Im(z) > 0\}/ < h >$. Then $A \subset T^2 = \mathbb{C}^*/ < h >$; $\partial_1 A$ is a curve on T. There is a nondegenerate holomorphic map $f : T \to \bar{\mathbb{C}}$; $f(\partial_1 A)$ is a smooth compact curve in $\mathbb{C}$. Suppose that $q : U \to T$ is a biholomorphic embedding where U is a closed neighborhood of ∞ in $\mathbb{C}$; $q(U)$ is a one-sided neighborhood N of $\partial_1 A$. Then $f(N)$ is relatively compact in $\mathbb{C}$. Thus, the function $q \circ f$ is bounded in U; then it extends holomorphically to ∞. Thus, $qf(U \cup \infty)$ is compact and contains $q(\partial_1 A)$. However, it means that $qf(\infty) \supset q(\partial_1 A)$ which is impossible. **QED of Non-example.**

Here is a way to construct a hyperbolic surface. Suppose that P is a convex closed polygon in $\mathbb{H}^2$ and we have some isometric identifications of its sides so that after gluing the total angle around each point is 2π and the result of gluing is a surface (without boundary). Then $S = P/\sim$ has a natural hyperbolic structure. Unfortunately, this structure can be incomplete.

Example 2.9. *Put $A = \{z \in \mathbb{C} : Im(z) > 0, 1 < Re(z) \leq 2\} \subset \mathbb{H}^2$. Let $g : z \mapsto 2z$; then identify the sides of A by the equivalence relation: $z \cong 2z$. The surface $A/\cong$ is not complete.*

Let's try to figure out the criterion of the completeness. First assume that P is compact. Then S is complete. Now consider closed finite-sided polygons P of finite area. Then, our problem is reduced to the consideration of the isolated vertices which can be the only source of incompleteness. Let Γ be a group generated by identifications.

Theorem 2.10. *(Criterion of completeness.) S is complete iff the stabilizer of each vertex is parabolic.*

In particular we proved now the following. Under conditions above (S is complete) the action of the group Γ can be identified with the fundamental group of S and P is the fundamental domain for Γ.

Theorem 2.11. *Let $(l_1, l_2, l_3) \in \mathbb{R}^3$ be nonnegative numbers. Then there exists a complete hyperbolic structure X with geodesic boundary on the pair of pants ($\mathbb{S}^2 \setminus$ 3 discs) such that lengths of boundary curves are (l_1, l_2, l_3).*

Proof: Use 3 disjoint mutually nonseparating geodesics in $\mathbb{H}^2$ such that hyperbolic distances between them are the numbers: $l_1/2, l_2/2, l_3/2$ (the continuity principle). Then connect the geodesics by orthogonal segments. This gives a hexagon Y in $\mathbb{H}^2$ with right angles such that lengths of 3 sides are $(l_1/2, l_2/2, l_3/2)$. Take a double of Y to obtain X. □

Remark 2.12. *We allow some l_j to be 0, the corresponding boundary curves degenerate to punctures in this case.*

Theorem 2.13. *(Existence theorem for hyperbolic structures). Each surface of the hyperbolic type has a complete hyperbolic structure.*

Proof: Each surface with punctures can be split along disjoint simple curves to a union of "pairs of pants". Find hyperbolic structures on each component such that punctures correspond to curves of 0-length, other curves have one and the same length (say 1). Finally glue these pairs of pants together via isometries of their boundary components. □

3 Quasiconformal maps

The main analytic tool for proof of the Uniformization Theorem and for all further discussion will be the theory of **quasiconformal maps**.

3.1 Smooth quasiconformal maps

An orientation preserving homeomorphism f of a domain $A \subset \bar{\mathbb{C}}$ is K-quasiconformal iff the function

$$H(z) = \lim sup_{r\to 0} \frac{\max |f(z + r^{i\phi}) - f(z)|}{\min |f(z + r^{i\phi}) - f(z)|} \tag{1}$$

is bounded in $A - \{\infty, f^{-1}\infty\}$ and $H(z) \leq K$ a.e. in A.

Suppose in addition that $u_x v_y - u_y v_x = |f_z|^2 - |f_{\bar{z}}|^2 = J_f(z) > 0$. Denote

$$\partial_\alpha f(z) = \lim_{r\to 0} \frac{f(z + re^{i\alpha}) - f(z)}{re^{i\alpha}} \tag{2}$$

Then $\partial_\alpha f(z) = \partial f + \bar{\partial} f e^{-2i\alpha}$, so

$$\max_\alpha |\partial_\alpha f(z)| = |\partial f(z)| + |\bar{\partial} f(z)| \tag{3}$$

$$\min_\alpha |\partial_\alpha f(z)| = |\partial f(z)| - |\bar{\partial} f(z)| \tag{4}$$

Recall that a function $\phi : \mathbb{R} \to \mathbb{R}^m$ is called *absolutely continuous* if it has measurable derivative ϕ' almost everywhere (in the domain D of ϕ) and for each subinterval $[a, b] \in D$ we have:

$$\phi(b) - \phi(a) = \int_a^b \phi'(x)dx$$

A function $\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ (defined on an open subset D) is called ACL (absolutely continuous on lines) if for almost every line L the restriction $\phi|L\mathcal{D}$ is absolutely continuous. Thus, each ACL function has measurable partial derivatives a.e. in D.

Suppose that f is ACL and (or has weak L^2 partial derivatives). Then, the K-quasiconformality is equivalent to the fact that the quotient

$$D_f = \frac{\max_\alpha |\partial_\alpha f(z)|}{\min_\alpha |\partial_\alpha f(z)|} = \frac{|\partial f(z)| + |\bar{\partial} f(z)|}{|\partial f(z)| - |\bar{\partial} f(z)|} \tag{5}$$

is finite and a.e. bounded by K.

This is the same as:

$$|\bar{\partial} f(z)| \leq \frac{K-1}{K+1} |\partial f(z)|$$

Suppose in addition that $J_f(z) > 0$. Then we can form the **complex dilatation of** f :

$$\mu(z) = \frac{\bar{\partial} f(z)}{\partial f(z)} \tag{6}$$

$$|\mu(z)| \leq \frac{K-1}{K+1} < 1 \tag{7}$$

The differential equation

$$\bar{\partial} f(z) = \mu(z) \partial f(z) \tag{8}$$

is called the *Beltrami equation*. If $\mu(z) = 0$ then it becomes the Cauchy-Riemann equation. Each solution of the latter equation is holomorphic.

Complex dilatation under the composition. Let $\zeta = g(z)$, then

$$\mu_{f \circ g^{-1}}(\zeta) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \left(\frac{\partial g(z)}{|\partial g(z)|}\right)^2 \tag{9}$$

Theorem 3.1. *(***Existence-Uniqueness Theorem***.) If f, g are quasiconformal in A with the same complex dilatation a.e. then $f \circ g^{-1}$ is conformal.*

For every measurable function μ in the domain A with $\|\mu\|_\infty < 1$ there exists a quasiconformal homeomorphism with the complex dilatation μ.

Definition 3.2. *A map f in D is K- quasiconformal if*

- *f is a homeomorphism ;*
- *f is AC (absolutely continuous) on a.e. coordinate line in D (ACL property);*
- *$|f_{\bar{z}}| \leq k|f_z|$ where $k = \frac{K-1}{K+1} < 1$.*

3.2 Properties of quasiconformal maps

1) Gehring- Lehto: quasiconformal maps f are differentiable a.e. in D.

2) Partial derivatives are locally in L^2. And vice- versa: if the partial derivatives are locally in L^2 then f is ACL.

3) Area is absolutely continuous function under q.c. maps. Thus, $f_z \neq 0$ a.e. in D.

4) **Mori's inequality**:
Let $\Omega \subset \bar{\mathbb{C}}$, $f : \Omega \to \Omega'$. Normalize f so that $f(\infty) = \infty$, $f(a_j) = b_j$, $j = 1, 2$.
Then for every compact subset $G \subset \Omega \cap \mathbb{C}$ we have the Holder inequality:

$$|f(z) - f(w)| \leq M_G |z - w|^{1/K} \tag{10}$$

5) **Convergence property**: Suppose that f_n be a sequence of K_n-q.c. mappings of $\overline{\mathbb{C}}$ so that :

(a) f_n fix three points: ∞ , $a_1 \neq a_2 \in \mathbb{C}$;
(b) $K_n \leq K < \infty$.

Then f_n has a subsequence which is uniformly convergent on compacts to a quasiconformal homeomorphism.

6) **Extension property.** Suppose that $f : \mathbb{H}^2 \to \mathbb{H}^2$ is a quasiconformal self-map of the upper half-plane. Then f extends to $\overline{\mathbb{C}}$ to a quasiconformal homeomorphism, whose restriction to the line $\partial \mathbb{H}^2$ is quasisymmetric if we normalize it by $f(\infty) = \infty$:

$$C^{-1} \leq \frac{\phi(x+t) - \phi(x)}{\phi(x) - \phi(x-t)} \leq C \tag{11}$$

However, the boundary value isn't necessarily AC. According to Ahlfors and Beurling the condition (11) is also sufficient for the extension of ϕ to a quasiconformal map of $\overline{\mathbb{C}}$.

See proofs in [Ah1].

3.3 The existence theorem

Consider the Beltrami equation:

$$f_{\bar{z}} = \mu f_z \tag{12}$$

where $\mu \in L_\infty$ and $\|\mu\| \leq k < 1$.

Recall that for f with L^1-derivatives we have:

$$f(\zeta) = -\frac{1}{\pi}(P.V.) \int_D \frac{f_{\bar{z}}}{z - \zeta} dxdy + \frac{1}{2i\pi} \int_{\partial D} \frac{f(z)}{z - \zeta} dz \tag{13}$$

(generalized Cauchy formula). Here P.V. means the principal value in the sense of Cauchy. In particular,

$$\bar{\zeta} = -\frac{1}{\pi}(P.V.) \int_{\Delta_R} \frac{1}{z - \zeta} dxdy \tag{14}$$

Consider the operator P on the functions $h \in L^p$, $p > 2$,

$$Ph(\zeta) = -\frac{1}{\pi}(P.V.) \int_{\mathbb{C}} h(z) \frac{\zeta}{z(z - \zeta)} dxdy = -\frac{1}{\pi} \int_{\mathbb{C}} h(z)(\frac{1}{z} - \frac{1}{z - \zeta}) dxdy \tag{15}$$

The integral is correctly defined as (P.V.) since $h \in L_p$ in a compact domain implies that $f \in L_1$.

Lemma 3.3. *Ph is continuous and satisfies the uniform Holder inequality with the exponent* $(p - 2)/(p - 1)$.

Proof: $h \in L_p$, $\zeta(z(z-\zeta))^{-1} \in L_q$ where $1/p + 1/q = 1$, $1 < q < 2$. Then the Holder inequality implies that:

$$|Ph(\zeta)| \leq \frac{1}{\pi}\|h\|_p\|\frac{|\zeta|}{z(z-\zeta)}\|_q \tag{16}$$

$$\int \frac{|\zeta|^q}{|z(z-\zeta)|^q}dxdy = |\zeta|^{2-q}\int_{\mathbb{C}}|z(z-1)|^{-q}dxdy = |\zeta|^{2-q}K_p \tag{17}$$

Then:

$$|Ph(\zeta)| \leq |\zeta|^{(p-2)/(p-1)}K_p\|h\|_p \tag{18}$$

and if $h_1(z) = h(z+\zeta_1)$ then:

$$Ph_1(\zeta_2 - \zeta_1) = Ph(\zeta_2) - Ph(\zeta_1) \tag{19}$$

so Ph is Holder with the exponent $(p-2)/(p-1)$. □

Remark 3.4. *The formula 16 implies that* $Ph(0) = 0$.

The operator T is defined for $h \in C_0^2$:

$$Th(\zeta) = \lim_{\epsilon\to 0} -\frac{1}{\pi}\int_{|z-\zeta|>\epsilon}\frac{h(z)}{(z-\zeta)^2}dxdy \tag{20}$$

This operator is called "Hilbert transformation". Notice that $Th(\zeta) = O(|\zeta|^{-2})$ as $\zeta \to \infty$, since

$$|Th(\zeta)| \leq (\int_{D_R} h)\cdot\sup_{D_R}\frac{1}{|z-\zeta|^2} = (\int_{D_R} h)\cdot|\zeta - R|^2 = O(|\zeta|^{-2}) \tag{21}$$

Lemma 3.5. *For $h \in C_0^2$, Th has class C^1 and:*

$$(Ph)_{\bar{z}} = h \ ; i.e. \quad \partial \circ P = id \tag{22}$$

$$(Ph)_z = Th \quad ; (Th)_{\bar{z}} = (Ph_z)_{\bar{z}} = h_z \tag{23}$$

$$\int |Th|^2 dxdy = \int |h|^2 dxdy \tag{24}$$

and moreover, $\|Th\|_p \leq C_p\|h\|_p$ for any $p > 1$ so that

$$\lim_{p\to 2} C_p = 1 \tag{25}$$

(Calderon- Zygmund inequality). Thus we can extend T to L_p.

Proof: We shall skip the proof of the Calderon- Zygmund inequality, the reader can find it in [Ah1].

(i) The generalized Cauchy formula (13) implies that

$$(Ph)_{\bar{z}} = -\frac{1}{\pi}\int_{\mathbb{C}}\frac{h_{\bar{z}}}{z-\zeta}dxdy$$

$$(Ph)_{\bar{z}} = -\frac{1}{\pi}\int_{\mathbb{C}}\frac{h_z}{z-\zeta}dxdy \tag{26}$$

Thus,

$$-\frac{1}{\pi}\int_{\mathbb{C}}\frac{h_{\bar{z}}}{z-\zeta}dxdy = -\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{dhdz}{z-\zeta}$$

$$= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{|z-\zeta|=\epsilon}\frac{hdz}{z-\zeta} = h(\zeta)$$

(ii)

$$P(h_z) = \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{dhd\bar{z}}{z-\zeta} = \tag{27}$$

$$\lim_{\epsilon\to 0}(-\frac{1}{2\pi i}\int_{|z-\zeta|=\epsilon}\frac{hd\bar{z}}{z-\zeta}) + \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{hdzd\bar{z}}{(z-\zeta)^2} = Th(\zeta) \tag{28}$$

Remark 3.6. *It follows that Ph is holomorphic near ∞ and $\cong a_0 + \frac{a_1}{z}$ as $z \to \infty$ since $\partial Ph = Th \cong z^{-2}$. Thus, if $a_0 = 0$, then*

$$Ph \in L^p(\{z \in \mathbb{C} : |z| \geq R\})$$

(iii) Now, let's prove that T is L_2-isometry.

$$\int_{\mathbb{C}}|Th|^2 dzd\bar{z} = \int_{\mathbb{C}}(Th)(\overline{Ph})_{\bar{z}}dzd\bar{z}$$

$$= \int(Th\overline{Ph})_{\bar{z}}dzd\bar{z} - \int(Th)_{\bar{z}}\overline{Ph}dzd\bar{z} = -\frac{1}{2i}\int h_z\overline{Ph}dzd\bar{z}$$

because $(Th)_{\bar{z}} = h_z$ and

$$\int(Th\overline{Ph})_{\bar{z}}dzd\bar{z} = \int_{\partial D_R}Th\overline{Ph}dz \to 0 \tag{29}$$

since $Th = O(|z|^{-2})$ as $|z| \to \infty$.

On another hand,

$$\int_{\mathbb{C}}h\bar{h}dzd\bar{z} = \int_{\mathbb{C}}h(\overline{Ph})_z = \int_{\mathbb{C}}(h\overline{Ph})_z - \int_{\mathbb{C}}h_z\overline{Ph} \tag{30}$$

the 1-st term is approximately equal to

$$\int_{\partial D_R}h\overline{Ph}d\bar{z} = 0 \tag{31}$$

since h has a compact support. So, T is an isometry. □

Theorem 3.7. *If μ has a compact support then there exists a unique solution f of the Beltrami equation such that $f(0) = 0$ and $f_z - 1 \in L_p$ where p is such that $C_p\|\mu\|_\infty \equiv C_p k < 1$. Such solution $f = f^{\mu}_{normal}$ is called "normal".*

Proof: (a) **Uniqueness**. Suppose that f is a solution. Then $f_{\bar{z}} \in L_p$ (because it has a compact support and locally it's roughly proportional to f_z) and there exists $P(f_{\bar{z}})$ so $P(f_{\bar{z}})(0) = 0$. Then the function

$$F = f - P(f_{\bar{z}}) \tag{32}$$

is analytic. Then $f_z - 1 \in L^p$ implies that $F_z - 1 = f_z - 1 - T(f_{\bar{z}})$ is in L^p since the last term has quadratic decay at infinity. Hence, $F_z = 1$ and $F = z$ since $F(0) = 0$.

Thus $f = P(f_{\bar{z}}) + z$ and $f_z = T(\mu f_z) + 1$. Let g be another solution. Then $f_z - g_z = T(\mu(f_z - g_z))$, hence

$$\|f_z - g_z\|_p \leq kC_p \|f_z - g_z\|_p \tag{33}$$

and $f_z = g_z$, $f_{\bar{z}} = g_{\bar{z}}$ so $f = g$.

(b) **Existence**. Consider the equation:

$$h = T(\mu h) + T\mu \tag{34}$$

The linear operator $h \mapsto T(\mu h)$ on L_p has norm $\leq kC_p < 1$. Then the series

$$h = T\mu + T\mu(T\mu) + T\mu(T\mu(T\mu))... \tag{35}$$

is convergent in L_p. This is a solution of (34). Then, for this h the function

$$f = P[\mu(h+1)] + z \tag{36}$$

is the solution since $\mu(h+1)$ is in L_p;

$$f_{\bar{z}} = \mu(h+1) \quad ; f_z = T[\mu(h+1)] + 1 = h + 1 \tag{37}$$

and $f(0) = 0$ and $f_z - 1 = h \in L_p$. □

Remark 3.8. *It follows from the formulas (37) that* $\partial f = T(\bar{\partial} f) + 1 = T(\mu \partial f) + 1$.

Lemma 3.9. *If* $\nu_n \to \mu$ *uniformly a.e. and supports are bounded; then*

$$\|\partial g^{\nu_n}_{normal} - \partial f^{\mu}_{normal}\|_p \to 0 \tag{38}$$

and $g^{\nu_n}_{normal} \to f^{\mu}_{normal}$ *uniformly on compacts. Moreover, since these functions are holomorphic near* ∞*, the convergence is uniform on* $\overline{\mathbb{C}}$.

Proof: Put $g_n := g^{\nu_n}_{normal}$. The Remark above implies that

$$\partial f - \partial g_n = T(\mu \partial f_z - \partial \nu g_n) \tag{39}$$

and hence

$$\begin{gathered} \|\partial f - \partial g_n\|_p \leq \|T(\nu_n(\partial f - \partial g_n))\|_p + \\ \|T(\mu - \nu_n)\partial f\|_p \leq kC_p \|\partial f - \partial g_n\|_p + C_p \|(\mu - \nu_n) f_z\|_p \end{gathered} \tag{40}$$

Thus, $\|\partial f - \partial g_n\|_p (1 - kC_p) \leq C_p \|(\mu - \nu_n) f_z\|_p \to 0$.

This implies the statement about convergence of derivatives (since $f_z \in L^p_{loc}$). The Beltrami equation implies the L^p convergence of the $\bar{\partial}$ -derivatives.

Now consider $f = P(\mu h_\mu + 1) + z = P(f_{\bar{z}}) + z$ Then $|f - g| = |P(f_{\bar{z}} - g_{\bar{z}})| \leq K_p \|f_{\bar{z}} - g_{\bar{z}}\|_p |z|^{2-q}$ which implies the last assertion. □

Lemma 3.10. *If μ has a compact support and distributional derivative $\mu_z \in L_p$ $(p > 2)$ then the normal solution $f \in C^1$ and is a homeomorphism.*

Proof: Let's try to determine λ such that the system:

$$f_z = \lambda \quad ; f_{\bar{z}} = \lambda\mu \tag{41}$$

has a solution (which will be the solution of the Beltrami equation). The necessary and sufficient condition is that

$$\lambda_{\bar{z}} = (\lambda\mu)_z = \lambda_z\mu + \lambda\mu_z \tag{42}$$

Or, for $\sigma := log\lambda$:

$$(\sigma)_{\bar{z}} = \mu(\sigma)_z + \mu_z \tag{43}$$

Consider the operator:

$$T_\mu : h \mapsto T(\mu h) \tag{44}$$

The L^p norm of it is less than 1, hence in L^p we have:

$$(T_\mu - 1)^{-1} = 1 + T_\mu + T_\mu^2 + ... \tag{45}$$

Therefore, we can find $q \in L_p$ such that

$$q = T(\mu q) + T(\mu_z) \tag{46}$$

Put $\sigma = P(\mu q + \mu_z) + const$ so that $\sigma \to 0$ as $z \to \infty$. Thus, σ is Holder continuous and

$$\sigma_{\bar{z}} = \mu q + \mu_z \quad ; \sigma_z = T(\mu q + \mu_z) = q \tag{47}$$

Hence $\lambda = \exp(\sigma)$ satisfies the equation and there is a solution f of the class C^1 and we can normalize $f(0) = 0$ so that $f_z = \lambda \to 1$ as $z \to \infty$. Then $\lambda - 1 \cong \sigma(z) = P(\mu q + \mu_z) + const$ as $z \to \infty$. Thus, we can use the Remark 3.6 to show that $\lambda - 1 \in L^p$ and f is a normal solution.

The Jacobian

$$|f_z|^2 - |f_{\bar{z}}|^2 = (1 - |\mu|^2)exp(2\sigma) \tag{48}$$

is positive, hence, f is a diffeomorphism. □

Corollary 3.11. *For any μ with compact support the normal solution is a homeomorphism.*

Proof: We can approximate any $\mu \in L_\infty$ by smooth μ_n with compact support, solutions $f^{\mu_n}_{normal}$ are diffeomorphisms and then $f^{\mu_n}_{normal}$ are convergent to f^{μ}_{normal} uniformly on compacts. Therefore, we can use the property (5) of q.c. maps (compactness property) to prove that the limit is a homeomorphism. □

Now we need a formula for composition of f with Moebius maps g.

Namely, (under assumption $\zeta = h(z)$) it follows from

$$\mu_{f \circ h^{-1}}(\zeta) = \frac{\mu_f(z) - \mu_h(z)}{1 - \mu_f(z)\overline{\mu_h(z)}}\left(\frac{\partial h(z)}{|\partial h(z)|}\right)^2 \tag{49}$$

that:

$$\mu_{g^{-1}fg}(\zeta) = \mu_f(g(\zeta))\frac{\overline{\partial g(\zeta)}}{\partial g(\zeta)} =: g_*\mu \tag{50}$$

Convention. Now, by f^μ we shall mean the solution which fixes $0, 1, \infty$. It is called the "normalized" solution.

Theorem 3.12. *For any μ on $\mathbb{C}$ with the norm $\|\mu\|_\infty < 1$ there exists a normalized solution of the equation $\bar\partial f = \mu \partial f$. This solution is a homeomorphism.*

Proof: (a) The solution exists if μ has a compact support, just adjust the normal solution by Moebius transformation.

(b) Suppose that μ vanishes in $\Delta_r(0)$. Then take $\nu = g_*\mu$ where $g(z) = z^{-1}$. Then ν has a compact support and there exists $h = f^\nu$. Now take $f = ghg^{-1}$ and $\mu_f = g_*^{-1}\nu = \mu$.

(c) Consider the general case. We can decompose μ as $\mu_1 + \mu_2$ where μ_1 has a compact support Δ and μ_2 has support outside Δ. Try to find the solution

$$f^\mu = f^\lambda \circ f^{\mu_2} \tag{51}$$

i.e. $g := f^{\mu_2}$,

$$f^\lambda = f^\mu \circ g^{-1} \tag{52}$$

Then, according to (9) we have:

$$\lambda = [(\frac{\mu - \mu_2}{1 - \mu\overline{\mu_2}})(\frac{\partial g}{|\partial g|})^2] \circ g^{-1} \tag{53}$$

Then, since $\mu - \mu_2 = \mu_1$ has compact support and, by convention, g fixes some neighborhood of ∞, the characteristic λ also has a compact support. So, there exists f^λ. Put $f = f^\lambda \circ f^{\mu_2}$; $\nu = \mu_f$. We have to show that $\nu = \mu$, and we can find f^μ from (51). Notice that μ and ν satisfy to one and the same equation (53). Suppose that $\mu \neq \nu$ on some set E of nonzero measure. The equation:

$$\frac{\mu - \mu_2}{1 - \mu\overline{\mu_2}} = \phi$$

is linear by μ, thus the nonuniqueness implies that on E we have: $\phi \equiv -\mu_2$ and $1 \equiv -\phi\overline{\mu_2}$; i.e. $|\mu_2|^2 \equiv 1$ on the set E which is impossible since $\|\mu_2\|_\infty < 1$. Thus, $\mu = \nu$ and $f = f^\mu$. □

3.4 Analytical dependence of f^μ on the complex dilatation

Theorem 3.13. *The normal and normalized solutions of the Beltrami equations depend holomorphically on μ which means that the map*

$$Belt : \mu \in B(1) \subset L_\infty \mapsto f^\mu \in C^0(\Delta, \mathbb{C})$$

has complex derivative for any disc $\Delta \subset \mathbb{C}$; where $C^0(\Delta, \mathbb{C})$ is the space of continuous $\mathbb{C}$-valued functions with supremum norm; $B(1) \subset L_\infty$ is the open unit ball.

