ADVANCES IN MATHEMATICS 54, 314–335 (1984)

Versal Families of Matrices with Respect to Unitary Conjugation

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INTRODUCTION

Let $M(n)$ be the space of square complex matrices. The complex unitary group $U(n)$ by conjugation acts in a natural way on $M(n)$: if $A \in M(n)$ and $P \in U(n)$ the action is ${}^t\bar{P}AP$. The first purpose of this paper is to classify the conjugacy classes (the orbits) in $M(n)$ with respect to that action. Set $O(A)$ for the orbit of any $A \in M(n)$. Actually in the first section we found an *algorithm* which yields:

(i) a map $\Delta: M(n) \to M(n)$ such that for every $A \in M(n)$:

(a) $\Delta(A)$ is upper triangular;

(b) $\Delta(A) = \Delta(A')$ iff $O(A) = O(A')$;

(c) $\Delta(A)$ belongs to $O(A)$.

(ii) the stabilizer of $\Delta(A)$, that is,

$$\mathrm{St}(\Delta(A)) = \{P \in U(n) \mid {}^t\bar{P}\Delta(A)P = \Delta(A)\}.$$

Roughly speaking $\Delta(A)$ is a normal formed representative for $O(A)$.

In the second section we consider the (*local*) *normal form* problem for families of matrices (smoothly or analytically) depending on parameters (clearly, with respect to unitary conjugation again). We solve the problem by constructing a *versal deformation* with a minimum number of parameters of each $\Delta(A)$; we obtain these deformations by means of another algorithm, parallel to the previous one, and they have the form

$$\Delta(A) + L(a_1,..., a_r) \qquad a_i \in \mathbb{R},\ L \text{ is } \mathbb{R}\text{-linear}.$$

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0001-8708/84 $7.50

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Here we describe briefly the structure of the first algorithm: let A be any matrix in $M(n)$. The first step consists in finding $A_1 \in O(A)$ upper triangular of a distinguished type (which depends merely upon the Jordan type of A, see Sect. 3.A) and $\mathrm{St}_1(A)$, that is, the subgroup of $U(n)$ preserving this upper triangular form. At step 2 one considers the action of $\mathrm{St}_1(A)$ on its A_1-obit $O_1(A_1) = \{{}^t\bar{P}A_1P \mid P \in \mathrm{St}_1(A)\}$ and finds $A_2 \in O_1(A_1)$ which gets a certain minor in a specialized form. Moreover one gets $\mathrm{St}_2(A)$ the subgroup of $\mathrm{St}_1(A)$ preserving this special form for the minor. At step p, $\mathrm{St}_{p-1}(A)$ acts on the A_{p-1}-orbit $O_{p-1}(A_{p-1})$, thus one finds $A_p \in O_{p-1}(A_{p-1})$ with a further specialized minor and $\mathrm{St}_p(A)$ the subgroup of $\mathrm{St}_{p-1}(A)$ preserving the new specialization. After a finite number of steps (say d) the process stabilizes and we may define $\Delta(A) = A_d$. It results also $\mathrm{St}(\Delta(A)) = \mathrm{St}_d(A)$. All specializations as above come from *four elementary moves* based on rather simple facts such as the rank of the minor, its Jordan type (if it is square), or its polar decomposition (if it is nonsingular).

Moreover, all steps of the algorithm depend only upon the orbit $O(A)$. Thus by collecting in the same bundle all orbits which determine formally the same types of steps (eventually with different "parameters": the eigenvalues...) one obtains a good real semialgebraic stratification of $M(n)$ in trivial fibre bundles which is naturally related to the action of $U(n)$ and is refinement of the Jordan bundles stratification (see [1]). We do this in the third section and we study some first properties of the stratification (by means of the versal deformations of 2). Also examples and remarks are in Section 3. We recall that the analogous program as in Section 2 and 3 is developed in [1] and [2] for the action of the general linear groups. At last we notice that the method seems easily adaptive to further actions: for instance, the conjugacy action of $U(n)$ on $M(n)^k$ or the conjugacy action of the real orthogonal group on $M(n, \mathbb{R})$. All arguments in proofs are elementary and refer to [3–6] or any other classical book on matrices theory.

1. Normal Forms

A. *Some Definitions*

We denote by $M(n, m)$ the set of all complex $n \times m$ matrices, and $M(n) = M(n, n)$; I_n is the identity of $M(n)$. If $A \in M(n, m)$, A^j is its jth column; we shall sometimes consider A as a linear map $A: \mathbb{C}^m \to \mathbb{C}^n$. We write $Gl(n) \subset M(n)$ for the complex general linear group and $U(n)$ for the complex unitary group. We fix on $\mathbb{C}$ a *lexicographic* total order relation $>$ and always consider $\mathbb{C}^n$ endowed with the canonical hermitian product. We say that A and $B \in M(n)$ are conjugated (or similar) if $A = P^{-1}BP$ for some $P \in Gl(n)$ and that they are unitary conjugated if the same holds for

$P \in U(n)$. For every $A \in M(n)$, $O(A)$ is its orbit under the conjugacy action of $U(n)$ on $M(n)$, and $\mathrm{St}(A) = \{P \in U(n) \mid {}^t\bar{P}AP = A\}$ is the stabilizer of A.

Let $n \in \mathbb{N}$, $n > 0$.

1.1. DEFINITION. Z_n^0 is the set of all finite ordered sequences of positive natural numbers $S = (s_1, ..., s_k)$ such that

$$\sum_{i=1}^{k} s_i = n.$$

Every S of Z_n^0 induces in a natural way a partition of any $A \in M(n)$ in minors $m(i, j, A)_S \in M(s_i, s_j)$, $j, i = 1, ..., k$, as the matrix shows (we set $m(i, A)_S = m(i, i, A)_S$):

$$A = \begin{array}{ccc} \overset{s_1}{\longleftrightarrow} & \overset{s_2}{\longleftrightarrow} & \\ \end{array} \begin{bmatrix} m(1, A)_S & m(1, 2, A)_S & \cdots \\ m(2, 1, A)_S & m(2, A)_S & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

1.2. DEFINITION. We define on Z_n^0 the following partial order relation: if $S, T \in Z_n^0$, we say that S is smaller than T, and write $S \lhd T$, iff the partition defined by S on any matrix $A \in M(n)$ is a refinement of the partition defined by T (i.e., every $m(i, j, A)_S$ is contained in some $m(p, q, A)_T$).

1.3. DEFINITION. We say that $A \in M(n)$ is of type $S \in Z_n^0$ if $m(i, j, A)_S = 0$ whenever $i \neq j$.

1.4. DEFINITION. Z_n is the set of all pairs (S, f), where $S = (s_1, ..., s_k)$ belongs to Z_n^0 and $f\colon \{1, ..., k\} \to \{0, 1, ..., p\}$ is a map such that: (i) if $f(i) \neq 0$, then there exists $j \neq i$ such that $f(i) = f(j)$; (ii) $f(i) = f(j) \neq 0$ implies $s_i = s_j$; (iii) the restriction of f to the set of minima

$\{\min f^{-1}f(i) \mid f(i) \neq 0\}$ is an increasing map and is onto $\{1, ..., p\}$.

We shall use the notation $(S, f) = (s_{1,f(1)}, ..., s_{k,f(k)})$ and we shall say that s_i and s_j (i, j) are coupled if $i = j$ or $i \neq j$ and $f(i) = f(j) \neq 0$.

1.5. DEFINITION. We say that $A \in M(n)$ is of type $(S, f) \in Z_n$ if:

(i) A is of type $S \in Z_n^0$;

(ii) $f(i) = f(j) \neq 0$ implies $m(i, A)_S = m(j, A)_S$.

For example, $P \in M(16)$ is of type $(3_0, 2_1, 3_2, 3_0, 3_2, 2_1) \in Z_{16}$ iff P is the diagonal block matrix $P = D(A, B, C, D, C, B)$, where $A, C, D \in M(3)$, $B \in M(2)$.

We remark that there exists a natural inclusion of Z_n^0 into Z_n identifying any $(s_1,\dots,s_k)$ with $(s_{1,0},\dots,s_{k,0})$. We want now to define a partial order relation on Z_n which is an extension of the relation $\lhd$ defined in Definition 1.2 and such that any matrix of type $(S,f)\in Z_n$ is a fortiori of type (T,g) whenever (S,f) is "smaller" than (T,g).

1.6. Definition. $(S,f)=(s_{1,f(1)},\dots,S_{k,f(k)})$ is smaller than $(T,g)=(t_{1,g(1)},\dots,t_{h,g(h)})$, and we write again $(S,f)\lhd(T,g)$, if: (i) $S\lhd T$ in Z_n^0; (ii) S induces on any pairs of coupled t_i and t_j the same partition (in the sense of Definition 1.2) with the natural couplings: $t_i=(s_{r,p},s_{r+1,p+1},\dots)$ and $t_j=(s_{u,p},s_{u+1,p+1},\dots)$, $p>0$ whenever $i\neq j$. For example: $(2_1,1_2,2_0,1_0,2_1,1_2)\lhd(3_1,3_0,3_1)$, $(1_1,1_2,1_3,1_3,1_1,1_2)\lhd(2_1,1_0,1_0,2_1)$, $(2_1,1_2,1_2,2_1)\not\lhd(3_1,3_1)$.

We are going to introduce a further notion. Let $n\in\mathbb{N}$ as above.

1.7. Definition. N_n is the set of all finite ordered sequences of positive natural numbers of the type

$$N=(n_1^1,n_2^1,\dots,n_{k_1}^1;\dots;\dots;n_1^h,\dots,n_{k_h}^h)$$

such that $n_1^j\geqslant n_2^j\geqslant\cdots\geqslant n_{k_j}^j$ and if $n^j=n_1^j+\cdots+n_{k_j}^j$ then one has $n^1+\cdots+n^h=n$.

Notice that there exists a natural forgetting map $N\to\hat{N}$ from N_n in Z_n^0. We use N_n to define a special class of upper triangular matrices as we do below.