For proof see [Ah1]. We shall need and prove much weaker statement:

Denote by $\Delta(r)$ the open disc of radius r in $\mathbb{C}$ with center at 0.

Theorem 3.14. *Let $\mu, \nu \in B(1)$ have compact support. Then for each $z \in \overline{\mathbb{C}}$, the function*

$$w \in \Delta(\epsilon = \frac{1 - \|\nu\|_\infty}{\|\mu\|_\infty}) \mapsto f^{w\mu+\nu}_{normal}(z)$$

is holomorphic.

Proof: Recall the representation for the normal solution (36 -37):

$$f(z) = P[(w\mu + \nu)(1 + T(w\mu + \nu) + T[(w\mu + \nu)(T(w\mu + \nu)] + ...)] + z =$$

$$P[(w\mu + \nu)(1 + A_1(w) + A_2(w) + ...)] = P[(w\mu(1 + A_1(w) + A_2(w) + ...)]+$$
$$P[\nu(1 + A_1(w) + A_2(w) + ...)] + z$$

this series is uniformly convergent for all $w \in \Delta(\epsilon)$ and each term is a holomorphic (in fact, polynomial) function on w. Then each $z \in \overline{\mathbb{C}}$, the limit depends holomorphically on w. □

Exercise 3.15. *Suppose that D is a domain in $\overline{\mathbb{C}}$ such that: each conformal automorphism of D is Moebius. Then any quasiconformal automorphism of D can be extended to $\overline{\mathbb{C}}$.*

4 Quasiconformal maps on Riemann surfaces

Let X be a Riemannian surface, ds^2 is a metric. Then locally we can write

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2$$

Put $dz = dx + idy$, $d\bar{z} = dx - idy$. Thus, $ds = \lambda|dz + \mu d\bar{z}|$,

where $\lambda^2 = (E + G + 2\sqrt{EG - F^2})$,

$$\mu = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}$$

Notice that

$$|\mu|^2 = \frac{E + G - 2\sqrt{EG - F^2}}{E + G + 2\sqrt{EG - F^2}} < 1$$

Theorem 4.1. *(Gauss' theorem on isothermal coordinates). For each Riemannian surface (S, ds^2) there exists a local system of coordinates such that $\mu = 0$. I.e. any Riemannian metric in dimension 2 is conformally- Euclidean.*

Proof: Suppose that $w = f^\mu(z)$ is the q.c. homeomorphism with the dilatation μ. Then $|dw| = |\partial w dz + \bar{\partial} w d\bar{z}| = |\partial w||dz + \mu d\bar{z}|$; thus $ds = \lambda|dw|/|\partial w|$. □

Thus, we should look more carefully on the μ_f as an object on Riemann surface. The formula (50) shows that $\mu dz/\overline{dz}$ is a differential of the type $(-1, 1)$ on the surface X, i.e. $\mu dz \otimes \partial/\overline{\partial z}$. Such differential will be called a **conformal structure** since (a) each conformal class of Riemannian metrics on S gives us some μ and (b) given μ we can solve the Beltrami equation and the maps f^μ will define a complex (conformal) structure:

For each point $z \in X$ we have a neighborhood U where we can solve the Beltrami equation $\bar\partial f = \mu \partial f$, where $f : U \to \mathbb{C}$ is a homeomorphism.

If $f_i, f_j : U \to \mathbb{C}$ are solutions of the Beltrami equation then by uniqueness of solution $f_i \circ (f_j)^{-1}$ is holomorphic.

As the result we get a conformal structure on X corresponding to μ. We will retain the notation μ for this conformal structure. Automorphisms of μ are the self-diffeomorphisms h of X such that iff $h^*(\mu) = \mu$. If μ is defined in some domain D in the plane $\bar{\mathbb{C}}$, then this condition means that:

$$\mu(z) \equiv \mu(hz)\overline{h'(z)}/h'(z) \tag{54}$$

where we assume $h \in Mob(\mathbb{C})$. Let f_μ be the solution of the B.e.: where $\mu = 0$ outside D. Then $f : (\bar{\mathbb{C}}, \mu) \to (\bar{\mathbb{C}}, can)$ is conformal. Therefore, since $h \in Aut(\mu)$, then $f \circ h \circ f^{-1}$ belongs to $Mob(\mathbb{C})$.

Corollary 4.2. *Suppose that $G \subset PSL(2, \mathbb{C})$; f_μ is a quasiconformal map of $\overline{\mathbb{C}}$ so that (54) holds for each $h \in G$. Then $f_\mu \circ h \circ f_\mu^{-1} \in PSL(2, \mathbb{C})$.*

Corollary 4.3. *Suppose that $f : A \subset \overline{\mathbb{C}} \to f(A) \subset \overline{\mathbb{C}}$ is a quasiconformal homeomorphism and G is a subgroup of conformal automorphisms of A and (54) holds for each $h \in G$. Then $f_\mu \circ h \circ f_\mu^{-1}$ is a conformal automorphism of $f(A)$ for each $h \in G$.*

Theorem 4.4. *Let X, Y be surfaces of the same type. Then there is a quasiconformal map $X \to Y$.*

Proof: Let $\bar X, \bar Y$ be conformal compactifications of X, Y; then there is a diffeomorphism $f : \bar X \to \bar Y$ which maps punctures to punctures. The restriction of f to $\bar X$ is the desired quasiconformal homeomorphism. □

Formula for transformation of the Beltrami differential under complex conjugation $g : z \to \bar z$. If $f = f^\mu$ and $h = g \circ f \circ g$ then

$$\mu_h = g \circ \mu \circ g \tag{55}$$

i.e. $\mu_h(z) = \overline{\mu(\bar z)}$.

5 Proof of the Uniformization Theorem for surfaces of finite type

Let X be a Riemann surface of finite type. Then we can construct a hyperbolic surface X_0 of the same type. denote by f a q.c. homeomorphism $f : X_0 \to X$ (which exists due to Theorem 4.4). Let μ be the dilatation of f lifted in Δ. Extend μ to $ext(\Delta)$ by the symmetry. Put $\mu = 0$ on $\partial\Delta$. Then the resulting differential Beltrami ν will be compatible with the action of the group $\Gamma_0 = \pi_1(X_0)$. Therefore, f^ν conjugate Γ_0 to a group $\Gamma \subset PSL(2, \mathbb{C})$ and since f^ν commutes with conjugation, $\Gamma \subset PSL(2, \mathbb{R})$.

However, the universal cover $\tilde{X}$ of the surface X is biholomorphic to (Δ, μ) which is conformally equivalent to Δ via f. Thus, we proved the Uniformization Theorem for surfaces of hyperbolic type. The proof for the elliptic type is essentially the same. Proof for the rational type is just a particular case of the existence theorem for quasiconformal maps. $\square$

6 Elementary theory of discrete groups.

6.1 Definitions

Let $\mathbb{S}^k$ be a round sphere in $\mathbb{S}^{k+1} = \overline{\mathbb{R}^{k+1}}$. Then the inversion J in $\mathbb{S}^k$ defined as follows. If $\mathbb{S}^k$ is an extended Euclidean plane, then J is just the Euclidean symmetry in $\mathbb{S}^k$. Otherwise, if O is the center of $\mathbb{S}^k$, $x \in ext(\mathbb{S}^k)$, L is the Euclidean line through x, O and K is the tangent cone from x to $\mathbb{S}^k$, then J maps x to the orthogonal projection of $K \cap \mathbb{S}^k$ to L.

It is easy to prove that each element $\gamma \in PSL(2, \mathbb{C})$ is a composition of even number of inversions.

Consider the group $G = PSL(2, \mathbb{C})$ acting on $\overline{\mathbb{C}}$. Extend this action in $\mathbb{R}^3_+ = \mathbb{H}^3$ using inversions. Namely, if $\gamma \in PSL(2, \mathbb{C})$ is a composition $J_1 \circ ... \circ J_s$ of inversions in the circles $\sigma_j \subset \overline{\mathbb{C}}$, then each J_k is extended canonically to the inversion $\tilde{J}_k$ in the Euclidean sphere Σ_j which contains σ_j and is orthogonal to $\overline{\mathbb{C}}$. Then, define the extension $\tilde{\gamma}$ as the product of extensions $\tilde{J}_1 \circ ... \circ \tilde{J}_s$. It's easy to see that this extension doesn't depend on the decomposition of γ in the product of inversions.

The extended complex plane $\overline{\mathbb{C}}$ can be identified with $\mathbb{S}^2$ via stereographic projection. This defines on $\overline{\mathbb{C}}$ the metric of constant positive curvature $|dz|/(1 + |z|^2)$. We can describe $\mathbb{H}^3$ as G/K where K is the maximal compact subgroup $SO(3)$. Three types of isometries of $\mathbb{H}^3$ can be distinguished by the matrices in $SL(2, \mathbb{C})$ representing these isometries:

elliptic: $Tr(g) \in (-2, 2)$;

parabolic: $Tr(g) = \pm 2$;

loxodromic: $Tr(g) \notin [-2, 2]$.

A special type of loxodromic elements are *hyperbolic* elements, which have real trace. They can be characterized as elements with invariant Euclidean discs in $\overline{\mathbb{C}}$.

The group G has the "Cartan decomposition": $G = KAK$ where

$$A = \{a : z \mapsto kz, k \in \mathbb{C}^*\}$$

One can prove the existence of this decomposition geometrically. Namely, $\mathbb{H}^3 = X = G/K$ and G acts on X transitively on the right. Let $x_0 \in X$ be the class of K; let $x \in X$ be any point. Then there is an element $k \cdot a \in KA$ such that $ak(x) = x_0$ (if γ is the invariant geodesic for A then there is an element $k \in K$ such that $k(x) \in \gamma$, then the action of A translates $k(x)$ into x_0). Therefore, for each $g \in G$ we can find k, a such that: $akgK = K$, since $gK = x, K = x_0$; thus $g \in KAK$.

6.2 The convergence property

A quasiconstant map $z_x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a map such that for $z, x \in \overline{\mathbb{C}}$ we have:

$$z_x(w) = z \ \text{ for each } \ w \neq x$$

We let $z_x^{-1} := x_z$. Each quasiconstant map naturally extends to $\mathbb{H}^3$. Let $\hat{G} := G \cup \{quasiconstants\}$

Topology. A sequence of elements $g_n \in \hat{G}$ is convergent to a quasiconstant z_x iff g_n converges to z_x uniformly on compacts in $\overline{\mathbb{C}} - \{x\}$.

Exercise 6.1. $g_n \to g$ *iff* $g_n^{-1} \to g^{-1}$.

Theorem 6.2. $\hat{G} = G \cup \{quasiconstants\}$ *is compact.*

Proof: For any sequence g_n we have $g_n = k_n a_n c_n$ where $k_n, c_n \in K$, $a_n \in A$. Up to subsequence we can assume that $k_n \to k$, $c_n \to c$, $a_n \to a$ where a is either element of A or quasiconstant ∞_0. Then g_n is convergent on $\overline{\mathbb{C}} - c^{-1}(0)$ to $k(\infty)$. Thus, $g_n \to a_b$ where $a = k(\infty), b = c^{-1}(0)$. □

6.3 Discontinuous groups

A subgroup Γ of $PSL(2, \mathbb{C})$ is called *elementary* if it has either an invariant point in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$ or invariant geodesic L in $\mathbb{H}^3$ (in the latter case Γ can change the orientation on L).

Examples of elementary groups: (i) Consider the group $B \subset SL(2, \mathbb{C})$ which consists of upper-triangular matrices. Let PB is the projection of B to $PSL(2, \mathbb{C})$. Then each subgroup of PB is elementary (since PB fixes the point ∞ of $\bar{\mathbb{C}}$.

(ii) Let Γ be a finite subgroup of $Isom(\mathbb{H}^n)$. Then Γ has a fixed point in $\mathbb{H}^n$ (hint: consider the Γ-orbit Γp of a point $p \in \mathbb{H}^n$, take the smallest metric ball D in $\mathbb{H}^n$ which contains Γp; then the center of D is Γ-invariant).

Definition 6.3. *Let* $\Gamma \subset G$. *Then* $x \in \overline{\mathbb{C}}$ *is a* point of discontinuity *for* Γ *if there is a neighborhood* U *of* x *such that* $U \cap gU \neq \emptyset$ *only for finitely many* $g \in \Gamma$. *Usually, this means that* $U \cap gU \neq \emptyset$ *for all* $G - \{1\}$*; the exceptional case is:* x *is a fixed point of a finite subgroup* $F \subset \Gamma$.

The *domain of discontinuity* $\Omega(\Gamma)$ consists of all points of discontinuity. Clearly this is an open subset of $\overline{\mathbb{C}}$. If g is an element of Γ which has infinite order then the fixed-point set of g is disjoint from $\Omega(\Gamma)$.

Definition 6.4. *A* **discontinuous** *(another name is* **Kleinian***) group is a subgroup of* $PSL(2, \mathbb{C})$ *with nonempty discontinuity domain.*

More general concept is a *discrete group*, i.e. a subgroup Γ of $PSL(2, \mathbb{C})$ which is a discrete subset in the induced topology (i.e. if $\Gamma \ni \gamma_n \to \gamma \in PSL(2, \mathbb{C})$ then $\gamma_n = \gamma$ for all but finite elements of the sequence $\{\gamma_n\}$.

Exercise 6.5. Γ *is discrete iff* 1 *is an isolated point of* Γ. *(Hint: if* Γ *isn't discrete, consider the sequence* $\gamma_{n+1}\gamma_n^{-1}$*).*

Clearly all Kleinian groups are discrete. Therefore all their elliptic elements have finite orders (however there is no a priori bound on these orders). One can show that if $\Gamma \subset PSL(2,\mathbb{R})$ is nonelementary then the absence of elliptic elements of infinite order is also a *sufficient* condition for discreteness. However, the discreteness doesn't imply that the group is Kleinian.

Exercise 6.6. *Consider $\Gamma := PSL(2,\mathbb{Z}[i])$ where $\mathbb{Z}[i]$ is the ring of "Gaussian integers" (i.e. complex numbers of the form $x+iy$ where $x,y\in\mathbb{Z}$). Show that Γ is discrete. Prove that the domain of discontinuity of Γ is empty. (Hint: show that each rational Gaussian number is a fixed point of a parabolic element of Γ.)*

Theorem 6.7. *(Schur's Lemma.) Suppose that Γ is a finitely generated* **torsion** *subgroup of $GL(n,\mathbb{C})$ (i.e. each element of Γ has finite order). then Γ is finite.*

This lemma immediately implies that if a discrete group Γ consists only of elliptic elements, then Γ is finite (and hence elementary).

However, there are examples of infinite (nondiscrete) subgroups $T\subset Isom(\mathbb{H}^5)$ such that each element $t\in T$ has a fixed point in $\mathbb{H}^5$ but the group T doesn't have a fixed point in $\mathbb{H}^5$. Nevertheless, such group necessarily has a fixed point in $\mathbb{H}^5\cup\mathbb{S}^4$. To construct such example, take a free 2-generated subgroup $<a,b>=H\subset SU(2)\subset SO(4)$ and $v,w\in\mathbb{R}^4-\{0\}$; $g(x)=ax+v, f(x)=bx+w$ for $x\in\mathbb{R}^4$. Then each element of the free group $T=<g,f>\subset Isom(\mathbb{E}^4)$ has a fixed point in $\mathbb{R}^4$ but there is no global fixed point for the action of T in $\mathbb{E}^4$. The extension of T in $\mathbb{H}^5$ provides the desired example. Such examples are impossible for $\mathbb{H}^k$, $k<5$.

Now we can define the *limit set* $\Lambda(\Gamma)$ of a Kleinian group Γ as the set of accumulation points for the orbit Γx for some $x\in\Omega(\Gamma)$ (i.e. $y\in\Lambda(\Gamma)$ iff there is an infinite sequence of (different) elements $\gamma_n\in\Gamma$ such that $\lim\gamma_n x=y$).

It is clear that both the domain of discontinuity and the limit set are Γ-invariant.

Remark 6.8. *More generally one can define the limit set for each subgroup $\Gamma\subset Isom(\mathbb{H}^n)$ as follows. Start with a point $x\in\mathbb{H}^n$, then consider the closure $cl(\Gamma x)$ of the Γ-orbit of x in $\mathbb{H}^n\cup\partial\mathbb{H}^n$. Finally let $\Lambda(\Gamma):=cl(\Gamma x)\cap\partial\mathbb{H}^n$.*

The Convergence Property implies that if $x\in\Omega(\Gamma)$ then $z_x\notin cl_{\hat{G}}(\Gamma)$ for any z. It follows that $\Lambda(\Gamma)$ does not depend on the choice of $x\in\Omega(\Gamma)$. Also: $\Lambda(\Gamma)\cap\Omega(\Gamma)=\emptyset$.

Theorem 6.9. *$\overline{\mathbb{C}}=\Lambda(\Gamma)\cup\Omega(\Gamma)$.*

Proof: Let $x\notin\Omega(\Gamma)$. Then there exist a sequence $g_n\in\Gamma$ and $z_n\to x$ such that

$$\lim_{n\to\infty}(g_n(z_n))=x \tag{54}$$

Then up to subsequence $g_n\to a_b$. Since $x\neq b$ then (54) implies that $x=a$ which is impossible. $\square$

Example: For each elementary Kleinian group the limit set consists of $0,1$ or 2 points.

Lemma 6.10. *For nonelementary groups $\Lambda(\Gamma)=cl(\Gamma x)$ for each $x\in\Lambda(\Gamma)$.*

Proof: Let $x, z \in \Lambda(\Gamma)$. There exists a sequence $g_n \in \Gamma$ such that $g_n(p) \to z$ for all $p \in \Omega(\Gamma)$. By taking a subsequence (if necessary) we can assume that $g_n \to z_w$. If $w \neq x$ then $g_n(x) \to z$ and we are done. Otherwise find $f \in \Gamma$ such that $f(x) \neq x$. Then $g_n f(x) \to z$. □

Corollary 6.11. *If* Γ *is nonelementary then* $\Lambda(\Gamma)$ *is the smallest nonempty* Γ*-invariant closed subset of* $\bar{\mathbb{C}}$.

Theorem 6.12. *If* Γ *is not elementary, then the loxodromic fixed points are dense in* $\Lambda(\Gamma)$.

Proof: First we need to find a loxodromic element in Γ. Indeed, the group Γ is infinite, hence it contains either a parabolic or loxodromic element g. If g is a loxodromic element we are done. Suppose that g is parabolic. The fixed point p of g is not fixed by the whole Γ, hence there is $h \in \Gamma$ such that $h(p) = q \neq p$. Then the element $f = hgh^{-1}$ is again parabolic and its fixed point is q. By conjugating Γ in $PSL(2, \mathbb{C})$ we can assume that $g : z \mapsto z + a$. Since g is parabolic there are two closed tangent (at p) round discs $D, D' \in \mathbb{C}$ such that $f(intD) = ext(D')$. Then, for large n we have: $E = g^{-n}(D) \cap D' = \emptyset$. It follows that $\alpha := f \circ g^n : intE \to extD'$. Thus, the iterations of α show that α is loxodromic and it has a fixed point $x \in \Lambda(\Gamma)$. Finally, by the previous lemma the Γ-orbit of x is dense in $\Lambda(\Gamma)$. For each $\gamma \in \Gamma$ the fixed-point set of $\gamma\alpha\gamma^{-1}$ is the γ-image of the fixed point set of α. Hence Γx consists of fixed points of loxodromic elements of Γ. □

Corollary 6.13. *For nonelementary group, loxodromic fixed pairs are dense in* $\Lambda(\Gamma) \times \Lambda(\Gamma)$.

Proof: Take disjoint open U and V which intersect the limit set. Then there are loxodromic p, q such that $p^n \to x \in U$, $q^n \to y \in V$. Find a lox element f with fixed points different from that of p and take $g = p^n f p^{-m}$. Then fixed point of g are in U and $g^n \to z_w$ where $z, w \in U$. Thus, for some large k we have: $h_n = q^k g^n \to v_w$ where $v \in V$. Thus, one fixed point of h_n is in U, another is in V. □

So, if Γ isn't elementary and Kleinian , then the limit set of G is perfect, closed and has empty interior.

7 Fundamental domains and quotient-surfaces

Let $\Gamma \subset PSL(2, \mathbb{C})$ be a Kleinian group. A subset $F \subset \Omega(\Gamma)$ is called *a fundamental set* of the group Γ if:

(i) $gF \cap F = \emptyset$ for all $g \in \Gamma - 1$ with the exception: $gF \cap F$ is a fixed point (or the set of two fixed points) of an elliptic $g \in \Gamma$, and

(ii)

$$\Gamma F \equiv \bigcup_{g \in \Gamma} gF = \Omega(\Gamma) \tag{55}$$

This is reasonable but too general definition. It allows to reconstruct the surface $S(\Gamma) = \Omega(\Gamma)/\Gamma$ as a set, but we would like also to recapture the topology of $S(\Gamma)$ as well.

Definition 7.1. *A* **fundamental domain** *D for a Kleinian group Γ is an open subset of $\Omega(\Gamma)$ such that:*

(1) "The fundamentality": There is a fundamental subset $F \subset cl(D)$, $D \subset F$ for the group Γ;

(2) The "side-pairing property": The boundary of D in $\Omega(\Gamma)$ is piecewise-smooth submanifold in $\Omega(\Gamma)$ and is divided in a union of smooth arcs which are called **sides** *(or* **edges**[1]*.); for each side s there another side s' and an element $g = g_{ss'} \in \Gamma - 1$ so that $gs = s'$ (g is called the "side-pairing transformation"); $g_{ss'} = g^{-1}_{s's}$.*

(3) The "finiteness condition": The action of Γ defines an equivalence relation on the boundary of D in $\Omega(\Gamma)$. We require the equivalence set of each vertex of D to be finite[2].

Notice that the property (2) implies that $\partial_{\Omega(\Gamma)} D$ is "locally finite" in $\Omega(\Gamma)$, i.e. each compact $K \subset \Omega(\Gamma)$ can intersect not more than finitely many sides of D.

There are several other conditions on D which imply (3):

(3') The orbit GD is *locally finite* in $\Omega(\Gamma)$, i.e. for each compact $K \subset \Omega(\Gamma)$ there is not more than finitely elements $g \in \Gamma$ such that $gD \cap K \neq \emptyset$.

Alternatively: (3") D has only finitely many components.

It is obvious that (3')$\Rightarrow$(3), the fact that (3")$\Rightarrow$(3') is less obvious, see Theorem 7.3 below.

Introduce the equivalence relation "$\cong$" on $\partial_{\Omega(\Gamma)} D$ generated by the equivalence: $x \cong y$ iff there is a "side-pairing transformation" g such that $gx = y$. The factor-space

$$E = cl_{\Omega(\Gamma)}(D)/ \cong$$

has the quotient topology. Denote by π the projection $\Omega(\Gamma) \to \Omega(\Gamma)/\Gamma = S(\Gamma)$.

Theorem 7.2. *The natural map $\theta : E \to S(\Gamma)$ is a homeomorphism, where $\theta : [x] \in E \mapsto Gx \in S(\Gamma)$.*

Proof: The projections $\tilde{\pi} : clD \to E$ and π are open which implies that θ is continuous. The map θ is surjective, thus we need to prove that θ is injective and open. It's easy to see that the restriction of θ to the complement to the projection of the set of vertices of ∂D is open and injective. Thus, our problem is the set of vertices. Let $x \in \partial_{\Omega(\Gamma)} D$ be any vertex, x_1 is the end point of a side s_1, then there is another side s'_1 and the pairing element g_1 such that $g_1(s_1) = s'_1$, $x_2 = g_1(x_1)$. Now, x_2 is the vertex for 2 sides: s'_1 and another side s_2. For s_2 we again find a pairing transformation g_2 and get $x_3 = g_2(x_2)$ etc. Thus, after finitely many steps we end up with some point $x_n = x_k$, $k < n$. In this case rename our sequence so that $x_1 = x_k, x_2 = x_{k+1}$ etc. The product $h = g_1^{-1} \circ ... \circ g_{n-1}^{-1}$ maps x_1 to x_1. Let U_1 be a small neighborhood of the point x_1 in clD, U_2 be small neighborhood of the point x_2 in clD (see the Figure 1) etc. Put $V = U_1 \cup g_1^{-1}(U_2) \cup ... \cup g_1^{-1} \circ ... \circ g_{n-2}^{-1} U_{k-1}$ and the element h maps the "free" boundary component s (of V) adjacent to x_1 and different from s_1 to another "free" boundary component $\sigma \subset g_1^{-1} \circ ... \circ g_{n-2}^{-1} U_{k-1}$. The element h is a priori nontrivial since G can have torsion. Put $W = V \cup hV \cup ... \cup h^q V$, where $q+1$ is the order of h. Then W is a neighborhood of x_1 and the images of U_j cover it without "overlaps".

[1]Their end-points are called the vertices

[2]This property is void if D is compact

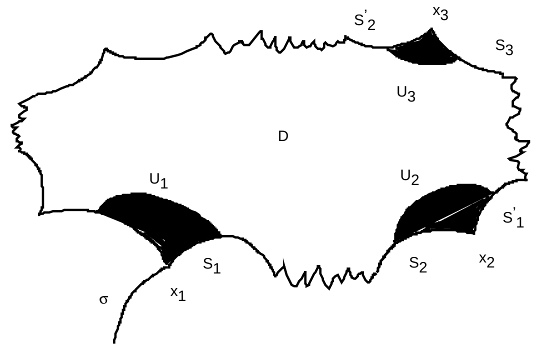


Figure 1:

This has two consequences. (1) If $g \in \Gamma$ is such that $gx = y$, $x = x_1$ and $y \in \partial D$, then $g^{-1}D \cap W$ should be $h^i \circ g_1^{-1} \circ ... \circ g_{n-j}^{-1}(D) \cap W$. Thus, $g^{-1} = h^i \circ g_1^{-1} \circ ... \circ g_{n-j}^{-1}(D)$ since D is a fundamental domain and so, $x \cong y$.

(2) Let $A \subset E$ be open, then there is B which is an open subset of Ω such that $clD \cap B = \tilde{\pi}^{-1}(A)$. Let Z be the G–orbit of $clD \cap B$. Then Z is a neighborhood of x according to the discussion above. However, $\pi(Z) = \pi(clD \cap B) = \theta \circ \tilde{\pi}(clD \cap B) = \theta(A)$. Thus, since $\tilde{\pi}$ is open, $\theta(A)$ is a neighborhood of $\theta(\pi(x))$. This is true for all vertices of ∂D. Thus, θ is open. □

Theorem 7.3. *Suppose that D satisfies (1) and (2) and has only finitely many connected components. Then D satisfies the local finiteness condition (3').*

Proof: Suppose that $\Omega \ni x \in \lim g_n D^0$, where D^0 is a connected component of D. There can be not more than countably many exceptional points $x \in \Omega(\Gamma)$; all are vertices of ∂D. If D^0 is relatively compact, then we are done. If not, then there is a point $z \in \partial \cap \Lambda(\Gamma)$. Let $w = \lim g_n z \in \Lambda(G)$. Let E be the set of points in $\overline{\mathbb{C}}$ whose neighborhood s intersect $g_n D^0$ infinitely many times (the set of "exceptional" points and the limit set are contained in E) . Then (since D^0 is connected) x and w belong to a common connected component of E which is impossible. □

Exercise 7.4. *Let Γ be the cyclic group generated by $z \mapsto 2z$. Construct example of an open connected domain $D \subset \Omega(\Gamma)$ which satisfies (1), has piecewise-smooth boundary, however $cl_{\Omega(\Gamma)}(D)/\Gamma$ is not compact.*

7.1 Dirichlet fundamental domain

Let Γ be a discrete subgroup of $PSL(2, \mathbb{C})$, suppose that $O \in \mathbb{H}^3 = X$ is not a fixed point of any nontrivial element of G. Then, define $D_g = \{x \in X : d(x, O) < d(x, gO)\}$ for $g \in \Gamma - 1$; $\tilde{D}_g = clD_g - \partial_X D_g$ where the closure is taken in $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \overline{\mathbb{C}}$. Define B_g to be the hyperbolic plane $\partial_X D_g$.

Definition 7.5. *The intersection*

$$D_O(\Gamma) = \bigcap_{g \in \Gamma - 1} D_g$$

is called the Dirichlet polyhedron of Γ *with center at* O. *The set*

$$\Phi_O(\Gamma) = \bigcap_{g \in G-1} \tilde{D}_g \cap \overline{\mathbb{C}}$$

is called the Dirichlet fundamental domain for Γ.

Theorem 7.6. $\Phi_O(\Gamma)$ *is a fundamental domain for the action of* Γ *in* $\Omega(\Gamma) \subset \overline{\mathbb{C}}$.

Proof: Let $x \in B_g \cap D$. Pick $h \neq g^{-1}$. Then $d(g^{-1}(x), h(O)) = d(x, g \circ h(O)) \geq d(x, O) = d(g^{-1}(x), O)$. This implies the "side-pairing property". The nontrivial statement is that for each $z \in \Omega(\Gamma)$ there is an element $g \in G$ such that $z \in cl(g(\Phi_O(G))$. If z doesn't belong to G-orbit of the closure of any face of D_O then the conclusion follows from the fact that there is a neighborhood V of x in $\mathbb{H}^3$ such that $V \subset g(D_O)$; thus $z \in g(\Phi)$. Suppose else. Then the set of such points form a nowhere dense subset E in $\Omega(G)$. Consider $p : \Omega(G) \to \Omega(G)/G = S$; $p(z) \in p(E)$, $p(z) = \lim p(z_n)$, $p(z_n) \in S - p(E)$. Thus, $z = \lim g_n(z_n)$, $z_n \in \Phi$. If g_n is relatively compact, then $g_n = g$, $z = \lim g(z_n) \subset clg(\Phi)$. Otherwise, $g_n \to a_b$. However, $diam(g_n(\Phi)) \to 0$ as $n \to \infty$. Thus, $\lim g_n(z_n) = \lim g_n(\Phi) = z$ which is impossible. $\square$

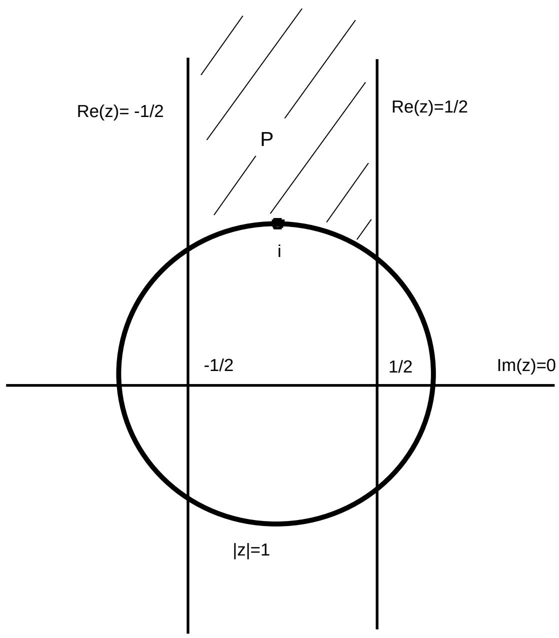


Figure 2:

Example. Let G be the *classical* or *elliptic* modular group $SL(2, \mathbb{Z})$ (later we shall deal also with "Teichmüller modular group). Denote by $P \subset \mathbb{H}^2$ be the open triangle bounded by $\{Re(z) = \pm 1/2\}$ and $\{|z| = 1\}$. Then, P is the Dirichlet polygon with center at $w = iv$, $1 < v \in \mathbb{R}$. Really,

(i) $f(z) = z + 1$ and $g(z) = -z^{-1}$ are in G. Then $P \subset D_w$. Thus, we need only to show that P has no equivalent points. Suppose that $z \in P, hz \in P$;

$$h(x) = \frac{ax + b}{cx + d}$$

Then $|cz+d|^2 = c^2|z|^2 + 2Re(z)cd + d^2 > c^2 + d^2 - |cd| = (|c| - |d|)^2 + |cd| = \alpha$. Then $\alpha \in Z$ and $\alpha = 0$ iff $c = d = 0$, thus $\alpha \geq 1$ and $|cz + d| > 1$. Then,

$$Im(hz) = \frac{Im(z)}{|cz+d|^2} < Im(z)$$

On another hand, we have $hz \in P$, so $Im(hz) > Im(z)$. This contradiction shows the absence of such point z. □

To be more precise, the Dirichlet fundamental domain for the action of G in $\overline{\mathbb{C}}$ is the union of P and it's image under inversion in $\partial\mathbb{H}^2$. The center O for this domain is contained in a copy of the hyperbolic plane in $\mathbb{H}^3$ which is invariant under G, O is a point on the geodesic L in $\mathbb{H}^3$ which connects the fixed points of f and gfg and lies between a fixed point of f and the fixed point j for the action of g on L.

Thus, $D_O(G)$ is bounded by 3 hyperbolic planes, two of them are tangent at the fixed point of f. See Figure 2.

7.2 Ford fundamental domain

Another example of the fundamental domain is so called *Ford fundamental domain*. Let $PSL(2,\mathbb{C}) \ni g : z \mapsto (az+b)/(cz+d)$, $g(\infty) \neq \infty$, i.e. $c \neq 0$; we assume that $ad - bc = 1$. Then, $g'(z) = (cz+d)^{-2}$ and define I_g to be the set of points in $\overline{\mathbb{C}}$ where g' is an Euclidean isometry i.e. $I_g = \{z \in \mathbb{C} : |g'(z)| = 1\} = \{z \in \mathbb{C} : |cz + d| = 1\}$. Then I_g is a circle which is called *isometric circle* of g. The center of this circle is $g^{-1}(\infty) = -d/c$ and $|c|^{-1}$ is the radius of I_g. Thus, the radius of I_g is the same as the radius of $I_{g^{-1}}$. Any Euclidean circle with the center at $g^{-1}(\infty)$ is mapped by g in a Euclidean circle with the center at $g(\infty)$ (since the last bunch of circles is described by the property that they are orthogonal to each line through $\infty, g(\infty)$).

However, the radius of I_g should be the same as the radius of $g(I_g)$, thus $g(I_g) = I_{g^{-1}}$ and $g(extI_g) = intI_{g^{-1}}$. Now, assume that $\infty \in \Omega(\Gamma)$ and it isn't a fixed point of any nontrivial element of the Kleinian group Γ, then

$$F(\Gamma) = \{z \in \mathbb{C} : |g'(z)| < 1, g \in \Gamma\} = \bigcap_{g \in \Gamma - 1} extI_g$$

is called the *Ford fundamental domain* of Γ.

It's possible to realize $F(\Gamma)$ as the degenerate case of the Dirichlet fundamental domain. To do this we need we need another point of view on B_g. namely, if $O \in \mathbb{H}^3$ then there is a unique ball $B(O,r)$ with center at O (and radius r) such that $\partial B(O,r)$ is tangent to $\partial(gB(O,r))$ at some point $x = x_g \in \mathbb{H}^3$. Then B_g is the unique hyperbolic plane which is tangent to both $\partial B(O,r)$, $\partial gB(O,r)$ at x. Now we can move the point O to the point s at the "infinity" $\overline{\mathbb{C}}$ of the hyperbolic space assuming that s isn't a fixed point of g), thus $r \to \infty$ and $B(O,r)$ degenerates to a horoball U_g with center at s and $gB(O,r)$ – to a horoball gU_g with center at gs. (See Figure 4 below where $s = \infty \in \overline{\mathbb{C}}$).

Then it's clear that $B_g \cap \overline{\mathbb{C}}$ is the isometric circle of g^{-1} since it has center at $g(\infty)$, the same (Euclidean) radius as $B_{g^{-1}}$ and $g(B_{g^{-1}}) = B_g$. Notice also that the Euclidean diameters of $I_{g(n)}$ tend to 0 for any infinite sequence of different elements $g(n) \in \Gamma$. Really, the centers of $I_{g(n)}$ belong to a compact subset K of $\mathbb{C}$ (since

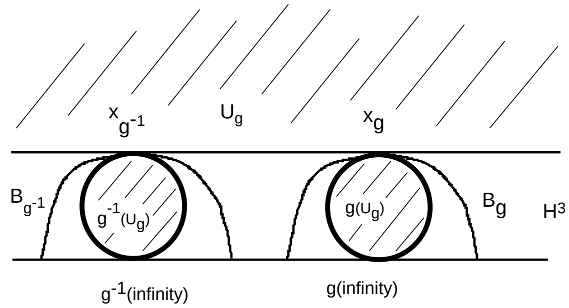


Figure 3:

$\infty \in \Omega(\Gamma)$) and the Euclidean distances of the points $x_n = B_g \cap U_g$ to $\mathbb{C}$ are bounded from above (since $\infty \in \Omega(\Gamma)$ and it isn't a fixed point of any element of $\Gamma - 1$). Thus, either infinitely many points x_n belong to a compact subset of $\mathbb{H}^3$ (which contradicts to the discreteness of Γ) or $dist(x_n, \mathbb{C}) = Rad(I_{g(n)}) \to 0$. The last implies that we can apply to $F(\Gamma)$ the same arguments as to $\Phi_O(\Gamma)$ to prove that it is fundamental.

7.3 Quasiconformal conjugations of Fuchsian groups

Theorem 7.7. *Suppose that S is a hyperbolic surface of finite area; $S = \mathbb{H}^2/G$. Then $\Lambda(G) = \mathbb{S}^1 = \partial\mathbb{H}^2$.*

Proof: Suppose that $x \in \mathbb{S}^1 \cap \Omega(G)$, then let $V_x = U_x \cap \mathbb{H}^2$, so that $gV_x \cap V_x = \emptyset$ for all $g \in G - 1$. Then, V_x projects isometrically to S; thus $\infty = Area(V_x) < Area(S)$. $\square$

Corollary 7.8. *Fixed points of loxodromic elements of G are dense on $\mathbb{S}^1$.*

Theorem 7.9. *(Extension Theorem). Each quasiconformal self-map of the unit disc Δ can be extended continuously to $ext(f) : \partial\Delta \to \partial\Delta$ (see Property 6 of quasiconformal maps).*

Moreover suppose that $f : \Delta \to \Delta$ is a quasiconformal homeomorphism such that $fgf^{-1} \in PSL(2, \mathbb{C})$ for all elements of some Kleinian group $\Gamma \subset PSL(2, \mathbb{R})$. Then we can extend f to a quasiconformal map of $\overline{\mathbb{C}}$ which conjugate Γ to a Kleinian group. To prove this, extend the complex dilatation μ of f to the Beltrami differential ν on $\overline{\mathbb{C}}$ using $j_*\mu$ (see (49)) where j is the inversion in $\partial\Delta$. Then, the solution of the Beltrami equation $\bar{\partial}\tilde{f} = \nu\partial\tilde{f}$ (after composition with some Moebius transformation) defines the desired extension of f (since $\tilde{f}$ is a self-map of Δ, the solution of the Beltrami equation in Δ is unique up to composition with a conformal automorphism). Notice that the coefficient of the quasiconformality of this extension is the same as that of f.

Theorem 7.10. *Suppose that $f_0, f_1 : X \to Y$. Then f_0 is homotopic to f_1 they induce "equivalent" isomorphisms of the fundamental group.*

Proof: The nontrivial implication is $\Leftarrow$. The isomorphisms are "equivalent" iff they differ by "tale" between $f_0(x)$, $f_1(x)$ where x is the base-point. Thus, there are lifts $\tilde{f}_j : \mathbb{H}^2 \to \mathbb{H}^2$ such that the induced isomorphisms of $\pi_1(X), \pi_1(Y)$ are equal to θ; i.e. $\theta(g) \circ \tilde{f}_j = \tilde{f}_j \circ g$ for all $g \in G = \pi_1(X)$. Define $\tilde{f}(z,t)$ to be the point of $[f_0(z), f_1(z)]$ which divides this segment as $t : (1-t)$. Consider $\tilde{f}(gz,t) =$ the point of $[f_0(gz), f_1(gz)]$ which divides as $t : (1-t)$; however, $[f_0(gz), f_1(gz)] = [\theta(g)f_0(z), \theta(g)f_1(z)]$, so $\tilde{f}(gz,t) =$ the point of $[\theta(g)f_0(z), \theta(g)f_1(z)]$ which divides as $t : (1-t)$. I.e. this is the same point as $\theta(g)\tilde{f}(z,t)$. Thus, $\tilde{f}(z,t)$ projects to the homotopy $X \to Y$ between f_0, f_1. □

Corollary 7.11. *Quasiconformal homeomorphisms $f_0, f_1 : X \to Y$ are homotopic iff the extensions of $\tilde{f}_j$ to $\partial\mathbb{H}^2$ coincide (for some choice of the lifts).*

Corollary 7.12. *Suppose that $f : X \to Y$ is a conformal automorphism homotopic to id. Then $f = 1$.*

Theorem 7.13. *(Theorem of Nielsen). Let X, Y be two surfaces of the same finite conformal type. Suppose that $\phi : \pi_1(X) \to \pi_1(Y)$ is an isomorphism, such that the image of "peripheral" (representing a loop around a puncture) element of $\pi_1(X)$ is again peripheral. homeomorphism. Then there exists a homeomorphisms $f : X \to Y$ which induces ϕ. If h is orientation- preserving, then f is homotopic to a quasiconformal homeomorphism.*

Corollary 7.14. *Suppose that G_1, G_2 be a pair of torsion-free Kleinian subgroups of $PSL(2,\mathbb{R})$ such that $\mathbb{H}^2/G_j$ have finite type and $\theta : G_1 \to G_2$ is a type preserving isomorphism (i.e. the image of parabolic element is parabolic etc.) Then, θ can be induced by a homeomorphism $\tilde{f}$ of the hyperbolic plane. If $\tilde{f}$ is orientation-preserving, then it can be chosen quasiconformal .*

We postpone the proof of this result. The isomorphisms that can be induced by quasiconformal maps are called *admissible*.

7.4 Finiteness of area versus finiteness of type

Theorem 7.15.

8 Teichmüller theory

8.1 Teichmüller space

Now, let's go back to Riemann surfaces. Our current goal- construction of Teichmüller space. Fix once and for all a Riemann surface S of finite type. The quasiconformal maps $f : S \to Y, g : S \to Z$ are equivalent if $f \circ g^{-1}$ is homotopic to a conformal map between Z and Y. The space of equivalence classes of the pairs (Y, g) is the *Teichmüller space* $T(S)$.

Several alternative definitions.

(i) Recall that each Riemannian metric on S defines a complex structure. Consider the space $R(S)$ of Riemannian metrics which have finite conformal type. The groups $Diff_0$ (of diffeomorphisms of S homotopic to identity) and $C^\infty(S, \mathbb{R}_+)$ act on $R(S)$ as $g \cdot ds = g^* ds$, and $C^\infty(S, \mathbb{R}_+) \ni \phi : ds \mapsto \phi ds$. Then

$$R(S)/Diff_0 \times C^\infty(S, \mathbb{R}_+) \cong T(S)$$

Namely, if $[X, g] \in T(S)$ then $g^{-1}(X)$ is a complex structure on S which corresponds conformally to a Riemannian metric of finite type on S. Conversely, if ds is a metric, let $f \in Diff_0$ be so that f^*ds near the punctures of S is conformally equivalent to the S. Then define the Beltrami differential μ as in the section 3. Then define the Beltrami differential μ as in the section 3. The supremum norm of μ is less than 1 and we can solve the Beltrami equation on S: $\bar{\partial}h = \mu Dh$, $h : S \to S$. then $(h(S, ds), h)$ projects to a point in $T(S)$.

If we consider the space $H(S)$ of complete metrics of constant curvature -1 then the quotient $H(S)/Diff_0 = F(S)$ is called "Fricke space" and it's diffeomorphic to $T(S)$.

(ii) Using the Uniformization identify $\pi_1(S)$ with a discrete subgroup

$$F \subset PSL(2, \mathbb{R})$$

Consider the space $Hom_a(F \to PSL(2, \mathbb{R}))$ consisting of *admissible* monomorphisms (which have discrete images, preserve the type of elements and "orientation"). Then take the quotient $T(F) = Hom_a(F \to PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$, where $\gamma \in PSL(2, \mathbb{R})$ acts on $r \in Hom_a(F \to PSL(2, \mathbb{R}))$ by conjugation $\gamma \cdot r(g) = \gamma r(g)\gamma^{-1}$, $g \in F$. This space is called the Teichmüller space of the group F.

The space $T(F)$ has a natural topology induced by the matrix topology of the group $PSL(2, \mathbb{R})$.

This space coincide with $T(F) = \{h : \Delta \to \Delta = \mathbb{H}^2 : h_*(g) \equiv hgh^{-1} \in PSL(2, \mathbb{R})$ for all $g \in F$, h is quasiconformal $\}/ \cong$, where $h_1 \cong h_2$ iff there exists an element $\gamma \in PSL(2, \mathbb{R})$ such that:

$$ext(h_1)|_{\partial\Delta} = \gamma \circ ext(h_2)|_{\partial\Delta}$$

We can avoid the composition with elements of $PSL(2, \mathbb{R})$ assuming that all quasiconformal maps $ext(h)$ fix 3 distinguished points p_j on $\partial\Delta$ (i.e. $ext(h)$ is normalized).

Teichmüller metric on $T(S)$. If $p, q \in T(S)$ then put

$$d_T(p, q) = inf\{\log K(f \circ g^{-1}) : f \in p, g \in q\} \tag{56}$$

The Convergence property for quasiconformal mappings implies that the Teichmüller distance is always achieved by some quasiconformal map. The triangle inequality is obvious since $K(f \circ g) \leq K(f)K(g)$; $d_T(p, q) = d_T(q, p)$ since $K(h) = K(h^{-1})$.

Several other definitions of $d_T(p, q)$. Let $\tau_1(p, q) = \log inf\{K(f \circ g_0^{-1}) : f \in [q])\}$; $\tau_2(p, q) = \log inf\{K(h) : h \in [f \circ g^{-1}]\}$. Then: $\tau_2 \leq d_T \leq \tau_1$; given h we can take $g = h^{-1}f$, thus $\tau_1 \leq \tau_2$.

Another point of view. Recall that for $z, w \in \Delta$ the hyperbolic distance

$$d(z, w) = \log \frac{1 + |z - w|/|1 - \bar{z}w|}{1 - |z - w|/|1 - \bar{z}w|} = \log \frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|} \tag{57}$$

where $ds = \frac{|dz|}{1-|z|^2} \cong \frac{|dz|}{Im(z)}$. Therefore (by (9)),

$$d_T(p, q) = inf \log \frac{1 + \|\mu - \nu\|/\|1 - \bar{\mu}\nu\|}{1 - \|\mu - \nu\|/\|1 - \bar{\mu}\nu\|} =$$

$$inf\{d(\mu(z), \nu(z)) : z \in \Delta, \mu \in [p], \nu \in [q]\}$$

Remark 8.1. *Let's compare two distances in the open unit ball $B(1)$ of complex characteristics. We have:*

$$\|\mu(f^{\mu_1} \circ f^{\mu 2})\| = \|\mu_1 - \mu_2\| / \|1 - \bar{\mu_1}\mu_2\| < 2\|\mu_1 - \mu_2\|$$

since $\|1 - \bar{\mu_1}\mu_2\| < 2$. On another hand, if $\|\mu_1\|$ or $\|\mu_2\| \le C < 1$ then

$$\|\mu(f^{\mu_1} \circ f^{\mu 2})\| \ge \|\mu_1 - \mu_2\| 2/(C+1).$$

Thus, for each $r < 1$ the metric on $B(r)$ defined by the supremum norm and the "Teichmüller metric" are equivalent.

Thus d_T is a metric. The space $T(S)$ is path-connected. If X, Y are quasiconformal equivalent then $T(X), T(Y)$ are canonically isomorphic.

8.2 The modular group

Define the *(Teichmüller) modular group Mod_S* as the quotient

$$Homeo_+(S)/Homeo_0(S)$$

where $Homeo_+(S)$ is the group of orientation preserving homeomorphisms of S and $Homeo_0(S)$ is the subgroup of homeomorphisms homotopic to id_S. Then Mod_S acts on $T(S)$ by the precomposition $Mod_S \ni g : [X, f] \to [X, f \circ g]$. The quotient $T(S)/Mod_S$ is called the *moduli space* of complex structures on S. The group Mod_S is isomorphic to

$$Out_+(S) = Aut_+(S)/Inn(S)$$

where $Aut_+(S)$ consists of "admissible" automorphisms "preserving the orientation".

The bijection $T(S) \to T(F)$ is continuous (with the Teichmüller topology on $T(S)$ and the "matrix" topology on $T(F)$) because of the continuous dependence of the solution of Beltrami equation on the dilatation. The fact that this an open map is more subtle. One way to prove it is to show that both are manifolds (as we shall see). Another way is to prove so called "quasiconformal stability" which is more general fact. The rough idea of the second approach is that small deformation of the representation leads to small deformation of the Dirichlet fundamental polygon and we can organize a quasiconformal diffeomorphism of the fundamental domains which is C^1 close to id and preserves the equivalence relation on the boundary of domains.

8.3 Teichmüller space of the torus

Let $\Gamma \subset Isom(\mathbb{C})$ be a torsion-free lattice, $\Gamma =< g_1, g_2 >\cong \mathbb{Z}^2$. Using conformal conjugations of representations $r : \Gamma \to Isom(\mathbb{C})$ we can assume that $g_1(z) = z$ and $g_2(z) = z + \tau$. Thus, $\tau \in \{t \in \mathbb{C} : Re(t) > 0\}$ is the only invariant of the conjugacy class of the admissible representation. Another way to describe this invariant is to embed $Hom_a(\Gamma, Isom(\mathbb{C}))$ in $\mathbb{C}^2$ so that $r \mapsto (t_1, t_2)$, where $r(g_j) : z \mapsto z + t_2$. Then, the conformal factorization of the space of representations is equivalent to the projectivization of $\mathbb{C}^2$. The point $(t_1 : t_2) \in CP(1)$ corresponds to $\tau = t_2/t_1$. Anyway, the Teichmüller space of the torus $T(T^2)$ is the upper-half plane $\mathbb{H}^2$. The modular group of the torus Mod_{T^2} acts on $T(T^2) \subset CP(1)$ as $SL(2,\mathbb{Z}) = Aut_+(S)$. This action isn't effective, so let $PSL(2,\mathbb{Z}) = Aut_+(S)/\{\pm I\}$.

Remark. The same happens with the surfaces of genus 2 which are hyperelliptic and the hyperelliptic involution is always induced by conformal map of the surface. But, anyway, the kernel is $\mathbb{Z}_2$. Now the action of $PSL(2,\mathbb{Z})$ on $\mathbb{H}^2$ is just the action of the "classical" modular group.

The next step is to calculate the Teichmüller distance, in particular to show the equivalence of the Teichmüller topology with the topology of $\mathbb{C}$. Suppose that r_1, r_2 are admissible representation with the normalization $r_j(g_1) = 1$; $r_1(g_2) : \tau \mapsto \tau + w$, $r_2(g_2) : \tau \mapsto \tau + z$, $\tau \in \mathbb{C}$.

Lemma 8.2. *There is an affine map of $\mathbb{C}$ which is an extremal quasiconformal map conjugating r_1 and r_2 (it's a particular case of the Teichmüller's analysis of the extremal maps which claims that moreover, the extremal map is unique).*

Proof: Let L be the lattice $\mathbb{Z} + w\mathbb{Z} \subset \mathbb{C}$, L' be the lattice $\mathbb{Z} + z\mathbb{Z} \subset \mathbb{C}$. denote by A the unique $\mathbb{R}$-linear map which transforms L to L', $A(1) = 1$, $A(w) = z$,

$$A(\tau) = \frac{z - \bar{w}}{w - \bar{w}} \cdot \tau + \frac{w - z}{w - \bar{w}} \cdot \bar{\tau} \tag{58}$$

$$\mu(A) = \frac{w - z}{z - \bar{w}} \tag{59}$$

Suppose that f is an extremal map. After normalization we can assume that $f(0) = 0$, $f(1) = 1$, thus $f(w) = z$. Therefore, restriction of f to L coincides with A. Denote by Φ the rectangle bounded by the segments $[0,1], [1, w+1], [w+1, w], [w, 0]$, which is a fundamental domain for Γ. For $k \in \mathbb{Z}_+$ put $f_k(z) = f(2^k z) \cdot 2^{-k}$. Then $K(f_k) = K(f)$ and they induce the same conjugation between r_1 and r_2. However, the restriction of f_k to $2^{-k} \cdot L$ coincide with A and $L_k = \Phi \cap 2^{-k} \cdot L \subset L_{k+1}$. Therefore,

$$L_\infty = \bigcup_{k \in \mathbb{Z}_+} L_k$$

is dense in Φ. On another hand, there is a uniform limit $f_\infty = \lim f_k$ and the restriction of f_∞ to L_∞ coincides with A. Thus, $f_\infty = A$ on $\mathbb{C}$, $K(f_\infty) = K(f)$ and thus A is an extremal map. $\square$

Thus we can calculate $K(A)$ as

$$K(A) = \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|} \tag{60}$$

and $d_T(r_1, r_2) = logK(A)$ is the hyperbolic distance between z and w. Really, on the imaginary axis it coincides with $\log w/z$ if $Im(w) \geq Im(z)$. The invariance of $K(A)$ under homotheties and translations in $PSL(2,\mathbb{R})$ is evident. The invariance under the inversion $j : \tau \mapsto 1/\bar{\tau}$ can be verified by a direct calculation.

Thus, in the case of the torus the Teichmüller and hyperbolic metrics coincide. This is a particular case of more general theorem of Royden that we shall (probably) discuss later.

Remark. Another simple case is the punctured torus $S_{1,1}$. In this case the Teichmüller space is the same as the Teichmüller space for the torus. Really, let

$[X, f] \in T(S_{1,1})$, f is quasiconformal. Then $\phi([X, f]) \in T(T^2)$ is just the extension of the complex structure and of the quasiconformal map to the puncture. This map is injective since the group of conformal automorphisms of the torus is transitive.

8.4 Simple example of the moduli space.

Let S be a sphere with $n + 3$ punctures. Then the moduli space of S is

$$(\mathbb{C} - \{0, 1\})^n - \text{diagonals}/S(n)$$

where $S(n)$ is the symmetric group on n symbols. The fact that the topology is the same as Teichmüller's just follows from the continuous dependence of solution of Beltrami equation on the dilatation and the fact that if $(z_1, ..., z_n) \in \mathbb{C}^n$ is close to $(w_1, ..., w_n) \in \mathbb{C}^n$ then there is a diffeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which is C^1 close to id such that $f(z_j) = w_j$.

8.5 Completeness of the Teichmüller space.

Theorem 8.3. *The space $T(S)$ is complete.*

Proof: Let $[X_n, f_n]$ be a Cauchy sequence in $T(S)$. $f_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a Cauchy sequence in $T(S)$. Without loss of generality we can assume that X_n are complex structures on one and the same smooth surface S. First fix f_i such that

$$inf\{\log K(f_{i+p} \circ f_i^{-1}), f_{i+p} \in [f_{i+p}]\} < 1/2$$

for all $p = 1, 2, ...$ Renumber the sequence, put $f_i = f_1$. Then choose f_2 such that $\log K_{f_2 \circ f_1} < 1/2$ and $d_T([X_i, f_i], [X_{i+p}, f_{i+p}] \leq 1/4$. Finally we get a sequence of maps $f_n : S \to X_n$ so that

$$\log K_{f_{n+1} \circ f_n^{-1}} < 2^{-n}$$

for each n. Then this Cauchy sequence in $T(S)$ has $\log K_{f_{n+p} \circ f_n} < 2^{-n+1}$ for each p. Thus,

$$\|\mu_{n+p} - \mu_n\|_\infty \leq 2\|(\mu_{n+p} - \mu_n)/(1 - \bar{\mu}_n \mu_{n+p})\| < 2 \exp \frac{2^{-n+1} - 1}{2^{-n+1} + 1} = 2 \tanh 2^{-n+1}$$

(See Remark 8.1) Thus μ_n is a Cauchy sequence in L_∞. Since L_∞ is complete, the limit μ exists. On another hand, $K(f_n) \leq K(f_1) 2^{-n+1} < C < \infty$. This estimate also implies that $\|\mu\| < 1$ and we can solve the Beltrami equation with μ. The solution-f^μ belongs to $T(S)$. The points μ_n are convergent to μ in the supremum norm, thus the points of the Teichmüller space are convergent in the Teichmüller topology. □

8.6 Real-analytic model of the Teichmüller space

We will also use another realization for $T(S)$.

Namely, let Γ be the Fuchsian group such that $\mathbb{H}^2/\Gamma = S$. Then consider the space $B(\Gamma)$ of all Beltrami differentials which are automorphic under Γ:

$$B(\Gamma) = \{\mu \in L_\infty(\mathbb{H}^2) : \|\mu\| < 1, \mu(z) = \mu(\gamma z)\overline{\gamma'(z)}/\gamma'(z), \gamma \in \Gamma\} \qquad (61)$$

Introduce the equivalence relation: $\mu \sim \nu$ iff f^μ and f^ν coincide on $\partial\mathbb{H}^2$ (q.c. maps are canonically normalized and the complex dilatation is extended to $\overline{\mathbb{C}} - \mathbb{H}^2$ by the inversion). Then f conjugate Fuchsian groups to Fuchsian groups. Denote by $\Psi : B(\Gamma) \to T(S)$ the corresponding projection.

8.7 Complex-analytic model of the Teichmüller space

We now extend the complex dilatation μ to $\overline{\mathbb{C}} - \mathbb{H}^2$ by 0. Denote by f^μ the normalized solution of the equation $\bar{\partial} f = \mu \partial f$ in $\overline{\mathbb{C}}$. Then (by formula (50)) we still have: $f^\mu_*(g) = f^\mu \circ g \circ (f^\mu)^{-1} \in PSL(2, \mathbb{C})$, $g \in \Gamma$. Thus, f^μ conjugates the group Γ to a Kleinian group $\Gamma' = f_*(\Gamma)$. The group Γ' has two simply connected components of the discontinuity domain- images of $\mathbb{H}^2$ and $\mathbb{H}^2_*$. Then the surface $f^\mu(\mathbb{H}^2_*)/\Gamma'$ is conformally equivalent to $\mathbb{H}^2_*/\Gamma'$. The surface $f^\mu(\mathbb{H}^2)/\Gamma'$ with the marking given by the isomorphism of fundamental groups $f^\mu_* : \Gamma \to \Gamma'$ gives the point of the Teichmüller space $T(S)$. This point is the same as the point of $T(\Gamma)$ given by the solution $f_\mu : \mathbb{H}^2 \to \mathbb{H}^2$ of the Beltrami equation $\bar{\partial} f = \mu \partial f$ in $\mathbb{H}^2$. Indeed, the map $g = f_\mu \circ (f^\mu)^{-1}$ of the domain $f^\mu(\mathbb{H}^2)$ is conformal (by uniqueness of solution of Beltrami equation) and g conjugates the representations f^μ_* and $f_{\mu *}$.

Thus we have the correspondence: $\Phi : [\mu] \in T(X) \mapsto f|_{\mathbb{H}^2_*}$ is a univalent (i.e. injective) holomorphic function in $\mathbb{H}^2_*$.

Theorem 8.4. *The map Φ is injective.*

Proof: If $f = g$ on $\mathbb{H}^2_*$ then their extensions to $\mathbb{R}$ also coincide. Thus, the induced homeomorphisms f_*, g_* of the group Γ to $PSL(2, \mathbb{C})$ are the same and $f(\mathbb{H}^2) = g(\mathbb{H}^2)$, therefore the surfaces $f(\mathbb{H}^2)/f_*(\Gamma)$, $g(\mathbb{H}^2)/g_*(\Gamma)$ are the same and their markings defined by f_*, g_* coincide. □

Thus, we obtained an embedding of $T(X)$ in the space of holomorphic functions in $\mathbb{H}^2_*$. Certainly, this map is very far from being surjective. Our aim is to describe somehow the image.

9 Schwarzian derivative and quadratic differentials

9.1 Spaces of quadratic differentials

Let $Q(\Gamma)$ be the space of all holomorphic in $\mathbb{H}^2_*$ functions ϕ such that ϕdz^2 is Γ-automorphic (i.e. $\phi(z) = f(\gamma z)\gamma'(z)^2$) and have finite norm:

$$\|\phi\| = \sup_{z \in \mathbb{H}^2_*} y^2 |\phi(z)| \tag{62}$$

where we realized $\mathbb{H}^2_*$ as the lower half plane. The norm $\|\phi\|$ is invariant under the precomposition with elements of Γ.

Notice that the condition (62) is equivalent to finiteness of the ds^2-norm of the projection of ϕ to $X = (S, ds^2) = \mathbb{H}^2_*/G$.

Later we shall show that the space $Q(\Gamma)$ is finite-dimensional. Its dimension is $3g - 3 + n$, where (g, n) is the type of X.

Schwarzian derivative of the holomorphic function f is

$$S_f \equiv \{f, z\} = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2 = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 \tag{63}$$

Suppose that $f \in PSL(2, \mathbb{C})$, then $S_f \equiv 0$. The inverse statement also holds for the holomorphic functions. Suppose that $S_f = \phi$. Put $h = \frac{f''}{f'}$, then $h' - h^2/2 = \phi$, if $h = -2\eta'/\eta = -2(\log \eta)'$ then for η we have the Riccati equation:

$$\eta'' + \phi\eta/2 = 0$$

If $\phi = 0$ then the only solution is: $\eta(z) = cz + d$. However, $h(z) = -2(\log \eta)' = \frac{-2c}{cz+d}$; $g := f'$ implies $(\log g)' = h$

$$\log g = \int h = -2 \int \frac{cdz}{cz + d} = \log((cz + d)^{-2}) + \log a$$

Thus, $f' = g = a/(cz + d)^2$,

$$f = a \int \frac{cdz}{(cz + d)^2} = \frac{\alpha\beta d + c\beta z}{cz + d} \in PSL(2, \mathbb{C})$$

Then $S_f = 0$ iff f is Moebius. Under the composition the Schwarzian derivative behaves as:

$$S_{f \circ g} = (S_f \circ g) g'^2 + S_g \tag{64}$$

If g is Moebius, then

$$S_{f \circ g} = (S_f \circ g) g'^2$$

in particular, if $F(z) = f(1/z)$ then $z^4 S_F(z) = S_f(1/z)$.

Thus, $S_{f \circ g} = S_f$ for all $g \in \Gamma$ iff S_f is Γ-automorphic quadratic differential.

Theorem 9.1. *The (holomorphic) solution of the equation*

$$S_f = \phi$$

exists and unique up to (left) composition with a Moebius transformation. This solution is locally injective.

Proof: Using the substitute $h = f''/f$, $h = -2\eta'/\eta$; as above we have the Riccati equation:

$$\eta'' + \phi\eta/2 = 0 \tag{65}$$

This equation has 2 linearly independent holomorphic solutions. (Proof: power series expansion.) Let η_1, η_2 be such solutions. Then $(\eta_1\eta_2' - \eta_2\eta_1')' = \eta_1\eta_2'' - \eta_2\eta_1'' = 0$. Thus, $\eta_1\eta_2' - \eta_2\eta_1' = \text{const} \neq 0$ (otherwise, $\partial(\log \eta_1) = \partial(\log \eta_2)$, $\eta_1 = a\eta_2$). Thus we can put const $= 1$. The function $f = \eta_1/\eta_2$ satisfies the equation $S_f = \phi$.

Notice that η_2 can have at most simple zeros (since if $\eta_2(z) = \eta_2'(z)$ then const $= 0$). Thus, f has at most simple poles. In other points:

$$f' = (\eta_1\eta_2' - \eta_2\eta_1')/\eta_2^2 = 1/\eta_2^2 \neq 0 \tag{66}$$

So, f is locally univalent.

Uniqueness of solution. It follows from (64) that

$$0 = S_{f\circ f^{-1}} \equiv (S_f \circ F^{-1})(f^{-1\prime})^2 + S_{f^{-1}}$$

and, since, $S_g = S_f$ we have:

$$0 = (S_f \circ F^{-1})(f^{-1\prime})^2 + S_{f^{-1}} = S_{g\circ f^{-1}}.$$

Thus, $g \circ f^{-1} \in PSL(2,\mathbb{C})$. □

Theorem 9.2. *Suppose that the domain of ϕ includes the point ∞ and ϕ has a simple pole at ∞. Then the solution of $S_f = \phi$ has at worst a simple pole at ∞.*

Proof: Suppose that $\phi(w) = \psi(w)w^{-4}$, let $F(z) = f(1/z = w)$. Then

$$z^4 S_F(z) = w^{-4}\psi(w) = z^4\psi(w) \tag{67}$$

and we have the equation $S_F(z) = \psi(1/z)$ where $\psi(1/z)$ is holomorphic near 0. Then we use Theorem 9.1 to conclude that the function F (and thus f) is meromorphic with at worst simple poles. □

Corollary 9.3. *Suppose that ϕ is Γ-automorphic. Then the solution of the equation $S_f = \phi$ defines a homomorphism $\rho : \Gamma \to PSL(2,\mathbb{C})$ such that $f \circ \gamma = \rho(\gamma) \circ f$ for each $\gamma \in \Gamma$.*

Proof: $S_{f\circ\gamma} = S_f$ and uniqueness implies that there is $\rho(\gamma)$ such that $f \circ \gamma = \rho(\gamma) \circ f$. □

Thus, any automorphic quadratic differential defines a complex projective structure on $\mathbb{H}^2/\Gamma$ which is "subordinate" to the initial complex structure. And vice versa, given a holomorphic developing map $d : \mathbb{H}^2 \to \overline{\mathbb{C}}$ which is Γ-equivariant, we have the automorphic quadratic differential S_d. Our problem is to determine, which quadratic differential correspond to the points of the Teichmüller space.

Theorem 9.4. *(Kraus- Nehari) Suppose that $f : \mathbb{H}^2_* \to \overline{\mathbb{C}}$ is a univalent function. Then*

$$|\{f, z\}Im(z)^2| \leq 3/2 \tag{68}$$

Proof: Step 1. We will need the following

Lemma 9.5. *Let $F(\zeta)$ be a univalent holomorphic function in $ext(\Delta) = \{|\zeta| > 1\}$ so that $F(\infty) = \infty$, $F(\zeta) = \zeta + b_1/\zeta + b_2/\zeta^2 + ...$ Then $|b_1| \leq 1$.*

Proof: Let $r > 1$; D_r be the complement to the image $F(\{|\zeta| \geq r\})$ then

$$Area(D_r) = \frac{1}{2i}\int_{|\zeta|=r} \bar{F} dF > 0$$

$$\frac{1}{2i}\int_{|\zeta|=r} \bar{F} dF = \frac{1}{2i}\int_{|\zeta|=r} [\bar{\zeta} + \bar{b}_2/\bar{\zeta}^2 + ...][1 - b_1/\zeta^2 - 2b_2/\zeta^3 - ...]d\zeta$$

However,

$$\int_{|\zeta|=r} \frac{d\zeta}{\zeta^s \bar{\zeta}^p} = 0$$

if $s \neq p+1$ and $= 2\pi i r^{1-s-p} = 2\pi i r^{-2p}$ if $s = p+1$. Thus, the hole integral is equal to:

$$\pi[r^2 - \sum_{p=1}^{\infty} \frac{p|b_p|^2}{r^{2p}}] > 0$$

If we put $r = 1$ then we will get in particular $|b_1| \leq 1$ □

Step 2. Now we can estimate $\{F, \zeta\}$. We have:

$$F' = 1 - b_1/\zeta^2 - ... \quad F'' = 2b_1/\zeta^3 + ...$$

$$F''' = -6b_1/\zeta^4 - ...$$

$$S_F = F'''/F' - 3(F''/F')^2/2 = (-6b_1/\zeta^4 - ...)(1 + b_1/\zeta^2 ...) -$$

$$\frac{3}{2}(2b_1/\zeta^3 + ...)^2(1 + b_1/\zeta^2 ...)^2 = -6b_1/\zeta^4 + O(1/\zeta^6)$$

Step 3. Finally, we consider the functions in the lower half-plane. Let $x_0 + iy_0 = z_0 \in \mathbb{H}^2_*$, then the transformation $\zeta = \gamma(z) = (z - \bar{z}_0)/(z - z_0)$ maps $\mathbb{H}^2_*$ to the exterior of Δ and $\gamma(z_0) = \infty$. Then put $f(\zeta) = f(z(\zeta))$; $f = F \circ \gamma$, thus

$$\{f, z\} = \{F, \zeta\}\gamma'(z)^2$$

$$\gamma'(z) = -\frac{2iy_0}{(z - z_0)^2}$$

and

$$\{f, z\} = (-6b_1 + \text{powers of } \ 1/\zeta)\zeta^{-4} \frac{-4y_0^2}{(z - z_0)^4}$$

$$\zeta^{-4} = \frac{(z - z_0)^4}{(z - \bar{z}_0)^4}$$

Therefore if $z = z_0, \zeta = \infty$ we are left with

$$|\{f, z_0\}| = |6b_1 \frac{1}{4y_0^2}| \leq \frac{3}{2y_0^2}$$

Theorem is proved. □

Theorem 9.6. *Suppose that $\|\phi\| \leq 1/2$, then the solution of the equation $S_f = \phi$ admits a quasiconformal extension to $\overline{\mathbb{C}}$ which is ρ-equivariant where ρ is the monodromy of the complex structure as above. The complex characteristic of this extension depends continuously on the norm of ϕ.*

Proof: We will not only establish the existence of the extension but also construct the *canonical* one. Let η_j be 2 linearly independent solutions of the Riccati equation $\eta'' = -\phi\eta/2$. Put

$$\hat{f}(z) = \frac{\eta_1(\bar{z}) + (z - \bar{z})\eta_1'(\bar{z})}{\eta_2(\bar{z}) + (z - \bar{z})\eta_2'(\bar{z})} = Q_1(z)/Q_2(z)$$

for $z \in \mathbb{H}^2$ and $\hat{f}(z) = \eta_1(z)/\eta_2(z) = f(z)$ for $z \in \mathbb{H}^2_*$. Then the function $\hat{f}(z)$ is smooth in its domain and $\bar{\partial}\hat{f}/\partial\hat{f}(z) = -2y^2\phi(\bar{z}) = \mu$ in $\mathbb{H}^2$ and $\bar{\partial}\hat{f} \equiv 0$ in $\mathbb{H}^2_*$ since in $\mathbb{H}^2$ we have: $\partial\hat{f} = 1/Q_2^2$, $\bar{\partial}\hat{f} = (z-\bar{z})^2\phi(\bar{z})/(2Q_2^2) = 2Im(z)^2\phi(\bar{z})/Q_2^2$, $Jac(\hat{f}) = |\partial\hat{f}|^2(1-|\mu|^2) \neq 0$ and $\|\mu\| < 1$. Therefore, $\hat{f}$ is a locally quasiconformal map holomorphic in $\mathbb{H}^2_*$. The complex dilatation μ of $\hat{f}$ is invariant under Γ: recall that $|Im(\gamma z)| = |\gamma'(z)Im(z)|$, thus

$$\mu(\gamma z) = 2|\gamma'(z)|^2|Im(z)|^2\phi(\gamma\bar{z}) = 2\gamma'(z)\overline{\gamma'(z)}\phi(\bar{z})\gamma'(\bar{z})^{-2} = \mu(z)\gamma'(z)/\overline{\gamma'(z)}$$

and $\mu(\gamma z)\overline{\gamma'(z)}/\gamma'(z) = \mu(z)$. So, our aim is to show that $\hat{f}$ is the restriction of a global quasiconformal homeomorphism of $\overline{\mathbb{C}}$. This will be done by some approximation.

Let g_n be a sequence of Moebius transformations such that $g_n \to id$ and $g_n(\mathbb{H}^2_*)$ is a relatively compact subset of $\mathbb{H}^2_*$. Form the functions: $\phi_n(z) = \phi(g_n z)(g_n'(z))^2$.

Then $|\phi_n(z)| = O(z^4)$ as $z \to \infty$ and ϕ_n is holomorphic near $\mathbb{R}$. Really,

$$|\phi(g_n z)(g_n'(z))^2| \leq Const|g_n'(z))^2|$$

in $\mathbb{H}^2_*$ since $g_n z$ belongs to a compact in $\mathbb{H}^2_*$. Then:

$$|g_n'(z)|^2 = |c_n z + d_n|^{-4} = O(z^{-4})$$

Lemma 9.7. *(Contraction property for holomorphic maps) Let $\mathbb{H}^2$ be a hyperbolic plane with the Poincaré metric $\rho(z)|dz|$, $\gamma : \mathbb{H}^2 \to \mathbb{H}^2$ is conformal. Then $d\gamma_z$ is "contracting" map of the hyperbolic metrics: $\|\xi\| > \|d\gamma(\xi)\|$ for each tangent vector at $z \in \mathbb{H}^2$ i.e. for $w = \gamma(z)$ we have: $\rho(w)|dw| \leq \rho(z)|dz|$, i.e. $\rho(\gamma z)|\gamma'(z)| < \rho(z)$.*

Proof: Assume that $\mathbb{H}^2$ is the unit disc with $\rho(z) = (1-|z|^2)^{-1}$, then let $h, g \in Isom_+(\mathbb{H}^2)$ be such that $g\gamma h(0) = 0$, $h(0) = z$. Since g, h preserve the metric its enough to prove Lemma for the point $z = 0$ and the conformal map $f = g\gamma h$. However, $f(\Delta) \subset \Delta$ and $f(0) = 0$, thus by Schwartz Lemma: $|f'(0)| < 1$; $\rho(0) = 1$, thus $\rho(0)|f'(0)| < \rho(0)$. □

The Lemma implies that $|g_n'(z)| < |Im(g_n(z)|/|Im(z)|$. Then

$$|y^2\phi_n(z)| = |y^2||\phi(g_n z)| \cdot |g_n'(z)|^2 < |\phi(g_n z)| \cdot |Im(g_n(z)|^2 \leq \|\phi\|$$

Thus, the hyperbolic norm of ϕ_n is bounded by $\|\phi\| < 1/2$.

Therefore, for the quadratic differentials ϕ_n we can construct the functions $\hat{f}_n$ which are locally injective and locally quasiconformal in $\mathbb{H}^2_* \cup \mathbb{H}^2$ and continuously extend to $\bar{\mathbb{R}}$ so that the boundary values coincide.

Moreover, $\hat{f}_n$ are holomorphic near $\bar{\mathbb{R}}$ and have at worst a simple pole at ∞; thus $\hat{f}_n$ is are quasiconformal homeomorphisms.

The complex dilatations μ_n of $\hat{f}_n$ have supremum norms bounded by $\|\phi\|$. On another hand, ϕ_n are convergent to ϕ uniformly on compacts, thus the normalized solutions of the Riccati equations

$$\eta'' = -\phi_n\eta/2$$

are uniformly convergent η_1, η_2. Now for each μ_n form the normalized solutions of the Beltrami equation. Then they are convergent uniformly to $\hat{f}$.

So, ϕ belongs to the image of the Teichmüller space. The bicontinuity of the correspondence $[\mu] \to S_f^\mu$ follows from the estimate on the norm of complex dilatation: if $\|\varphi - \psi\| \le \epsilon$, then $\|\mu - \nu\| \le \epsilon$, where $\mu(z) = 2Im(z)^2\varphi(\bar{z})$.

□

The correspondence

$$T(S) \ni [\mu] \to S_{f^\mu}$$

is called the "Bers embedding".

Theorem 9.8. *(1) Consider the projection $p : \mathbb{H}^2_* \to X = \mathbb{H}^2_*/\Gamma$. Then the image of $Q(\Gamma)$ under p_* is the space $Q(X)$ of holomorphic quadratic differentials on X with at worst simple poles at the punctures of X.*

(2) The Bers map is continuous.

Proof: The elements of $Q(\Gamma)$ project to quadratic differentials on X since they are Γ-invariant. Suppose that ∞ is a parabolic fixed point of Γ stabilized by the group $A =< z \mapsto z + 2\pi >$. Then as the conformal parameter near the puncture on X corresponding to the point ∞ we can choose $w = \exp(iz)$. Denote by D small neighborhood of 0 in $\overline{\mathbb{C}}$ which is in the image of local parameter near the puncture. Let $\phi(z)dz^2$ be Γ-invariant, then ϕ is invariant under A and on D we have: $p_*(\phi(z)dz^2) = \Phi(w)dw^2 = -\Phi(w = \exp(iz))w^2$. Suppose that $\Phi(w) = w^n\Psi(w)$ where $\Psi(w)$ is holomorphic and $\Psi(0) \neq 0$. Then $\phi(z) = -\Psi(w)w^{n+2}$; $Im(z) = \log(|w|)$ and $|\log(|w|)\Psi(w)w^{n+2}|$ is bounded as $w \to 0$ iff $n + 2 \ge 1$, i.e. $n \ge -1$, which means that $\Psi(w)$ has at worst a simple pole at zero. A priori there is also case when $\Phi(w)$ has essential singularity ar zero. However, in such case $w^k\Phi(w)$ would be unbounded in D for every k, thus $|\log(|w|)\Phi(w)$ is unbounded as well. This finishes the proof of (1).

To prove (2) its enough to show that for each $\phi_n, \phi_0 \in Q(\Gamma)$ if $\phi_n \to \phi_0$ uniformly on compacts in $2h_*$, then $\|\phi_n - \phi_0\| \to 0$ as $n \to \infty$. Let $\phi_k(z) = \Phi_k(w)w^2$, for $w = \exp(iz)$, where $\Phi_k(w) = w^{-1}\Psi_k(w)$, Ψ_k are holomorphic in D and $\Psi_k \to \Psi_0$ uniformly on compact, thus by the Maximum Principle, $\Psi_k \to \Psi_0$ uniformly in D. Now,

$$|y^2\phi_n(z) - y^2\phi_0(z)| = \log^2(|w|)|w|^2|w|^{-1}|\Psi_n(w) - \Psi_0(w)| \le$$
$$\le |\Psi_n(w) - \Psi_0(w)|$$

This implies that $\|\phi_n - \phi_0\| \to 0$ as $n \to \infty$. □

Remark. The property that X has finite type is essential in the proof. Each injective holomorphic function f in Δ can be uniformly on compacts approximated by functions f_n with quasiconformal extension to $\overline{\mathbb{C}}$ (trick: take $f_n(z) = f(z\cdot(1-1/n))$). Thus, S_{f_n} are convergent to S_f uniformly on compacts in Δ. However, Thurston constructed examples of functions f so that there is no any sequence of injective holomorphic functions f_k with the property:

$$\lim_{k\to\infty} \|S_{f_k} - S_f\| = 0$$

Theorem 9.9. *The dimension of $Q(X)$ is equal to $3g - 3 + n$ where g is the genus of X and n is the number of punctures.*

Proof: Denote by $\bar{X}$ to conformal compactification of X, let P be the divisor given by the set of punctures. Let k be the canonical divisor of X, $L(k^{-2} \cdot P)$ = set of holomorphic functions on X which have divisors at $k^{-2} \cdot P$ at least of order $k^{-2} \cdot P$. Then, by Riemann-Roch theorem, $r(L(k^{-2} \cdot P)) = deg(k^2/P) - g + 1 + r(k^2/(Pk) = kP^{-1})$. However, $deg(kP^{-1}) > 0$, thus $r(kP^{-1}) = 0$. On another hand, $deg(k^2/P) = deg(k^2) - deg(P) = 2(2g-2) - (-n) = 4g - 4 + n$; thus $r(L(k^{-2} \cdot P)) = 3g - 3 + n$. The dimension $r(L(k^{-2} \cdot P))$ of $L(k^{-2} \cdot P)$ is equal to the dimension of $Q(X)$ since $f \in L(k^{-2} \cdot P)$ iff $f\omega^2 \in Q(X)$, where $\omega \in \Omega(\bar{X})$ is the canonical class. $\square$

Theorem 9.10. *Teichmüller space is a manifold of the dimension* $3g - 3 + n$.

Proof: We already proved that the Bers map is a homeomorphism on some neighborhood $U(X)$ of $[X, id]$ in $T(X)$. Thus, $U(X)$ is a manifold of the dimension $3g - 3 + n$. Now, let $[Y, f]$ be any other point of $T(X)$, then we consider the homeomorphism α between $T(X)$ and $T(Y)$ given by : $[Z, h] \in T(Y)$ maps to $[Z, h \circ f]$; thus $\alpha[Y, id] = [Y, f]$. However, some neighborhood V of the point $[Y, id]$ in $T(Y)$ is also a manifold; thus the neighborhood αV of $[Y, f]$ is again a manifold. Therefore, $T(X)$ is a manifold. $\square$

Corollary 9.11. *The Bers' map is a homeomorphism on its image.*

10 Poincaré theta series

Let $A(\mathbb{H}^2)$ be the space of all holomorphic functions f in $\mathbb{H}^2$ which is realized as the unit disc in $\overline{\mathbb{C}}$. If Γ is a discrete torsion- free lattice in $PSL(2, \mathbb{R})$ then $A(\Gamma)$ is the space of all holomorphic functions φ in $\mathbb{H}^2$ such that :

(i) $\varphi(\gamma z)\gamma'(z)^2 = \varphi(z)$ for all $\gamma \in \Gamma$;

(ii) $\|\varphi\|_1 = \int_D |\varphi(z)| dxdy < \infty$ where D is a fundamental domain for the action of Γ in $\mathbb{H}^2$.

Such quadratic differentials are called "cusp forms" and their projections on $X = \mathbb{H}^2/\Gamma$ are L^1-integrable holomorphic quadratic differentials with at worst simple poles at the punctures. Thus, $A(\Gamma) = Q(\Gamma)$ as linear spaces, but they are different as the normed spaces.

Now, define the operator $\Theta : A(\mathbb{H}^2) \to A(\Gamma)$ by the formula:

$$\Theta(f)(z) = \sum_{\gamma \in \Gamma} f(\gamma(z))\gamma'(z)^2$$

Theorem 10.1. *(1) The series $\Theta(f)$ is convergent absolutely and uniformly on compacts in $\mathbb{H}^2$;*

(2) $\|\Theta\| \leq 1$;

(3) The operator Θ is surjective.

Proof: First we recall the "mean value" theorem for holomorphic functions:

$$|\varphi(w_0) = \frac{1}{2\pi r^2} | \int_{D(w_0, r)} \varphi| \leq \frac{1}{2\pi r^2} \int_{D(w_0, r)} |\varphi|$$

for each holomorphic function in the disc $D(w_0, r)$ with center at w_0 and radius r.

This theorem can be proved for example via Taylor expansion for φ with center at w_0.

Now, we can prove the assertion (1). Let $z_0 \in \mathbb{H}^2$ and $D(z_0, 2r)$ be the Euclidean disc which is contained in $\mathbb{H}^2$. Then there is a fundamental domain D for the group Γ such that $D(z_0, 2r) \subset cl(D)$. Thus, for each $z \in D(z_0, r)$ we have:

$$2\pi r^2 \sum_{\gamma\in\Gamma} |f(\gamma)| \cdot |\gamma'(z)^2| \leq \sum_{\gamma\in\Gamma} \int_{D(z,r)} |f \circ \gamma| \cdot |\gamma'^2| \leq$$

$$\leq \sum_{\gamma\in\Gamma} \int_D f \circ \gamma| \cdot |\gamma'^2| = sum_{\gamma\in\Gamma} \int_{\gamma D} |f| = \|f\|_1 < \infty$$

Now we can prove (2). Again,

$$\|\Theta f\|_1 = \int_D |\Theta f| \leq \sum_{\gamma\in\Gamma} \int_D |f \circ \gamma||\gamma'|^2$$

$$= \sum_{\gamma\in\Gamma} \int_{\gamma D} |f| = \|f\|_1$$

Remark. Curt McMullen in [Mc] proved the old conjecture due to I. Kra that the norm of the Theta operator is always strictly less than 1.

We skip completely the proof of the most interesting statement (3) since it will lead us too far from the main subject (to the theory of Poisson kernel). You can find the proof for example in [G]. □

11 Infinitesimal theory of the Bers map.

Consider the Beltrami differential μ with the support in the unit disc $\mathbb{H}^2$; denote by $f = f^{t\mu}$ the normal solution of the Beltrami equation:

$$\bar{\partial} f^{t\mu} = t\mu \partial f^{t\mu}$$

where t is sufficiently small. Then we recall that $f_z = h + 1$ where $h = Tt\mu + Tt\mu(Tt\mu) + ...$:

$$f(z) = z + \sum_{n=1}^{\infty} a_n(z) t^n$$

where

$$a_1(z) = P\mu(z) = -\frac{1}{\pi} \int_{\mathbb{H}^2} \frac{z\mu(\zeta)}{\zeta(\zeta - z)} d\xi d\eta$$

Therefore,

$$f'(z) = 1 + \sum_{n=1}^{\infty} a'_n(z) t^n$$

$$f''(z) = \sum_{n=1}^{\infty} a''_n(z) t^n$$

$$f'''(z) = \sum_{n=1}^{\infty} a_n'''(z) t^n$$

Now, our aim is to calculate the Schwarzian derivative of the function f in the complement to the hyperbolic plane. First,

$$\lim_{t \to 0} \frac{f'''}{tf'} = \lim_{t \to 0} \frac{a_1'''(z) + O(t)}{1 + O(t)} = a_1'''(z)$$

Then,

$$\frac{1}{t}(\frac{f''}{f'})^2 = \frac{1}{t}(\frac{a_1''(t)t + ...}{1 + ...})^2 = \frac{t^2}{t}\frac{(a_1''(t) + ...)^2}{(1 + ...)^2} = O(t)$$

Thus,

$$\lim_{t \to 0} \frac{1}{t} S(f^{t\mu}) = a_1'''(z) = \frac{d^3}{dz^3}(-\frac{1}{\pi}\int_{\mathbb{H}^2} \frac{\mu(\zeta)}{\zeta - z} d\xi d\eta =$$

$$= -\frac{6}{\pi}\int_{\mathbb{H}^2} \frac{\mu(\zeta)}{(\zeta - z)^4} d\xi d\eta$$

This is the formula for the derivative of the Bers' map in the direction μ:

$$\dot{\Phi}(0)[\mu](z) = -\frac{6}{\pi}\int_{\mathbb{H}^2} \frac{\mu(\zeta)}{(\zeta - z)^4} d\xi d\eta = \sum_{n=0}^{\infty} c_n z^{-(n+4)} \int_{\mathbb{H}^2} \mu(z)\zeta^n d\xi d\eta$$

where $c_n \neq 0$ for all n.

Theorem 11.1. *Let $\Gamma \subset PSL(2, \mathbb{R})$ be a torsion-free lattice with the fundamental domain D in $\mathbb{H}^2 = \Delta$. Then Beltrami differential μ belongs to the kernel of $\dot{\Phi}(0)$ iff $\int_D \mu\varphi = 0$ for all $\varphi \in A(\Gamma)$. In other words, the variation of the complex structure on $X = \mathbb{H}^2/\Gamma$ is infinitesimally trivial along $\mu \in L_\infty(\mathbb{H}^2, \Gamma)$ if and only if μ belongs to the orthogonal complement $A(\Gamma)^\perp$ of $A(\Gamma)$ in $L_\infty(\mathbb{H}^2, \Gamma)$.*

Proof: Let $\theta_n = \Theta(z^n)$; then $\int_\Delta \mu(\zeta)\zeta^n d\xi d\eta = \int_D \mu(\zeta)\theta_n(\zeta)$. Thus, if $\mu \in A(\Gamma)^\perp$ then $\int_D \mu(\zeta)\theta_n(\zeta) = 0$ and $\int_\Delta \mu(\zeta)\zeta^n d\xi d\eta = 0$ for all n, therefore, $\dot{\Phi}(0)[\mu] = 0$.

Conversely, if $\int_\Delta \mu(\zeta)\zeta^n d\xi d\eta = 0$ for all n then we can use the Carlemann's density theorem:

polynomial functions are dense (in L^1 norm) in the space of L^1 holomorphic functions in the unit disc.

Therefore, μ is orthogonal to all holomorphic functions f in Δ and

$$0 = \int_\Delta \mu f = \int_D \mu\Theta(f)$$

However, the operator Θ is surjective, thus μ is orthogonal to each quadratic differential: $\mu \in A(\Gamma)^\perp$. $\square$

Remark 11.2. *This theorem is one of fundamental facts of the Teichmüller theory.*

12 Teichmüller theory from the Kodaira-Spencer point of view

Definition (Kodaira- Spencer): A holomorphic family is a complex manifold $V = V^{m+1}$ and a holomorphic map $\pi : V \to M = M^m$ where M^m is a complex manifold and all preimages $\pi^{-1}(t)$ are Riemann surfaces ($t \in M$).

In our case, $M = \Phi(T(X)) \subset Q(X)$ and $m = 3g - 3 + n$. Each point $t \in M$ corresponds to a quadratic differential $\phi_t \in Q(X)$ and to a group $\Gamma_t \subset PSL(2, \mathbb{C})$; for each $A \in \Gamma$, A_t depends holomorhically on t. Riemann surfaces of our family will be: $S(t) = \Omega_t \Gamma_t$;

$$V = \cup_{t\in M} S(t)$$

where Ω_t is a component of $\Omega(\Gamma_t)$ which is the image of the upper half plane under quasiconformal map (thus, the variation of the complex structure on $S(t)$ isn't trivial). Points of the space V are the orbits $\Gamma_t z$ where $z \in \Omega_t, t \in M$. The projection is the obvious map $\pi : V \to M$. Now we need topology and a complex structure for the space V. Consider a point $\Gamma_{t_0} z_0 \in V$. Then there is a neighborhood $N = N(z_0)$ of the point z_0 such that $clN \subset \Omega_{t_0}$ and $\gamma_{t_0} clN \cap clN = \emptyset$ for all nontrivial γ in Γ.

Now, the neighborhood $N(\epsilon, z_0, t_0)$ of $\Gamma_{t_0} z_0 \in V$ consists of all $\Gamma_t z$ such that:

$$\|\phi_t - \phi_{t_0}\| < \epsilon \ , \ z \in N$$

Here ϵ is so small number that clN doesn't meet it's $\Gamma_t - \{1\}$ - orbit (such ϵ exists since $f_t \to f_{t_0}$ uniformly on compacts. Define the map h on $N(\epsilon, z_0, t_0)$ as: $h : \Gamma_t z \mapsto (z, t)$ where $z = \Gamma_t z \cap N$.

The neighborhood s $U = N(\epsilon, z_0, t_0)$ define the base of topology on V and the maps h are coordinate maps for the complex structure on V. The transition maps are holomorphic since A_t are holomorphic functions on t: on U_1, U_2 we have:

$$h_1(\Gamma_t z) = (z, t); h_2(\Gamma_t z) = (\gamma_t z, t)$$

The projection map π locally is given by: $(w, t) \mapsto t$, so this is a submersion. Therefore, V is a holomorphic family.

It's much easier to visualize this construction for the case of the Teichmüller space of the torus. Namely, let $\mathbb{H}^2 = T(T^2)$, $\tilde{V} = \mathbb{H}^2 \times \mathbb{C}$. For each $t \in \mathbb{H}^2$ the lattice $\mathbb{Z}^2$ acts on $\mathbb{C}$ as a lattice Γ_t. Now, V is the quotient of $\tilde{V}$ by this action of the group Z^2. As the result we have the fibre bundle over $\mathbb{H}^2$ with the fiber T^2 (which have variable complex structure). Now we can even consider the quotient of V by the modular group $PSL(2, \mathbb{Z})$. The resulting variety U is fibered over the modular curve $\mathbb{H}^2/PSL(2, \mathbb{Z})$; U is called the universal elliptic curve.

With some success we can repeat the same in the case of hyperbolic surfaces; however instead of one and the same space $\mathbb{C}$ we have to consider Ω_t as the fiber of $\tilde{V}$; Ω is a domain in $\overline{\mathbb{C}}$; the base of $\tilde{V}$ is the Teichmüller space; V is the quotient of $\tilde{V}$ by the action of $\pi_1(X)$ which acts as Γ_t in each fiber. Again, we can take the next quotient $V/Mod(X)$ to obtain the universal Teichmüller curve which has the moduli space as the base and the surface X (with variable complex structure) as the fiber.

Actually, the relation between Teichmüller and Kodaira-Spencer theory is much deeper.....

13 Geometry and dynamics of quadratic differentials

13.1 Natural parameters

Let X be a compact Riemann surface , and ϕ be a holomorphic quadratic differentialon X which is different from zero. Throughout this section ϕ is assumed to be fixed. A point $p \in X$ is said to be **regular** with respect to ϕ if $\phi(p) \neq 0$ and **critical** if $\phi(p) = 0$. It's easy to see that these definitions do not depend on the choice of local coordinates on X. Critical point of ϕ form a finite set $C(\phi)$. Let p be any regular point and $q \mapsto h(q) = z$ is a local coordinate near p such that $h(p) = 0$. Since $\phi(p) \neq 0$ then there is a small neighborhood of 0 where who branches of $\sqrt{\phi(z)}$ are single valued. For a fixed branch of square root every integral

$$z \mapsto \Phi(z) = \int_0^z \sqrt{\phi(w)}dw \tag{44}$$

is also a single-valued function in some simply-connected neighborhood of 0 and uniquely determined up to an additive constant.

On another hand, $\Phi'(0) = \sqrt{\phi(0)} \neq 0$ and thus, Φ is locally injective near 0. It follows that the system of maps $z \mapsto w = \Phi(z = h(q))$ is a holomorphic atlas on $X - C(\phi)$. In these local coordinates $\phi(z)dz^2 = dw^2$. The coordinate Φ is called a **natural parameter** near p. An arbitrary natural parameter near p has the form $\pm z + const$. This means that the natural parameters define a very special kind if Euclidean structure on $X - C(\phi)$ (which is called F-structure, where F stands for the "foliation").

There are natural parameters at the critical points as well. Suppose that $p \in X$ is a zero of order n for ϕ. Again, let $q \mapsto h(q) = z$ be a local parameter near p. Then there is a disc $D = D(0, r)$ where $\phi(z) = z^n\psi(z)$ with $\psi(z) \neq 0$. We fix a single-valued branch of $\sqrt{\psi}$ in D. If n is odd then we cut D along $\mathbb{R}_+$ and fix a branch of $z \mapsto z^{n/2}$ in $D' = D - \mathbb{R}_+$; if n is even we don't need any cut. In any case,

$$z \mapsto \Phi(z) = \int_0^z \sqrt{\phi(w)}dw = z^{(n+2)/2}(c_0 + c_1 z + ...) = z^{(n+2)/2}\omega(z) \tag{45}$$

(where $c_0 \neq 0$) is a single-valued function in D'. Moreover, the function

$$z \mapsto \omega(z) = \Phi(z)z^{-(n+2)/2} \tag{46}$$

is single-valued at some neighborhood of 0. This,

$$\zeta : z \mapsto \Phi(z)^{2/(n+2)} = z\omega(z)^{2/(n+2)} \tag{4}$$

is single-valued near 0 since $\omega(z) \neq 0$; and has nonzero derivative at 0:

$$\zeta'(0) = \omega(0)$$

We call $\zeta : q \mapsto \Phi(z)^{2/(n+2)}$ to be the *natural parameter* at p. In terms of this natural parameter we have:

$$\phi dz^2 = (\frac{n+2}{2})^2\zeta^n d\zeta^2 \tag{5}$$

since:

$$d\zeta = \frac{2}{n+2}\Phi^{\frac{2}{n+2}-1}(z\Phi'(z)dz = \frac{2}{n+2}\Phi^{\frac{-n}{n+2}}\sqrt{\phi(z)}dz$$

$$\phi dz^2 = (\frac{n+2}{2})^2\Phi^{\frac{2n}{n+2}}d\zeta^2 = (\frac{n+2}{2})^2\zeta^n d\zeta^2$$

Define the differential $|\phi(z)|^{1/2}|dz|$. This is a Riemannian metric outside the critical set of ϕ. This metric is locally Euclidean on $X - C(\phi)$ and is called ϕ-metric. The natural parameter is the local isometry between this metric and the Euclidean metric on $\mathbb{C}$. The surface X has a finite diameter and area with respect to this singular metric. We shall return to this metric later.

13.2 Local structure of trajectories of quadratic differentials

Horizontal (or vertical) trajectories of ϕ correspond to the (maximal) horizontal (or vertical) Euclidean line in $\overline{\mathbb{C}}$ under the natural parameter (on $X - C(\phi)$). Another way to define these trajectories is as follows. Let $\gamma : [-1, 1] \to X - C(\phi)$ be a smooth path, take the pull-back

$$\phi(\gamma(t))(\gamma'(t))^2 dt^2 \tag{6}$$

of the form dz^2 on $[-1, 1]$. Then the curve γ is called a straight line if the argument $arg(\phi(\gamma(t))(\gamma'(t))^2) = \theta$ is constant. The trajectory is called horizontal if $\theta = 0$ and vertical if $\theta = \pi$.

Near singular points the trajectories are more complicated. Let $\zeta = w^{2/(n+2)}$ be the natural parameter near a critical point p; then w is a natural parameter outside of p.

If p is zero of order n for ϕ then there are $n+2$ horizontal rays emanating from p; in the natural parametrization the angles between them are $2\pi/(n+2)$. See Figure 5.

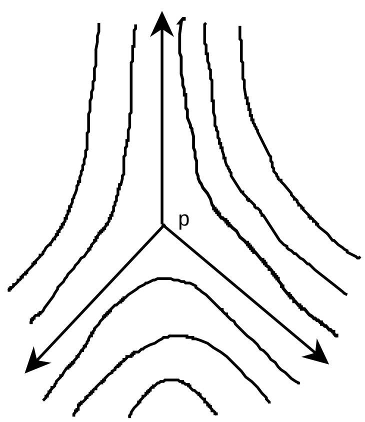


Figure 5. The differential has a simple pole at the point p.

Trajectory is called *critical* if it contains one of critical points. From now on we shall consider only horizontal trajectories of ϕ.

Examples on the torus. Suppose that X is a torus obtained by identification of sides of a parallelogram $P \subset \mathbb{C}$; denote by ϕ the projection on X of the quadratic differential dz^2. In this case $C(\phi) = \emptyset$. Then the natural parameter on X is the inverse to the universal covering and restriction of it to P is the identity map. The horizontal trajectories of ϕ are projections on X of the horizontal lines in $\mathbb{C}$. Suppose that P is a rectangle. Then all horizontal trajectories of ϕ are closed parallel geodesics on X. However, in the generic case, the trajectories of ϕ are irrational lines which are dense on X.

Now we consider the case of surface of general type. To simplify the discussion we shall assume that X is compact. First notice that $C(\phi) \neq \emptyset$ since $\chi(X) \neq 0$. We have the following classes of trajectories:

(a) Periodic trajectories. Let γ be a periodic trajectory of a quadratic differential. We shall see that there is a maximal open annulus A on X which contains γ and which is foliated by closed trajectories of ϕ and which has no critical points.

(b) Critical trajectories. These are trajectories γ such that at least one ray of γ end in a critical point of ϕ. There is only a finite number of such trajectories.

(c) Nonperiodic noncritical (**spiral**) trajectories γ. We shall see that they are recurrent in positive and negative direction. This means that γ is contained in the limit set for both rays γ_+ and γ_-.

Now, let's discuss the trajectories in more details.

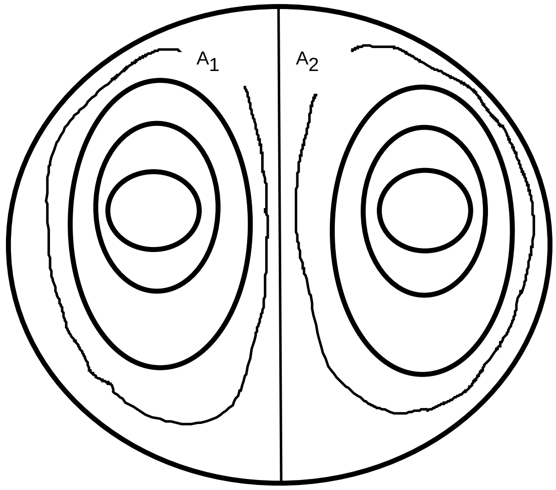
13.3 Dynamics of trajectories of quadratic differential

Suppose that γ is a (horizontal) trajectory of ϕ. Let $p \in \gamma$, $\Phi(p) = 0$, $\Phi(\gamma) \subset \mathbb{R} \subset \mathbb{C}$. Let $I = [a, b]$ be the maximal open interval on $\mathbb{R}$ which contains 0 such that the inverse to Φ is defined there. Denote the inverse by f.

(a) First suppose that I is bounded and $\Phi^{-1}(a) = \Phi^{-1}(b)$. This implies that γ is a periodic trajectory of ϕ. In this case we can consider the maximal horizontal strip $]a, b[\times]x, y[\subset \mathbb{C}$ where f is defined and injective. The image of $\mathbb{R}\times]x, y[$ is an annulus A in X foliated by trajectories of ϕ. If there are points on $I \times \{x\}$ and $I \times \{y\}$ which have the same image under f then X is a torus. So we can assume that f is an injective holomorphic function on $I \times [x, y]$. However, we can't extend f through $I \times \{x\}$ and $I \times \{y\}$ which implies that both sides of A contain critical trajectories. Example of this type of behavior on a pair of pants is shown on the Figure 6. The maximal annulus A has the geometric invariant- height (i.e. $|x - y|$). There is a class of differentials which have only periodic and critical trajectories (of finite length). These differentials are called **Strebel differentials**. Moreover, given a maximal collection of pairwise disjoint nonhomotopic loops γ_j on X and a collection of positive numbers h_j there is a unique Strebel quadratic differential ϕ on X such that the maximal annuli A_j of ϕ are homotopic to γ_j and have the prescribed heights h_j (see [G]). Strebel differentials are dense in $Q(X)$.

(b) Consider the case when $I \neq \mathbb{R}$ and the points on the boundary of I have different image under f. Then γ is a critical trajectory and if z is a boundary point of I then $w = f(z)$ is a critical point of ϕ. If z is the right end of I then the positive ray of γ has unique limit point w.

(c) The most interesting case is when (say) the positive ray γ_+ of γ has more than one limit point and γ isn't periodic. Then γ_+ has necessarily infinite (Euclidean) length. Really, if $q \in L(\gamma_+)$ then γ_+ intersects any neighborhood of q infinitely many



Trajectories foliate the annuli A_1 and A_2 on the pair of pants.

There are exactly two critical points and 3 critical trajectories.

Figure 4:

times , thus the length of intersection of some neighborhood of q is bounded away from zero and γ_+ has infinite length. Suppose that γ isn't critical; then both rays γ_+ and γ_- have infinite length and the map f is defined and injective on the whole line $\mathbb{R}$.

Theorem 13.1. *If γ is the trajectory of the type (c) which is infinite in the positive direction then the ray γ_+ is recurrent.*

Proof: Let $p \in \gamma$ and β is a vertical interval with endpoint in p. We can assume that β is so small that no critical (positive) ray intersects this interval. Suppose that for all positive rays α emanating from points of β , $\alpha \cap \beta$ is the origin of α. Then the infinite horizontal strip S with base at β is embedded in X (since trajectories form a foliation on $X - C(\phi)$. However, S has infinite area. This contradicts the finiteness of the area of X with respect to the metric $|\phi(z)|^{1/2}|dz|$. □

Corollary 13.2. *The intersection J of γ_+ with a small interval β above is dense in β.*

Proof: The closure of the intersection above is a perfect subset. On another hand, let z be a boundary point of $Cl(J)$ in β. Denote by σ a positive ray emanating from z. We can assume that σ isn't critical, thus, it intersect infinitely many times each small interval adjacent to z on β. But σ is contained in the closure of γ_+, thus the points of J accumulate to z from 2 sides on β. This contradicts to the assumption that z is a boundary point. Thus, $cl(J) \supset \beta$. □

Now, consider the closure of γ. The arguments of Theorem above imply that $A = Cl(\gamma)$ has nonempty interior A^0 which is called a maximal spiral domain in X. As in the case of periodic trajectories this domain is swept by parallel trajectories.

Lemma 13.3. *The boundary of A consists of finite critical trajectories of ϕ and their limiting points.*

Proof: Let $\alpha \subset \partial A$ is infinite in the positive direction and $P \in \alpha$ be a regular point. Then for some small vertical interval β with center at P the ray α_+ intersects β on dense subset (see Corollary above). Thus A contains a neighborhood of P which contradicts to our assumption. $\square$

13.4 Examples of spiral trajectories.

As you can see, it's rather difficult to construct examples of spiral trajectories. To do this we shall use the construction of "interval exchange" transformations. Take the horizontal rectangle S in the complex plane : $\{z : 0 \leq Im(z) \leq 1, -1 \leq Re(z) \leq 1\}$. On the segment $\{0\} \times [0, 1]$ we choose the intervals: $I_1^+ = [0, x]$, $I_2^+ =]x, y]$, $I_3^+ =]y, 1]$ and:

$$I_1^- = [0, 1-y], I_2^- =]1-y, 1-x], I_3 =]1-x, 1]$$

Now, to each pair of intervals I_k^+, I_k^- we glue a rectangle S_k of the width 1 in orientation preserving way. We identify the "adjacent" rectangles by 1/3 of their width. In the bifurcation points we introduce the local complex coordinates using the square root. The result is a Riemann surface Y with boundary, then we can take the double X of Y by reflection. The surface X has the quadratic differential ϕ which is just the projection of dz^2 from the complex plane. The critical points are the points of bifurcation. Generically the trajectories of ϕ are recurrent or critical. (Figure 7)

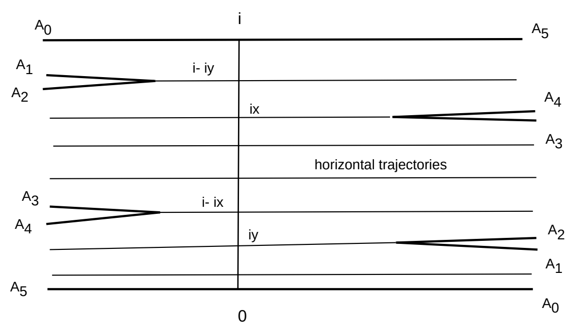


Figure 5:

13.5 Singular metric induced by quadratic differential

Define the differential $|\phi(z)|^{1/2}|dz|$. This is a Riemannian metric outside the critical set of ϕ. This metric is locally Euclidean on $X - C(\phi)$ and is called ϕ-metric. The ϕ-length of curve is also defined in the case when the curve is passing through a critical point. the total angle around the critical point of order $n > 0$ is equal to $(n+2)\pi > 2\pi$. This metric is so called $CAT(0)$.

Consider the lift of the ϕ-metric to the universal cover of $X = \mathbb{H}^2$, suppose that P is a polygon in $\mathbb{H}^2$ with geodesic sides E_j. Denote by $\tilde{\phi}$ the lift of ϕ. Let z_j denote a zero of $\tilde{\phi}$ on P where the interior angle (in the hyperbolic metric) between adjacent edges is θ_j, n_j denote the order of z_j (it can be zero). Denote by w_i zeros inside P with orders m_i. Let m be the number of zeros inside of P. Then we have

Lemma 13.4. *(Teichmüller-Gauss-Bonnet). In the notations above we have:*

$$\sum_j (1 - \frac{\theta_j(n_j+2)}{2\pi}) = 2 + \sum_i m_i = m + 2 \tag{7}$$

(This is a combinatorial Gauss-Bonnet formula for the ϕ-metric).

Proof: Along the sides E_j we have: $arg(\phi dz^2) = const_j$. Let $z = h_k(t)$ be a parametrization of E_k. Then

$$arg(\phi dz^2) = arg[\phi(h_k(t))(h_k'(t))^2] = arg[\phi(h_k(t))] + 2arg[h_k'(t)]$$

therefore,

$$\frac{d}{dt}\arg\phi(z(t)) = -2\frac{d}{dt}arg[h_k'(t)]$$

abusing notations we can write this as:

$$\frac{d}{dz}\arg\phi(z) = -2\frac{d}{dz}arg[dz]$$

along P. However the increment of $arg[dz]$ along P is equal to

$$2\pi - \sum_j(\pi - \theta_j)$$

(if curve would be smooth then the total increment is 2π, in each corner we arrive with angle smaller by $\pi - \theta_j$ than the expected vector- tangent to the next arc).

Thus we have:

$$\frac{1}{2\pi}\int_P darg[\phi(z)] = -\frac{2}{2\pi}\int_P darg[dz]$$

The last integral is equal to:

$$-2 + \sum_j(1 - \frac{\theta_j}{\pi})$$

According to the argument principle the first integral is:

$$\sum_i m_i + \sum_j \frac{n_j\theta_j}{2\pi}$$

Therefore,

$$\sum_i m_i + \sum_j \frac{n_j\theta_j}{2\pi} = -2 + \sum_j(1 - \frac{\theta_j}{\pi})$$

This implies the lemma. □

Corollary 13.5. *The ϕ-geodesic between two points of $\mathbb{H}^2$ is unique.*

Proof: Suppose that we have a geodesic bigon P in the ϕ-metric. The Lemma 13.4 implies that at least 3 summands in the left side of (7) are positive since the left side is ≥ 2. But this means that we have at least 3 points with $\theta_j < \frac{n_j+2}{2\pi}$. Thus, we have at least one point on the side of bigon such that the angle at this point is less than $\frac{n_j+2}{2\pi}$. Therefore,this side of the bigon isn't geodesic. $\square$

Now, let's count the sum of angles in a geodesic triangle in our metric. We have:

$$\sum_{j=4}^{k}(2\pi - \theta_j(n_j+2)) + 6\pi - (\theta_1+\theta_2+\theta_3) = 2\pi(n+2) \tag{8}$$

The summands $(2\pi - \theta_j(n_j+2)) = \epsilon_j$ are nonpositive since we have only 3 vertices, thus:

$$\theta_1+\theta_2+\theta_3 = 2\pi(n-1)+\epsilon \tag{9}$$

where $\epsilon \leq 0$. Therefore, $m = 0$, sum of angles in the triangle is $\leq 2\pi$ and the triangle can't contain any critical points inside.

Thus, the radius of the inscribed disc in any geodesic triangle is bounded from above by the diameter of (X, ϕ). (It's impossible to find a bound independent on ϕ.)

13.6 Deformations of horizontal arcs.

Lemma 13.6. *(Teichmüller). Let X be a compact Riemann surface , $f : X \to X$ be a homeomorphism homotopic to identity and α is a horizontal arc. Then there exists a constant M independent on α such that:*

$$l(f(\alpha)) \geq l(\alpha) - 2M \tag{10}$$

where $l(.)$ is the ϕ-length.

Proof: Let f_t be the family of continuous maps such that $f_0 = id$, $f_1 = f$. For each point $p \in X$ consider the displacement function: $d_f(p) = l_\phi(\gamma_p)$ where γ_p is the ϕ-geodesic connecting p and $f(p)$ which belongs to the homotopy class of the path $f_t(p), t \in [0,1]$. The function $d_f(p)$ is continuous, denote its maximum on X by M. Now, connect the endpoints p, q of the horizontal arc to $f(p), f(q)$ by the geodesic segments γ_p, γ_q. We have : $l(\gamma_p), l(\gamma_q) \leq M$,

$$length((\gamma_q)^{-1} \cdot f(\alpha) \cdot (\gamma_p)) \leq l(\alpha)$$

thus $l(f(\alpha)) + 2M \geq l(\alpha)$. $\square$

Corollary 13.7. *Under conditions above:*

$$\lim_{l(a)\to\infty} l(f(\alpha))/l(\alpha) \geq 1$$

13.7 Orientation of the horizontal foliation

Suppose that the monodromy group of the natural parameter Φ consist only of translations. Then the horizontal foliation of $X - C(\phi)$ admits an orientation which is just the pull-back of the orientation of horizontal lines in $\mathbb{C}$. In general however there exists a nontrivial character $\rho : \pi_1(X - C(\phi)) \to U(1)$ given by the linear part of the monodromy of Φ. Then $Ker(\rho)$ is a subgroup of index 1 or 2 in $\pi_1(X - C(\phi))$. The 2-fold ramified covering $q : X_0 \to X$ corresponding to this subgroup is called the orienting covering of X. The pull-back $q^*(\phi) = \psi$ is a quadratic differential on X_0 which has only even zeros and whose horizontal foliation admits a global orientation.

14 Extremal quasiconformal mappings

14.1 Extremal maps of rectangles

We shall use the proof of the following theorem as the model for proof of the extremality theorem in the general case.

Recall that

$$K_f(z) = \frac{|\partial f| + |\bar{\partial} f|}{|\partial f| - |\bar{\partial} f|}$$

and the coefficient of quasiconformality of f is

$$K_f = esssup_z K_f(z)$$

Theorem 14.1. *(Grötch). Let R, R' be rectangles: $a \times b$ and $a' \times b'$. Suppose that $f : R \to R'$ be a diffeomorphism. Then $K_f \geq K_0 = (a'/a) \cdot (b/b')$ and equality is achieved only on affine maps.*

Proof: Put $z = x + iy$. Let α be a horizontal line in R, then

$$a' \leq \int_\alpha |f_x| dx \tag{11}$$

Taking integral over y we have:

$$a'b \leq \int_R |f_x| dxdy \tag{12}$$

However, $f_x = f_z + f_{\bar{z}}$ and Jacobian of f is

$$J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \tag{13}$$

Therefore,

$$(|\partial f| + |\bar{\partial} f|)^2 = K_f(z) J_f(z)$$

Then

$$a'b \leq \int_R |f_z + f_{\bar{z}}| dxdy \leq \int_R \sqrt{K_f(z)} \sqrt{J_f(z)} dxdy \tag{14}$$

The last integral is estimated as

$$\sqrt{\int_R K_f(z)}\sqrt{\int_R J_f(z)} \leq \sqrt{K_f}\sqrt{ab}\sqrt{a'b'} \tag{15}$$

Finally we have:

$$(a'b)^2 \leq K_f aba'b'$$

i.e.

$$a'b \leq K_f ab'$$

and we are done. The equality here is achieved only under conditions:
(a) $K_f = (a'/b')/(a/b) = K_0$
(b) $Im(f)_x = 0, Re(f)_y = 0$
(c) $J_f = Re(f)_x Im(f)_y = c \cdot K_f = cRe(f)_x/Im(f)_y$.
Thus, f is a linear function. □

15 Teichmüller differentials

Let $\phi \in Q(X)$ be a nonzero quadratic differential. Then for each $0 \leq k < 1$ we define

$$\mu = k\frac{\bar{\phi}}{|\phi|} \tag{16}$$

It's easy to see that μ is a Beltrami differential on X. Then the Riemann surface Y with complex structure determined by μ is quasiconformally equivalent to X. The new Riemann surface has natural marking and thus defines a point in $T(X)$. This deformation of the original complex structure on X is called *Teichmüller deformation*.

Let's look at this deformation in terms of the natural parameter near regular points. Suppose that z is the natural parameter , then $\phi = dz^2$ and $\mu(z) = k$. Solutions of the Beltrami equation with the characteristic μ which fix the points ∞ are affine maps. The push-forward of ϕ under f^μ is $\psi \in Q(Y)$. In terms of the natural parameters z, ζ corresponding to ϕ and ψ we have:

$$\psi = (d\zeta)^2, f(x+iy) = Kx + iy = \zeta$$

where $K = (1+k)/(1-k)$. The affine maps f as above form a new F-structure on Y since the group $\mathbb{R}^2 \times O(1)$ is normal in $Aff(\mathbb{R}^2)$ and therefore, the transition maps are as above. Therefore, the map f is an affine horizontal stretch in terms of the natural parameters. For the Teichmüller mapping f the quadratic differentials ϕ, ψ are called *initial* and *terminal* respectively. In such case $(Y, \psi) = f(X, \phi, k)$.

16 Stretching function and Jacobian

Suppose that we have (X, ϕ) and $(Y, \psi) = f(X, \phi, k)$. Suppose that $g : X \to Y$ be any quasiconformal homeomorphism. Then, if $z = x + iy$ is the natural parameter on X, and $w = g(z)$ be the natural parameter on Y, then

$$\lambda_{g,\phi,\psi}(z) = \lambda_g(z) = |w_x| \tag{17}$$

In terms of the conformal structures on X, Y we have:

$$\lambda_g(p) = |\frac{\partial g(p)}{\phi(p)^{1/2}} + \frac{\partial \bar{g(p)}}{\bar{\phi}(p)^{1/2}}| \cdot |\psi(g(p)|^{1/2} \tag{18}$$

Then $\lambda_g(p)$ is a function on X. Let α be any horizontal arc on X. Then for all but finitely many of trajectories we have:

$$\int_\alpha \lambda_g |\phi|^{1/2} = length(g(\alpha)) \tag{19}$$

The Jacobian of g in terms of the natural parameter is:

$$J_g(z) = |\partial w|^2 - |\bar{\partial} w|^2 \tag{20}$$

this is the same as

$$J_g(z) = (|\partial w|^2 - |\bar{\partial} w|^2) \frac{|\psi(w(z))|}{|\phi(z)|} \tag{21}$$

which is a function on X. Therefore,

$$Area_\psi(Y) = \int_X J_g |\phi| \tag{22}$$

17 Average stretching

Theorem 17.1. *Let $g : X \to X$ be a quasiconformal homeomorphism homotopic to identity and $\phi \in Q(X)$. Define λ_g for $\phi = \psi$. Then:*

$$\int_X \lambda_g |\phi| \geq Area_\phi(X) \tag{23}$$

Proof: First we define a 1-dimensional average. Let α be a subarc of a horizontal trajectory with midpoint p and of length $2a$. We set:

$$\lambda_a(p) = \frac{1}{2a} \int_\alpha \lambda |\phi|^{1/2} \tag{24}$$

Assume for a moment that (X, ϕ) has oriented trajectory structure. Let X_0 be the union of noncritical trajectories. This set has full measure on X. We define a flow on X_0. Let p be a point on a horizontal trajectory $\alpha \subset X_0$, $t \in \mathbb{R}$. Let $\chi(p, t)$ be the horizontal translation of p to the time t. Then $\chi(., t)$ is isometry for each t. It follows that $\lambda^t = \lambda \circ \chi(., t)$ is a measurable function on X_0 and

$$\int_X \lambda^t |\phi| = \int_{X_0} \lambda^t |\phi| = \int_X \lambda |\phi| \tag{25}$$

Hence,

$$\int_X \lambda |\phi| = \frac{1}{2a} \int_{-a}^{a} (\int_X \lambda^t |\phi|) dt = \int_X (\frac{1}{2a} \int_{-a}^{a} \lambda^t dt) |\phi| = \int_X \lambda_a |\phi|$$

Therefore,

$$\int_X \lambda|\phi| = \int_X \lambda_a|\phi| \tag{27}$$

If the trajectory system isn't orientable, then we pass to the 2-fold branched covering of X, where all integrals double, thus the identity (27) is valid in this case as well. Then, according to Teichmüller's lemma,

$$\lambda_a(p) \geq 1 - M/a \tag{28}$$

almost everywhere and

$$\int_X \lambda|\phi| \geq (1 - M/a)Area_\phi(X) \tag{29}$$

Finally, letting $a \to \infty$ we obtain

$$\int_X \lambda|\phi| \geq Area_\phi(X) \tag{30}$$

□

Corollary 17.2. *If g is as above and $\|\psi\|_{L_1} \geq 1$ then*

$$Area_\psi(Y) \leq \int_Y \lambda^2_{g,\psi,\psi} dA_\psi \tag{31}$$

Proof: According to Theorem 17.1 and Schwarz inequality we have:

$$Area_\psi(Y) \leq \int_Y \lambda_{g,\psi,\psi} dA_\psi \leq \int_Y \lambda^2_{g,\psi,\psi} dA_\psi$$

□

17.1 Teichmüller's uniqueness theorem

Theorem 17.3. *(Teichmüller's uniqueness theorem). Let*

$$f : X \to (Y, \psi) = f(X, \phi, k)$$

be a homeomorphism of X homotopic to identity. Then

$$K(f) \geq K_0 = (1 + k)/(1 - k) \tag{32}$$

The equality takes place only if $f = id$.

Proof: Without loss of generality we can assume that $Area_\psi(Y) = \|\psi\|_{L_1} = 1$. Denote by id the identity map of X. First we notice that:

$$\lambda_{f,\phi,\psi} = K_0 \lambda_{f,\psi,\psi} \tag{33}$$

$$J_{id,\phi,\psi} = K_0 \quad , \quad dA_\psi = K_0 dA_\phi \tag{34}$$

Claim 17.4.

$$\lambda^2_{f,\phi,\psi} \le K(f) J_{f,\phi,\psi} \tag{35}$$

Proof: Let the natural parameter on X be $z = x + iy$, and on Y: $w = u + iv = f(z)$. Then in the local coordinates we have:

$$K_f(z)J_f(z) = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|}|f_z|^2 - |f_{\bar z}|^2 = (|f_z| + |f_{\bar z}|)^2 \ge (|f_z + f_{\bar z}|)^2 = \lambda^2_{f,\phi,\psi} \tag{36}$$

This proves the claim. □

Now, applying (33- 35) we get:

$$\int_Y \lambda^2_{f,\psi,\psi} dA_\psi = K_0^{-2} \int_Y \lambda^2_{f,\phi,\psi} dA_\psi = K_0^{-1} \int_X \lambda^2_{f,\phi,\psi} dA_\phi \le$$

$$\frac{K[f]}{K_0} \int_X J_{f,\phi,\psi} dA_\phi = \frac{K[f]}{K_0} \int_Y dA_\psi = \frac{K[f]}{K_0} Area_\psi(Y) \tag{37}$$

However, according to Corollary 17.2,

$$Area_\psi(Y) \le \int_Y \lambda^2_{g,\psi,\psi} dA_\psi$$

Thus, $K[f] \ge K_0$. Therefore, we proved (32).

Suppose now that in (32) we have the equality, in particular,

$$|\bar\partial w + \partial w|^2 = \lambda^2_{f,\phi,\psi} = K_0 J_{f,\phi,\psi} = K_0(-|\bar\partial w|^2 + |\partial w|^2) \tag{38}$$

$$|\bar\partial w| = k_0 |\partial w| \tag{39}$$

Denote ∂w by $r(z)e^{i\theta(z)}$, then $\bar\partial w = k_0 r(z) e^{i\nu(z)}$ and (38) can be written as:

$$|e^{i\theta} + k_0 e^{i\nu}|^2 = K_0(1 - k_0^2) = (1 + k_0)^2 \tag{40}$$

i.e.

$$|1 + k_0 e^{i(\nu - \theta)}| = 1 + k_0$$

which is possible only if $\nu = \theta$. This means that

$$\bar\partial w = k_0 \partial w$$

and f is a conformal mapping from Y to Y. Therefore, $f = id$. □

Corollary 17.5. *If $Y = f(X, \phi, k) = f(X, \psi, t)$ then $\phi/\psi \in \mathbb{R}$, $t = k$.*

17.2 Teichmüller's existence theorem

Denote by $T_e(X)$ the space of pairs (ϕ, k), where $\phi \in Q(X)$ has norm 1, $k \in (0, 1)$, and the pair $(0, 0)$. This space has natural topology given by supremum norm on $Q(X)$ and is homeomorphic to the open unit ball of dimension $3g - 3$ in $Q(X)$. Denote by F the natural map from $T_e(X)$ to $T(X)$:

$$F : (\phi, k) \mapsto f^\mu, \mu = k\bar\phi/|\phi|$$

Theorem 17.6. *The map F is a homeomorphism on the Teichmüller space.*

Proof: Then this map is injective according to Corollary 17.5. This map is continuous since solution of Beltrami equation depends continuously on the characteristic. We have:

$$\lim_{n\to\infty} d_T[F(0) = X, F(\phi, k_n = (1-1/n))] = \lim_{n\to\infty} \log\frac{1+k_n}{1-k_n} = \infty$$

and thus for any bounded subset $C \subset T(X)$ its preimage $F^{-1}(C)$ is relatively compact in T_e. On another hand, the map F is open since the spaces $T(X)$ and $Q(X)$ are manifolds of the same dimension $3g-3$. Therefore, F is a surjection on the connected component of $T(X)$. But as we know, $T(X)$ is connected, thus F is a surjection. Therefore F is a homeomorphism. □

17.3 Teichmüller geodesics

We recall that $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc model of the hyperbolic plane.

Theorem 17.7. *For each $\phi \in Q(X) - \{0\}$ the map*

$$h_\phi : \Delta \to T(X)$$

defined by the formula

$$h_\phi : t \mapsto [f^\mu] \quad , \quad \mu = t\bar{\phi}/|\phi|$$

is an isometry of the hyperbolic plane into the Teichmüller space.

Proof: We know that

$$d_T(0, [f^\mu]) = \log\frac{1+|t|}{1-|t|}$$

since Teichmüller maps are extremal. On another hand, the hyperbolic distance between 0 and t in Δ is equal to

$$\log\frac{1+|t|}{1-|t|}$$

Thus, the map h_ϕ preserves the distance from 0 to t. To prove the assertion in general case we need

Lemma 17.8. *Suppose that $\psi = t\phi \in Q(X) - \{0\}$. Then the composition*

$$f^{k\bar{\psi}/|\psi|} \circ (f^{r\bar{\phi}/|\phi|})^{-1}$$

is again a Teichmüller map.

Proof: First we notice that if $f : (X, \phi, k) \to (Y, \psi)$ is a Teichmüller map then f^{-1} is the Teichmüller map $(Y, -\psi, k) \to (X, -K^2\phi)$. Really, ...

Now, let's prove the assertion of Lemma. Denote $f^{k\bar{\psi}/|\psi|}$ by f_2 and $f^{r\bar{\phi}/|\phi|}$ by f_1. Denote by ζ, ζ^* the natural parameters for ϕ, ψ and ζ_1, ζ_2 the natural parameters for the terminal differentials of f_1, f_2. Consider the map $f_2 \circ (f_1)^{-1}$ in terms of ζ_1, ζ_2. It can be presented as composition $A \circ C \circ B$ where $B : \zeta_1 \mapsto \zeta$, $C : \zeta \mapsto \zeta^*$, $A : \zeta^* \mapsto \zeta_2$. Here A, B are stretchings and C is a conformal map $\zeta \mapsto \zeta \cdot \sqrt{\psi/\phi} = \zeta a e^{-i\theta/2}$, where $a > 0, 0 \le \theta < 2\pi$. □

18 Discreteness of the modular group

Our aim is to prove the following

Theorem 18.1. *The action of Mod_X on $T(X)$ is properly discontinuous.*

We already know that this action is isometric. Thus, it's enough to prove that any orbit of Mod_X has no accumulation points.

Now, let $[X]$ be the origin of $T(X)$, $[Y, f] \in T(X)$. Then the "length spectrum" of $[Y, f]$ is the function

$$\mathcal{L} : \gamma \mapsto length_Y[f(\gamma)]$$

where $[f(\gamma)]$ is the geodesic in the homotopy class of $f(\gamma)$, $\gamma \in \pi_1(X)/Inn(X)$. The image of the function $\mathcal{L}$ is called the **length spectrum** of Y and denoted by $L(Y)$.

Lemma 18.2. *If Y be a compact Riemann surface , then $L(Y)$ is discrete.*

Remark 18.3. *The statement is still true for surfaces of finite type but the proof is slightly more complicated and we restrict ourself to the compact case.*

Proof: We realize $\pi_1(Y)$ as a discrete group Γ acting in $\mathbb{H}^2$ with the (relatively) compact fundamental domain F. Suppose that Y contains infinitely many closed geodesics of the length not greater than C. Then we can lift them to segments $[a_n, b_n]$ in $\mathbb{H}^2$ which intersect the domain F. Then $\gamma_n(a_n) = b_n$ for some (different) elements of Γ. Therefore, $\gamma_n(F)$ intersect the C-neighborhood of F. This contradicts to discreteness. □

Theorem 18.4. *There is a finite number γ_j of elements of $\pi_1(X)$ such that any point $(Y, f) \in T(X)$ is determined by their length spectrum.*

Remark 18.5. *Y isn't determined by $L(Y)$ as it was shown by M.F. Vigneras.*

Proof: We identify each (Y, f) with the conjugacy class of admissible representation

$$\rho : \Gamma = \pi_1(X) \to PSL(2, \mathbb{R})$$

Then

$$Tr^2(\rho(\gamma)) = 4\cosh^2 \frac{\mathcal{L}(\gamma)}{2}$$

Algebraic proof. We will use the fact that Γ (and thus ρ) can be lifted in $SL(2, \mathbb{R})$. Our proof follows [Mag]. Let $g_1, ..., g_n$ be any system of generators of. We can assume that $\rho(g_1)$ is the diagonal matrix with given eigenvalues α, α^{-1}. Denote by t_{ij} the trace of $\rho(g_i g_j)$, etc. After conjugation we can assume that $\rho(g_2)$ is the matrix:

$$\begin{pmatrix} r & rs - 1 \neq 0 \\ 1 & s \end{pmatrix}.$$

Then $t_1 = \alpha + \alpha^{-1}, t_2 = r + s, t_{12} = \alpha r + \alpha^{-1} s$. From this linear system we can find α, r, s. Now, let

$$\rho(g_3) = \begin{pmatrix} \beta & \gamma \\ \delta & \epsilon \end{pmatrix}$$

when $\beta\epsilon - \gamma\delta = 1$. Then we know that

$$\alpha\beta + \alpha^{-1}\epsilon = t_{13}$$

$$r\beta + \delta(rs-1) + \gamma + s\epsilon = t_{23}$$

And using t_{123} we can find the forth equation on $\beta, \gamma, \epsilon, \delta$. One can check that from these equations we can find the coefficients of the matrix $\rho(g_3)$. Therefore, it's enough to take as the finite subset of Γ:

$$g_1, ..., g_n, g_1 g_k, g_2 g_j, g_1 g_2 g_s$$

Geometric proof.

I recall that last quarter we proved that for every triple $(2a, 2b, 2c) \in (\mathbb{R}_+)^3$ there exists a hyperbolic pair of pants P with the lengths of boundary loops given by this triple. Now we want to prove that P is unique.

We split P into union of 2 "all right" hexagons X_1, X_2. Denote by

$$a_1, \alpha, b_1, \beta, c_1, \gamma$$

the lengths of edges of X_1, then there is the hyperbolic cosine formula relating these numbers:

$$\cosh c_1 \sinh a_1 \sinh b_1 = \cosh a_1 \cosh b_1 + \cosh \gamma$$

For proof see [Be].

This means that α, β, γ determine a_1, b_1, c_1, thus $a = a_1, b = b_1, c = c_1$. On another hand, X_j is uniquely determined by a, b, c. This proves that P is uniquely determined by a, b, c. $\square$

Let a_j, b_j , $j = 1, ..., g$ be the canonical basis of Γ, d_i be as on Figure 8.

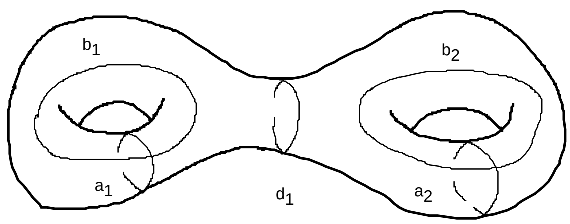


Figure 6:

We shall assume that the traces of the elements above and their double and triple products are preserved by the representation ρ.

We can assume that $\mathcal{L}(\rho)$ is the same as $\mathcal{L}(id)$ and

$$\rho(a_1) = a_1, \rho(d_1) = d_1, \rho(a_1 d_1) = a_1 d_1$$

since the surface X contains a pair of pants corresponding to $a_1, d_1, a_1 d_1$. Now, the image $\rho(a_2)$ is obtained by conjugating a_2 via some isometry g which commutes with d_1 (since we can consider now the pair of pants corresponding to a_2, d_1). Denote by $\alpha, \beta, \gamma, \delta$ the axes of the elements $a_1, a_1 d_1, a_2, d_1$ in $\mathbb{H}^2$. Then g is a translation along δ. We have to have: $dist(\alpha, \gamma) = dist(\alpha, g\gamma)$ and $dist(\beta, \gamma) = dist(\beta, g\gamma)$ since the restrictions of ρ on $< a_1, a_2 >, < a_1 d_1, a_2 >$ are conjugations. But this means that

$g = 1$, since we have to have: if g isn't trivial then it's a symmetry in the geodesic which is orthogonal to δ, α, β.

So, we conclude that $a_2 = \rho a_2$. The same argument can be applied to b_2. Now, we can use the fact that a_2, d_1 are fixed by ρ to prove that the element b_1 is fixed by ρ (applying the same arguments as above). We can continue this process to prove that all a_j, b_j are fixed by ρ.

$\square$

Remark 18.6. *It is known that for closed manifolds of nonpositive curvature the equality of marked length spectrums is equivalent to the existence of time preserving conjugation of the geodesic flows. In the dimension 2 it was proven independently by J.-P. Otal [O] and C.Croke [Cr] that surfaces of nonpositive curvature are uniquely determined by their marked length spectrums, see also [CFF]. In higher dimensions this is an outstanding research problem. It was later proven by U.Hamenstädt [H] that, if M, N are closed manifolds of negative curvature with conjugate geodesic flows so that N has constant sectional curvature, then M, N are isometric.*

We shall need the following fact:

Theorem 18.7. *(See for instance [FK]). If $G \subset PSL(2, \mathbb{R})$ is a discrete group then the area of fundamental domain of $\mathbb{H}^2/G$ is bounded from below by $\pi/21$.*

Actually, for us it will be enough to now the existence of some nonzero lower bound which we shall prove later.

Corollary 18.8. *The order of the group of conformal automorphisms of any Riemann surface Y of genus g is note greater than $42(2g-2) = 84(g-1)$.*

Proof: The area of Y is $2\pi(2g-2)$. This and Theorem 18.7 imply Corollary. $\square$

Corollary 18.9. *The kernel of the action of Mod_X on $T(X)$ is finite.*

Remark 18.10. *Actually, the kernel is nontrivial only if $g = 2$ in which case the kernel is $\mathbb{Z}_2$. For any generic surface Y of genus > 2 the group $Aut(Y)$ is trivial.*

Lemma 18.11. *Let $[Y, f] \in T(X)$ have the stabilizer H in Mod_X. Then H is isomorphic to the group of conformal automorphisms of Y.*

Proof: Let $h \in H$, then there is a conformal automorphism c_h of Y such that $f \circ h$ is homotopic to $c_h \circ f$. Thus, we have a homomorphism $c : H \to Aut(Y)$. This homomorphism is injective since the only element of $Aut(Y)$ homotopic to id is id; and it is onto since for any automorphism $a \in Aut(Y)$ defines a homeomorphism h of X by the formula $h = f^{-1}af$. $\square$

Now we can start the proof of Theorem 18.1. Suppose that there exists a sequence $g_n \in Mod_X$ such that

$$\lim_{n\to\infty} g_n[X, id] = [Y, f]$$

Then, the corresponding monodromy representations in

$$Hom(\pi_1(X), SL(2, \mathbb{R}))/SL(2, \mathbb{R})$$

are convergent. Therefore, the length spectrum of $g_n[X, id] = [X, (g_n)^{-1}] = p_n$ is convergent to the length spectrum of $[Y, f]$. But the unmarked length spectrum of $g_n[X, id]$ is the same as $\mathcal{L}(X)$. Therefore, the discreteness of $\mathcal{L}(X)$ implies that for each γ_j in Theorem 8 there exists a number n_j such that for all $n > n_j$ we have: $\mathcal{L}_{\gamma_j}(p_n) = \mathcal{L}_{\gamma_j}([Y, f])$. Therefore, for all large n we have: $\mathcal{L}(p_n) = \mathcal{L}[Y, f]$, therefore, $[X, id]$ is a fixed point of the sequence $g_n \in Mod_X$. However, we know that the stabilizer of any point in $T(x)$ is finite (Lemma 18.11, Corollary 18.9). Therefore the sequence g_n is finite. This contradiction proves the Theorem. □

Corollary 18.12. *The moduli space $M(X) = T(X)/Mod_X$ is Hausdorff.*

Below is an alternative (analytical) proof of discontinuity of the modular group.

Theorem 18.13. *The action of Mod_S on $T(S)$ is properly discontinuous.*

Proof:

Lemma 18.14. *Let S be a Riemann surface of finite hyperbolic type. Then the group of conformal automorphisms $Aut(S)$ of the surface S is finite.*

Proof: Suppose that $g_n \in Aut(S)$ is an infinite sequence convergent to a certain $g \in Aut(S)$. Then for large n the elements g_n are homotopic to each other. Recall that if $g \in Aut(S)$ is homotopic to the identity then $g = id$. Thus all but finitely many elements in the sequence g_n which shows that $Aut(S)$ is discrete. If S is compact this immediately implies finiteness of $Aut(S)$. So we consider the case when S is noncompact. Lift $Aut(S)$ into the hyperbolic plane $\mathbb{H}^2$. The lift is a group N which equals the normalizer of $\Gamma = \pi_1(S)$ in $PSL(2, \mathbb{R})$. The group N is discrete since $Aut(S)$ is. Notice that $N/\Gamma \cong Aut(S)$, thus our goal is to show that $|N : \Gamma| < \infty$. Consider the coset decomposition of N:

$$N = g_0\Gamma \sqcup g_1\Gamma \sqcup g_2\Gamma....$$

where $g_0 = 1$. As we know, each discrete subgroup of $PSL(2, \mathbb{R})$ has a fundamental domain, let D be a (closed) fundamental polygon for N. Then

$$P := D \cup g_1 D \cup g_2 D...$$

is a fundamental domain for Γ.

$$\infty > Area(S) = Area(P) = \sum_i Area(g_i D)$$

Since $Area(g_i D) = Area(D)$ we conclude that the sum is finite, which in turn implies that $|N : \Gamma| < \infty$. □

We now continue with the proof of discontinuity.

The Teichmüller space $T(S)$ is a proper metric space (metric balls in $T(S)$ are compact). Thus proper discontinuity of Mod_S is equivalent to discreteness of Mod_S in $Isom(T(S))$. Suppose that Mod_S is not discrete. Then there exists a sequence of **distinct** elements $[f_n] \in Mod_S$ such that $\lim_n [f_n] = [id]$. In particular, if $[S]$ is the origin in $T(S)$ then $\lim_n [f_n]([S]) = [S]$, i.e. $\lim_n d([S, id], [S, f_n]) = 0$. Hence we can choose quasiconformal representatives $f_n : S \to S$ in $[f_n]$ such that

$$\lim_n K(f_n) = 1$$

Each f_n extends quasiconformally to the conformal compactification $\bar{S}$ of the surface S:

$$\hat{f}_n : \bar{S} \to \bar{S}$$

and $K(\hat{f}_n) = K(f_n)$.

Lemma 18.15. *The sequence $\hat{f}_n$ is subconvergent to a conformal self-map of $\bar{S}$.*

Proof: . There are several cases depending to the type of $\bar{S}$, I will consider only the cases when $\bar{S}$ is rational and hyperbolic, the elliptic case is left to the reader.

(a) Suppose that $\bar{S}$ is rational. Then $\bar{S}$ is the sphere with $p \geq 3$ punctures. Therefore $\hat{f}_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is subconvergent on the set of at least 3 points (corresponding to the punctures). We let $\hat{f}$ be the limit of a subsequence, $f := \hat{f}|S$. Then $K(f) = 1$ and hence f is a conformal automorphism of S.

(b) Suppose that $\bar{S}$ is hyperbolic. Let $\tilde{f}_n$ be lifts of $\hat{f}_n$ to the universal cover $\tilde{S} \cong \mathbb{H}^2$ of $\bar{S}$; $\bar{S} = \mathbb{H}^2/\Gamma$. We retain the notation $\tilde{f}_n$ for the extension of $\tilde{f}_n$ to the closed hyperbolic plane $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. The map $\tilde{f}_n$ conjugates the group Γ into itself. Pick a triple of distinct points $x_1, x_2, x_3 \in \partial\mathbb{H}^2$. Then there exists a sequence $\gamma_n \in \Gamma$ such that

$$\lim_n \gamma_n \tilde{f}_n(x_j) = y_j$$

(up to a subsequence) and the points $y_j, j = 1, 2, 3$ are mutually distinct. Therefore the sequence of quasiconformal maps $\gamma_n \tilde{f}_n$ is subconvergent which implies that the sequence f_n is subconvergent as well. Let $f : S \to S$ be the limit of a subsequence. Similarly to the case (a) this limit is a conformal self-map of S. □

Now we can finish the argument. We choose a convergent subsequence in $\{f_n\}$ (and retain the notation $\{f_n\}$ for this subsequence). The maps f_n are homotopic to the conformal map $f = \lim_n f_n$ for sufficiently large n. This contradicts the assumption that all the members of the sequence $[f_n] \in Mod_S$ are distinct. □

19 Compactification of the moduli space

Our first aim is to prove the Mumford's compactness theorem for the moduli space (which is the reminiscence of the Mahler's compactness criterion):

Theorem 19.1. *For any $\epsilon > 0$ the subset of the moduli space $M(X)$ consisting of surfaces with the injectivity radius $\geq \epsilon$ is compact.*

Lemma 19.2. *Let X be a Riemann surface of finite type and α_2 be a simple closed geodesic on X. Then for any geodesic loop α_1 intersecting α_2 we have:*

$$\exp(l(\alpha_2)) \geq (\exp(2l(\alpha_1) + 1)/(\exp(2l(\alpha_1) - 1)^2$$

Proof: Let $X = \mathbb{H}^2/\Gamma$. Then we can assume that α_2 corresponds to the transformation

$$\gamma_2 : z \mapsto \lambda^2 z$$

where $l(\alpha_2) = 2\log(\lambda) > 0$ and the axis is $\ell =< 0, \infty >$; and α_1 corresponds to

$$\gamma_1 : z \mapsto \frac{(B-k)z + B(k-1)}{(1-k)z + (kB-1)}$$

where γ_1 has the fixed points $1, B$ and $l(\alpha_1) = \log(k) > 0$. Since α_1 intersects α_2 we conclude that $B < 0$. We have:

$$\gamma_1(\infty) = (B-k)/(1-k) > 0$$

and

$$1 > \gamma_1(0) > 0$$

Similarly, since $g_1(\ell) \cap g_2 g_1(\ell) = \emptyset$,

$$\gamma_2\gamma_1(0) > \gamma_1(\infty) > 0$$

which means:

$$\lambda^2 B(k-1)/(kB-1) > (B-k)/(1-k)$$

Therefore,

$$-\lambda^2 B(k-1)^2 > (B-k)(kB-1) > -k^2 B - B$$

and

$$\lambda^2 > (k^2+1)/(k-1)^2$$

$\square$

Remark 19.3. *If $l(\alpha_2) = l$ and α_1 intersects α_2 then $l(\alpha_1) \geq f(l)$, where*

$$\lim_{l \to 0} f(l) = \infty$$

Corollary 19.4. *Suppose that α_1, α_2 are simple loops on X such that:*

$$l(\alpha_1) \leq 1, l(\alpha_2) \leq 1$$

Then α_1, α_2 are disjoint.

Proof: If $\alpha_1 \leq 1$ then the left side of the inequality (*) is $> (e^2+1)/2 \geq e$. $\square$

20 Zassenhaus discreteness theorem.

Let G be a Lie group, $[,] : G \times G \to G$. It's easy to see that the first derivative of this map at the point (e, e) is zero. Therefore, there exists a compact neighborhood U of e in G such that $[,]$ is a contracting map.

Theorem 20.1. *(Zassenhaus). Suppose that Γ is a discrete subgroup of G and $x, y \in \Gamma \cap U$. Then the group generated by x, y is nilpotent.*

Proof: Form the sequence $x_0 = x, x_n = [x_{n-1}, y]$. Then $x_n \in \Gamma \cap U$ for all n and $\lim_{n\to\infty} x_n = e$. Therefore, the discreteness of Γ implies that for sufficiently large n we have $x_n = e$.

Application. Consider the case $G = Isom(\mathbb{H}^n)$. Then any infinite nilpotent subgroup of G is almost Abelian.

Suppose that $G = Isom_+(\mathbb{H}^n)$, K is the maximal compact subgroup of G, $X = \mathbb{H}^n = G/K$. We can assume that G has left-invariant Riemannian metric which is also right-invariant under K. Then we shall identify X with a Borel subgroup P of G. We can assume that U in the Zassenhaus theorem is ϵ-neighborhood $U_\epsilon(1)$ with respect to the metric on G, we shall denote the number ϵ by ϵ_Z (Zassenhaus constant).

Lemma 20.2. *There exist numbers $\mu < \epsilon_1 < \epsilon$ and an integer N such that: if $g_1, ..., g_k$ generate a discrete group Γ so that for $x = K \in G/K = X$ we have $d(x, g_j x) \le \mu$ then:*

(1) K has a $\epsilon_1/2$–net of N elements;

(2) each word $w = w(g_1, ..., g_k)$ of the length $\le N$ has the property: $d(x, wx) \le \epsilon_1$;

(3) for each w as above we have:

$$w(U_{3\epsilon_1}(1))w^{-1} \subset U_\epsilon(1)$$

Proof: We start with $\epsilon_1 = \epsilon/5$. Then we can find N such that (1) is satisfied, then we can choose μ such that (2) is correct. Now, w above is the product pk where $d(p, 1) \le \epsilon_1$ (according to (2)), therefore, if $\delta \in U_{3\epsilon_1}$ then we put $\tau = k\delta k^{-1}$ then $\tau \in U_{3\epsilon_1}$ since the metric on G is biinvariant under G, thus

$$d(1, pk\delta k^{-1}p^{-1}) = d(1, p\tau p^{-1}) \le 2d(1, p) + d(1, \tau) \le 5\epsilon_1$$

Denote by Γ' the subgroup of Γ generated by all elements $\gamma \in \Gamma \cap U_\epsilon(1)$ (a priori this subgroup can be trivial). Denote by

$$\Gamma = \bigcup_{j=1}^{\nu} \gamma_j \Gamma'$$

the coset decomposition of Γ.

Lemma 20.3. *In the decomposition above $\nu < \infty$.*

Proof: Let $\gamma_j = w = g_{i_1}....g_{i_M}$ be a word of the length $M > N$. Then $w = w_1w_2 = w_3w_4$ where $l(w_2) < l(w_4) \leq N$, $w_j = p_jk_j$ and $d(k_2, k_4) \leq \epsilon_1$. Therefore,

$$\delta = w_4w_2^{-1} = p_4k_4k_2^{-1}p_2^{-1}$$

However, $d(1, k_4k_2^{-1}) \leq \epsilon_1$. On another hand,

$$\epsilon_1 \geq d(w_jx, x) = d(p_jx, x)$$

Therefore, $d(1, \delta) \leq 3\epsilon_1$. This implies that

$$w = w_3\delta w_2 \quad , l(w_3w_2) < M$$

Consider the element

$$(w_3w_2)^{-1}w = w_2^{-1}w_3^{-1}w_3\delta w_2 = w_2^{-1}\delta w_2$$

Then the property (3) implies that this element belongs to $U_\epsilon(1) \cap \Gamma \subset \Gamma'$. Thus, w_3w_2 and w belong to the same coset (mod Γ'), but the length of w_3w_2 is strictly smaller than M. The induction argument thus imply that for all cosets (mod Γ') we can find representatives such γ_j that $l(\gamma_j) \leq N$. This implies that ν is finite. $\square$

However, according to Zassenhaus theorem, the group Γ' is almost Abelian, therefore, Γ is a finite extension of an Abelian group.

Thus we proved

Theorem 20.4. *For each $\mathbb{H}^n$ there exists a constant $\mu = \mu_n$ such that: for any $x \in \mathbb{H}^n$ and any elements $g_1, ..., g_k \in Isom(\mathbb{H}^n)$ which generate a discrete group Γ and $d(x, g_j(x)) \leq \mu$ the group Γ is almost Abelian and is a finite extension of a subgroup $\Gamma' \subset \Gamma$ which is generated by elements $\gamma \in U_\epsilon(1)$– Zassenhaus ϵ_Z–neighborhood of e in $SO(n, 1)$.*

Remark 20.5. *According to [Mart] one can take as μ the number:*

$$9^{-(2+[n/2])}$$

Remark 20.6. *Actually we proved that for any $\epsilon < \epsilon_Z$ we can find $\mu(\epsilon) < \epsilon$ (increasing function on ϵ) such that in each discrete group $\Gamma_{\mu(\epsilon)}(x)$ the almost Abelian subgroup $\Gamma'_\epsilon(x)$ has finite index. Here $\Gamma_{\mu(\epsilon)}(x)$ is any discrete group generated by elements g_j such that $d(g_j(x), x) \leq \mu(\epsilon)$ and $\Gamma'_\epsilon(x)$ is the maximal subgroup in $\Gamma_{\mu(\epsilon)}(x)$ generated by elements in $U_\epsilon(1)$. The function $\mu(\epsilon)$ is invertible and $\epsilon = \epsilon(\mu)$. Moreover,*

$$\lim_{\mu \to 0} \epsilon(\mu) = 0$$

Corollary 20.7. *There is an increasing function $f(\lambda)$ defined for all $\lambda < \mu$ such that:*

(a) $\lim_{\lambda \to 0} f(\lambda) = \infty$;

(b) for any $h, g \in SO(n, 1)$ and $x \in \mathbb{H}^n$ such that $d(x, h_x) \leq \lambda < \mu$ such that $< g, h >$ is discrete and nonelementary, we have:

$$d(x, gx) > f(\lambda)$$

Proof: Let $\psi_R(\delta)$ be an increasing function such that:

$$pU_\delta(1)p^{-1} \subset U_{\psi_R(\delta)}(1)$$

for all $p \in P \cap U_R(1)$ and for fixed δ

$$\lim_{\epsilon=\psi_R(\delta)\to 0} R = \infty$$

For given R denote by δ_R the number such that $\psi_R(\delta_R) = \epsilon_Z-$ Zassenhaus constant. Now, let λ be $\mu(\delta_R)$ where μ is the function from the **Remark** above.

Then the group $\Gamma'_{\epsilon(\lambda)}(x)$ has finite index in $\Gamma =< h >$ and $g\Gamma'_{\epsilon(\lambda)}(x)g^{-1} \subset U_{\epsilon_Z}(1)$. Therefore, according to Zassenhaus Theorem, the group generated by g and $\Gamma'_{\epsilon(\lambda)}(x)$ is almost Abelian and elementary, and thus the group $< g, h >$ is elementary.

The function $\lambda = \lambda(R)$ is increasing and we can find the inverse $R = f(\lambda)$. This function has the property that if $d(x, gx) \leq f(\lambda)$ then either $< g, h >$ isn't discrete or is elementary (property (b)). We have to verify the property (a):

$$\lim_{\lambda \to 0} f(\lambda) = \infty$$

Really, $\mu(\lambda) \to 0$ as $\lambda \to 0$ and $R \to \infty$ as $\delta_R \to 0$. □

Suppose now that X is a hyperbolic of finite type with totally geodesic boundary and $\alpha \subset \partial X$, $l(\alpha) = l$. Then there are three functions $A(l), W(l), L(l)$ such that:

Lemma 20.8. *(1) α has a regular neighborhood U of the width $W(l)$;*

(2) the length of the second (different from α) component α^* of ∂U is $L(l)$ and the area of U is $A(l)$;

(3) $\lim_{l\to 0} W(l) = \infty$, $L(l)^2 \leq Area(X)^2 + l^2$, $\lim_{l\to 0} L(l) \leq Area(X)$.

Proof: Let Y be the double of X. Then, according to Lemma 8 the geodesic α on Y has the normal injectivity radius at least $W(l) = f(l)/4$. If we lift U in the hyperbolic plane, then the preimage of β is a hypercycle $\tilde{\beta}$ which makes the angle ψ with $\tilde{\alpha}$. Now the hyperbolic trigonometry and integration in polar coordinates imply that:

$$\cosh(W) = 1/\cos(\psi)$$

$$A(l) = Area(U) = l \cdot \tan(\psi)$$

$$length(\alpha^*) = l/\cos(\psi)$$

However, $Area(U) < Area(X)$. Thus,

$$Area(U)/l = \sqrt{L^2/l^2 - 1}$$

and

$$L^2 = Area(U)^2 + l^2 \leq Area(X)^2 + l^2$$

□

Theorem 20.9. *Let X be a surface of finite type with totally geodesic boundary. Then there exists a geodesic decomposition of X on pants such that the lengths of the decomposing loops are bounded from above by some constant C depending only on the topology of X and the lengths of boundary curves.*

Proof: If X is a pair of pants then we are done. Let $\alpha_1, ..., \alpha_m$ be the boundary curves of X. Denote by W_i the width of the regular collar around α_i given by the Lemma 8. Denote by ϵ the minimum of all $W_i/2$. denote by $C(X)$ the union of ϵ-collars of α_j and put $X^* = X - C(X)$. If we can find on X^* a homotopically nontrivial simple loop α which is not homotopic to boundary which has length less than ϵ, then we are done. Suppose that such loops do not exist. If two collars intersect then the distance between two loops α_j^* and α_i^* is zero and the length of these loops is bounded by $Area(X) + length(\partial X)$. Thus, we can find a nontrivial curve β on X as on the Figure below. So, we assume that the collar $C(X)$ of ∂X is embedded. Cover $X^* = X - C(X)$ by a maximal set of disjoint discs $D(P_j, \epsilon) \subset X$. If the number of these discs is n then their joint area is at least $n\epsilon^2$ and every point $z \in X$ has the property:

$$d(z, \partial(X^* - \bigcup_{i=1}^{n} D_i(z, \epsilon)) \ < \epsilon$$

Thus $n < Area(X)/\epsilon$. Draw a graph $\mathcal{G}$ on X^* by connecting any two points P_i, P_j such that $d(P_i, P_j) \leq 4\epsilon$.

Lemma 20.10. *The graph $\mathcal{G}$ is connected.*

Proof: Suppose that $\mathcal{G}$ consists of two components Z_1, Z_2. Consider the ϵ- neighborhood s U_1, U_2 of the unions of discs with centers at Z_1, Z_2 respectively. Then $U_1 \cup U_2 \supset X^*$ which means that their intersection isn't empty. This implies that there are vertices in Z_1, Z_2 such that the distance between them is $\leq 4\epsilon$. This contradiction shows that $\mathcal{G}$ is connected. □

Thus, for every $z, w \in X^*$ we have:

$$d(z, w) \leq 4n\epsilon + l(\partial X^*) \leq Area(X)/\epsilon + l(\partial X^*) \leq$$

$$Area(X) + Area(X)/\epsilon + length(\partial X) = c$$

and the diameter of X^* is bounded by a constant c which depends only on topology of X and the length of ∂X. If we have at least two different boundary components α_j^*, α_i^* of X^* then we connect them by a shortest arc γ and thus find a loop

$$\beta = \alpha_j^* \cdot \gamma \cdot \alpha_i^* \cdot g^{-1}$$

which has bounded by $2cArea(X)$ length and not boundary-homotopic. If we have only one boundary component α_j^* then choose a shortest arc γ with endpoints on α_j^* such that $\gamma \in \pi(X^*, \partial X) \neq 0$. The length of this arc is bounded by $2diam(X)$ (use 2-sheeted covering over X which has 2 boundary components). Let α_1^* be one of components of $\alpha - \gamma$. Then take the loop

$$\beta = \alpha_1^* \cdot \gamma$$

The length of β is again bounded and this loop is homologically nontrivial, and therefore- nonparallel to the boundary.

□

Corollary 20.11. *Suppose that the injectivity radius of a closed surface X is bounded from below by ϵ. Then the diameter of X is bounded from above by*

$$c(\epsilon) = Area(X)/\epsilon$$

On another hand, the area of X in general is growing exponentially with the growth of $diam(X)$.

Now we can prove Theorem 20.9. There are only finitely many nonequivalent pants decomposition of X (the number $\nu(X)$ of trivalent graphs with $2g-2$ vertices.) We fix $\nu(X)$ decompositions of X with hyperbolic metrics: $X_1, ..., X_\nu \in T(X)$. Let Y be any point of the subset $M(X)_\epsilon$ of $M(X)$ where $RadInj(Y) \geq \epsilon$. Then there is a marking (Y, f) on Y such that $d_T([Y, f], X_i) \leq b(\epsilon, g)$ where g is the genus of Y. Therefore, the set $M(X)_\epsilon$ is bounded and therefore- compact.

Another remark. Suppose that $S \subset M(X)$ doesn't belong to any $M(X)_\epsilon$ for any $\epsilon > 0$. Then S is unbounded. Really, for any choice of canonical generators of $\pi_1(Y)$ we will have: $a_j \cap \gamma \neq \emptyset$ where $length(\gamma) < \epsilon$. Therefore, $length(a_j) \to \infty$ as $\epsilon \to 0$. This means that the sequence of surfaces is not relatively compact in $M(X)$. □

Corollary 20.12. *There exists a number $q > 0$ so that for each (in particular non-compact) hyperbolic surface X there is a point p such that $RadInj_p(X) \geq q$.*

Proof: Continuity method. Let X_μ be the subset of all points in X where $2RadInj \leq \mu$- Margulis constant. This is a disjoint union of annuli which at worst can be tangent. In the worst case they decompose X to the union of pairs of pants. Consider this worst case. Then in the universal covering we have a union of hypercycles which are at worst tangent one to another. Then we can find a disc D in $\mathbb{H}^2$ which doesn't intersect interiors of hypercycles and tangent at least to three of them. I claim that the diameter of this disc is bounded from below by some universal constant. Suppose not, we can assume that the center of this disc is the point 0. Then our configuration has a limit where hypercycles degenerate to some discs which intersect or tangent the boundary of $\mathbb{H}^2$. But these discs are at worst tangent and do not contain 0. □

Theorem 20.13. *Suppose that X_n is a sequence in $M(X)$ such that*

$$\lim_{n\to\infty} [RadInj(X_n) = \epsilon_n] \to 0$$

Then the sequence X_n is not relatively compact in the moduli space $M(X)$.

Proof: Let $\gamma_n \subset X_n$ be the sequence of geodesic loops such that $l(\gamma_n) = \epsilon_n$. Then for any choice of canonical basis of $\pi_1(X_n)$ there is a loop α_n in this basis which has nonzero intersection number with γ_n. Therefore $l(\alpha_n) \to \infty$ as $n \to \infty$. This means that the sequence X_n is divergent in $M(X)$ with respect to the Teichmüller metric. □

21 Fenchel–Nielsen coordinates on Teichmüller space.

Consider a closed hyperbolic Riemann surface X and fix a pants decomposition D of X. For each pair of pants P_j in D there are 3 boundary loops C_{j1}, C_{j2}, C_{j3}. We connect them by the disjoint oriented geodesic arcs α_{ji} orthogonal to ∂P_j. Each arc has the end-point ζ_{ji}. Suppose that C is a common boundary loop for two pairs of pants P_j, $P_{j'}$ and ζ_{ji}, $\zeta_{j'i'}$ are end-points of the arcs α, α'. The loop C is oriented, so we can define

$$\theta_{ij} = 2\pi d(\zeta_{ji}, \zeta_{j'i'})\lambda(C_{ji})$$

where we calculate the distance in the positive direction. The number θ_{ij} is defined (mod 2π) and is called the "angle of gluing". Therefore, we have a continuous function

$$\hat{F} = (L, \Theta) : T(X) \to \mathbb{R}^{3g-3} \times \mathbb{S}^{3g-3}$$

which associates with a point $Y \in T(X)$ the logarithms of geodesic lengths of the loops C and Θ consists of the coordinates θ_{ji}. This map is obviously continuous and onto.

Lemma 21.1. *The map $\hat{F}$ is a covering.*

Proof: If $\hat{F}(p = [Y, h]) = \hat{F}(q = [Z, g])$ then the surfaces Y, Z are isometric as unmarked Riemann surfaces. Therefore, $q = f(p)$ where $f \in Mod_X$. The element of the modular group f must preserve D. Denote the subgroup of Mod_X which preserve D by $Mod_X(D)$. Then, for each $f \in Mod_X(D)$ and each $p \in T(X)$ we have:

$$\hat{F}(p) = \hat{F}f(p)$$

therefore, $\hat{F}$ is a covering with the covering group $Mod_X(D)$. Let's describe this group. This group is isomorphic to Z^{3g-3}. Therefore, the Dehn twists along the loops C in the pants decomposition generate $Mod_X(D)$.

Now, the lift of $\hat{F}$ to $\mathbb{R}^{6g-6}$ is denoted by F and is a homeomorphism of $T(X)$ which is called the Fenchel- Nielsen coordinates on the Teichmüller space. □

22 Riemann surfaces with nodes.

The Riemann surface with nodes is a complex space modelled on $\mathbb{C}$ and $\{zw = 0\} \subset \mathbb{C}^2$ subject to the following topological restriction: each surface with nodes can be obtained from a nonsingular Riemann surface S^* by pinching to points some system of simple disjoint nonparallel homotopically nontrivial loops. We can consider any surface with nodes as a (disconnected in general) Riemann surface of finite type with a finite number of punctures together with the identification pattern of the punctures. The space of Riemann surface with nodes forms a compactification of the moduli space.

Consider now the space $\hat{M}(X)$ - the space of Riemann surfaces with nodes obtained by pinching some loops on elements of $M(X)$. We already have the topology on

$M(X) \subset \hat{M}(X)$. The **horocyclic neighborhood** of a point $S \in \hat{M}(X) - M(X)$ is defined as follows. If $A = \{\alpha_1, \alpha_2, ..., \alpha_q\}$ are loops on X to be pinched on S then enlarge A to a pants decomposition D of X. Convention: we shall think that the loops α_j have zero length on S. Then the base of topology at S is given by neighborhoods U_ϵ which consists of Riemann surface with nodes Y such that:

(1) for pinched loops α_j the angles of gluing are arbitrary;

(2) the differences between all other Fenchel-Nielsen coordinates of S, Y are less than ϵ.

Theorem 22.1. *The space $\hat{M}(X)$ with the topology defined above is compact, Hausdorff and has countable base of topology.*

Proof: The proof of compactness just repeats the proof of Theorem 20.9. We'd like only to note that the Fenchel-Nielsen coordinates "extend" to the compactification $\hat{M}(X)$, but different stratums of $\hat{M}(X) - M(X)$ correspond to different pant decompositions of X (so we have several coordinate systems which cover $\hat{M}(X)$). Two other statements are obvious. □

There are several ways to construct the complex structure on the space $\hat{M}(X)$, one- using algebraic geometry [Mu], another- using Kleinian groups and automorphic functions [B].

23 Boundary of the space of quasifuchsian groups

I recall the definition of the Bers' embedding of the Teichmüller space:

Let $X = \mathbb{H}^2/F$ is a compact surface of genus $g > 1$, and given $[\mu] \in T(X)$ we lift μ to the hyperbolic plane $\mathbb{H}^2$ (the upper half plane) and extend by zero to the whole complex plane, denote by ν the result. Take $f = f^\nu$ to be solution of the Beltrami equation with the complex characteristic ν which fixes the points $0, 1, \infty$. The map f is conformal in the lower half-plane $\mathbb{H}^2_*$ and we can take the Schwarzian derivative $S(f)$ of the restriction of f to $\mathbb{H}^2_*$. Then $S(f) \in Q(X)$ which we identify with the space of F-invariant holomorphic quadratic differentialon $\mathbb{H}^2_*$. The quasiconformal map f defines the representation $\rho : F \to PSL(2, \mathbb{C})$ so that $f(g(z)) = \rho(g)f(z)$ for all $g \in F$. The correspondence $\Phi : [\mu] \mapsto S(f)$ is called the "Bers' embedding", its image is a bounded domain D in $Q(X)$.

We can assume that $0, 1, \infty$ are fixed points of three elements g_0, g_1, g_∞ of the group F. Therefore the assumption that $\rho(g_j)$ has the fixed point j $(= 0, 1, \infty)$ gives us a slice on a Zariski open subset of $Hom(F, PSL(2, \mathbb{C}))$ to the projection $\pi : Hom(F, PSL(2, \mathbb{C})) \to Hom(F, PSL(2, \mathbb{C}))//PSL(2, \mathbb{C}) = R(F)$.

On another hand, for each $\phi \in Q(X)$ we have the monodromy homomorphism ρ_ϕ of the Schwarzian equation $S(f) = \phi$ on the lower half plane $\mathbb{H}^2_*$. The map $hol :$ $Q(X) \to Hom(F, PSL(2, \mathbb{C}))$ defined by the formula $hol(\phi) = \rho_\phi$ is a holomorphic map of $Q(X)$ to $R(F)$. It's easy to show that the restriction of this map to D is an embedding. Really, suppose that $\phi, \psi \in Q(X)$ be such that $hol(\phi) = hol(\psi)$. Then we have two holomorphic maps $f_1, f_2 : \mathbb{H}^2_* \to \mathbb{C}$ which are extendable to the boundary of the half-plane and the restrictions of f_1, f_2 to $\bar{\mathbb{R}}$ coincide (because the fixed points of $G = hol(\phi)(F)$ are dense on $f_i(\bar{\mathbb{R}}) = \Lambda(G)$ -limit set of the group G.

Therefore, $f_1 = f_2$ and $\phi = \psi = S(f_i)$. Moreover, one can prove that the map *hol* is injective immersion on the whole space $Q(X)$ (Poincaré's lemma). Moreover, the space $hol(Q(X))$ is an complex-analytic subvariety in $R(F)$ (i.e. is a solution of an equation $H(z) = 0$ for a vector-function $H : R(F) \to \mathbb{C}^{3g-3}$.

For each $\phi \in D$ the image of the representation $hol(\phi)$ is a "quasifuchsian group" G , the limit set of this group is a topological circle in $\overline{\mathbb{C}}$. However, we will be interested in the image of the boundary of D under *hol*. This is a compact subset of $R(F)$ and the images of the representations in $hol(\partial D)$ are called *b*-groups (boundary groups). For each $\phi \in \partial D$ the solution of the equation $S(f) = \phi$ is injective holomorphic function in $\mathbb{H}^2_*$ (by the continuity reasons: uniform on compacts limit of injective functions on $\mathbb{H}^2_*$ is again injective). Therefore, $f(\mathbb{H}^2_*) = \Omega_0(G)$ is invariant under G which implies that $\Omega_0(G)$ is a subset of the domain of discontinuity $\Omega(G)$. Thus, each *b*-group G is Kleinian. The compactness of $X = \Omega_0(G)/G$ implies that the boundary of $\Omega_0(G)$ consists of limit points of G. On another hand, $\Omega_0(G)$ is G-invariant, thus the whole limit set of G coincides with the boundary of G. Another remark is that $\Omega_0(G)$ is simply-connected, and $\rho : F \to G$ must be an isomorphism (since it is induced by the injective conformal conjugation f on $\mathbb{H}^2_*$).

Now we have to understand the topology of other components of $\Omega(G)$ (if there are any !). I recall that the group G is finitely generated, therefore the Ahlfors' finiteness theorem can be applied to G as follows:

Theorem 23.1. *(Ahlfors' finiteness Theorem.) The quotient $Y = \Omega(G)/G$ of any finitely generated Kleinian group $G \subset PSL(2, \mathbb{C})$ consists of a finite union of Riemann surfaces of finite conformal type. Each puncture p on Y corresponds to a parabolic element of G (i.e.if you lift a loop around p to $\Omega(G)$ then it is stabilized by a cyclic parabolic subgroup of G.*

For proof see [Ah2], [Kra].

I don't have any time to discuss the proof of this central fact of theory of Kleinian groups, hopefully we shall do it next year.

From now on we shall denote by Y_0 the quotient $\Omega_0(G)/G$.

Suppose now that some component O of $\Omega(G)$ is not simply connected. Consider the projection Z of O to Y. Then Z is a boundary surface of the 3-manifold $M(G) = (\mathbb{H}^3 \cup \Omega(G))/G$ and the induced homomorphism $i : \pi_1(Z) \to \pi_1(M)$ isn't injective. Therefore, according to Dehn's lemma, we can find a simple closed curve γ on Z which isn't trivial on Z but bounds an embedded disc B in $M(G)$. Therefore, the fundamental group of $M(G)$ (isomorphic to G) splits into a nontrivial free product. But it contradicts to our assumption that $F \cong G$ is the fundamental group of the closed surface X. Therefore, all components of $\Omega(G)$ are simply-connected. Thus, if Z is any component of Y, then $\pi_1(Z)$ is a finitely generated subgroup of $\pi_1(X)$. Suppose that the fundamental group H of some component Z of $Y \neq Y_0$ has a finite index in G. Then H has the same limit set as G and has at least two invariant components of the domain of discontinuity: Ω_0, Ω_1. Thus, the manifold $(M(H) - Y) \cup (\Omega_0/H) \cup Z$ is compact and is properly homotopy equivalent to $Z \times [0, 1]$. Theorem of Stallings implies that $M(H)$ is homeomorphic to $Z \times [0, 1]$ and there exists a quasiconformal map $\psi : \mathbb{H}^3 \to \mathbb{H}^3$ conjugating H to a Fuchsian group. This homeomorphism extends to the boundary $\overline{\mathbb{C}}$ of $\mathbb{H}^3$. That contradicts to our assumption that G is a boundary group.

On another hand, we know that all subgroups of $\pi_1(X)$ of infinite index are free. Therefore, if Z is any component of $Y - Y_0$, then Z is noncompact and cusps of Z correspond to parabolic elements of G.

Conclusion. Suppose that G has no parabolic elements. Then $\Omega(G) = \Omega_0(G)$, i.e. it's connected and simply connected.

Kleinian groups with such "pathological property" are called "totally degenerate Kleinian groups". Our next goal is to prove the existence of such monsters.

For each $\gamma \in F - \{1\}$ we consider the polynomial function T_γ on the representation variety $R(F)$:

$$T_\gamma([\rho]) = Trace^2(\rho(\gamma))$$

Thus the subset $S_\gamma = T_\gamma^{-1}(4) \subset R(F)$ is a complex-analytic subvariety as well as the preimage $hol^{-1}S_\gamma$. Now, consider the set E of all real rays R with origin at 0 in $Q(X)$ so that

$$R \cap \cup_{\gamma \in F - \{1\}} hol^{-1} S_\gamma = \emptyset$$

Almost every ray in $Q(X)$ belongs to E and for each $R \in E$ the groups $G = hol(R \cap \partial D)$ have no parabolic elements. Therefore, "almost every group" G on the boundary of Teichmüller space $hol(D)$ is totally degenerate.

24 Examples of boundary groups

Let $\{c_1, ..., c_k\} = C \subset X$ be a union of simple closed disjoint nonparallel geodesics on X. Lift C to the universal cover $\Delta = \mathbb{H}^2$ of X. Denote the preimage by L. Now, consider the following equivalence relation on $\overline{\mathbb{C}}$:

$x \sim y$ if and only if they belong to the closure of one and the same geodesic in L.

It follows from theorem of C.Moore that the quotient $\overline{\mathbb{C}}/\sim$ is homeomorphic to the sphere $\mathbb{S}^2$. The action of the group F on $\overline{\mathbb{C}}$ projects to the action of a group of homeomorphisms G on $\mathbb{S}^2$. It can be proven that this action is conformal in some conformal structure on $\mathbb{S}^2$, thus G becomes a Kleinian group and this is a b-group.

The limit set Λ of the group G is the projection of the boundary of $\mathbb{H}^2$, obtained by " pinching " the geodesics in L. Projections of these geodesics are fixed points of parabolic elements of G. The group G has simply-connected invariant component Ω_0 - projection of $\mathbb{H}^2_*$. There are also some non-invariant components of the domain of discontinuity; $\Omega(G)/G$ is homeomorphic to

$$X \cup (X - C = X_1 \cup ... \cup X_s),$$

where the component $X = X_0$ is covered by Ω_0. This follows from the fact that $\mathbb{S}^2 - L$ is equivariantly homeomorphic to $\overline{\mathbb{C}} - (\Lambda(F) \cup L)$. The curves $X_1, ..., X_s$ are obtained from X by "pinching" along C. The limit set of the group G looks like an infinite union of "bubbles" : boundary curves of the domains covering $X_1, ..., X_s$, two bubbles can be tangent at the fixed point of a parabolic element.

Now, let me try to give you an idea how the action of a totally degenerate group looks like. Let ϕ be a quadratic differential on X so that horizontal trajectories of ϕ are never periodic. Lift the horizontal foliation of ϕ to a "foliation" $\mathcal{F}$ on Δ.

[Technically speaking, this is a foliation on a complement to some discrete subset-preimage of zeros of ϕ]. Each leaf of $\mathcal{F}$ is a "quasigeodesic" or a "quasigeodesic ray" in $\mathbb{H}^2$ (it's located in a finite distance from a geodesic) so its closure on $\partial\Delta$ consists of 2 or 1 points. Now, again consider the equivalence relation $\sim$:

two points x, y on $\Delta \cup \partial\Delta$ are equivalent if they belong to two leaves L_1, L_2 of $\mathcal{F}$ so that $Cl(L_1) \cap Cl(L_2) \neq \emptyset$.

It turns out that the quotient of $\overline{\mathbb{C}}$ by $\sim$ is again a topological sphere and the action of F projects to a topological action G on $\mathbb{S}^2$. Under some choice of ϕ this action is conformal in a conformal structure on $\mathbb{S}^2$. The discontinuity domain of G consists of a single simply connected component Ω_0 - projection of $\mathbb{H}^2_*$. The limit set Λ of G looks like an infinite tree which isn't locally finite. This tree is "dual" to the foliation $\mathcal{F}$. The points of branching of this tree are projections of the singular points of the foliation and they a dense on Λ.

An example of ϕ can be given as follows. Let $h : X \to X$ is a homeomorphism with the property: for any $\gamma \in F$ and for any $n \in \mathbb{Z} - \{0\}$ the elements $\gamma, h_*(\gamma)$ are not conjugate in F. Such maps are called irreducible. Let h_0 is the extremal (Teichmüller) quasiconformal map in the homotopy class of h. Then ϕ the quadratic differential corresponding to h_0.

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