1.8. Definition. $A\in M(n)$ is called upper triangular of type $N\in N_n$ if:

(i) it is upper triangular and has h distinct eigenvalues $\lambda_1>\cdots>\lambda_h\in\mathbb{C}$ of multiplicity $n^1,\dots,n^h$;

(ii) it has along the main diagonal the blocks: $\lambda_1 I_{n_1^1}$, $\lambda_1 I_{n_2^1},\dots,\lambda_1 I_{n_{k_1}^1};\dots;\dots,\lambda_h I_{n_{k_n}^h}$, as in the matrix A

$$A=\begin{bmatrix}
\lambda_1 I_{n_1^1} & B_2^1 & & & & & & \\
 & \lambda_1 I_{n_2^1} & & & & & * & \\
 & & \ddots & & & & & \\
 & & & \lambda_1 I_{n_{k_1-1}^1} & B_{k_1}^1 & & & \\
 & & & & \lambda_1 I_{n_{k_1}^1} & & & \\
 & & & & & \ddots & & \\
 & 0 & & & & & \lambda_n I_{n_{k_n-1}^n} & B_{k_n}^n \\
 & & & & & & & \lambda_n I_{n_{k_n}^n}
\end{bmatrix};$$

(iii) every minor $B_i^j \in M(n_{i-1}^j, n_i^j)$ of A lying above $\lambda_j I_{n_i^j}$, $2 \leqslant i \leqslant k_j$, has maximal rank n_i^j.

1.9. DEFINITION. A is called upper triangular of distinguished type $N \in N_n$ if it is as in Definition 1.8 and, more, every B_i^j has the first $n_{i-1}^j - n_i^j$ rows zero.

B. *Elementary Operations*

Let $G = U(n) \times U(m)$ and $H = \{(P, Q) \in G \mid P^{-1} = Q\}$. G acts on $M(n, m)$ and H acts on $M(n)$ in the natural way: $(P, Q) \cdot A = P \cdot A \cdot Q$. Let $A \in M(n, m)$; we shall consider four kinds of action of G or H on A, which we call elementary operations, which put A into a more simple form.

(I) Suppose $m \leqslant n$, $\operatorname{rank}(A) = r < n$. We say that $(P, Q) \in G$ performs on A an elementary operation of type I iff the first $n - r$ columns of P lie in the orthogonal of the image of A in $\mathbb{C}^n$. Given A, with an elementary operation of type I, we get a matrix PAQ of rank r whose first $n - r$ rows are zero, which we call in normal form of type I. This normal form is preserved exactly by the subgroup of G:

$$\mathrm{St}_{\mathrm{I}, n-r} = \{(P, Q) \in G \mid P \text{ is of type } (n - r, r) \in Z_n^0\}.$$

(II) Suppose $m > n$, $\operatorname{rank}(A) = r$. We say that $(P, Q) \in G$ performs on A an elementary operation of type II iff the first $m - r$ columns of Q lie in $\operatorname{Ker} A$. With an elementary operation of type II we get a matrix PAQ of rank r whose first $m - r$ columns are zero, which we call in normal form of type II. This normal form is preserved exactly by the subgroup of G:

$$\mathrm{St}_{\mathrm{II}, m-r} = \{(P, Q) \in G \mid Q \text{ is of type } (m - r, r) \in Z_m^0\}.$$

(III) Assume $m = n = \operatorname{rank}(A)$.

1.10. LEMMA. *In situation* III *there exist* $(P, Q) \in G$ *and an unique real diagonal matrix* A' *having* $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n > 0$ *along the main diagonal, such that*: $PAQ = A'$. *Moreover, if* $d_1 > \cdots > d_p$ *are the distinct eigenvalues of* A' *and have multiplicity* $n_1, \ldots, n_p$, *then* $(P, Q) \in G$ *and* $PA'Q = A'$ *iff* Q *is of type* $(n_1, \ldots, n_p) \in Z_n^0$ *and* $P = {}^t\bar{Q}$.

Proof. It is well known (polar decomposition) that every nonsingular matrix A can be decomposed as the product $A = UK$, U being unitary and K being hermitian positive definite; moreover, such a decomposition is unique (actually $U = AR^{-1}$, $K = R$, where $R^2 = {}^t\bar{A}A$). Let $T \in U(n)$ such that ${}^t\bar{T}KT = A'$. Setting $P = {}^t\bar{T}{}^t\bar{U}$ and $Q = T$, clearly $PAQ = A'$. The unicity of A' and the last point of the statement easily follow from the unicity of the polar decomposition.

If $(P, Q) \in G$ is chosen as in Lemma 1.10 we say that it performs an A an elementary operation of type III and the resulting normal form (of type III) is preserved exactly by the subgroup of G: $\mathrm{St}_{\mathrm{III},(n_1,\ldots,n_p)} = \{({}^t\bar{Q}, Q) \in H \mid Q$ is of type $(n_1,\ldots, n_p)\}$.

In the next case we use the following

1.11. *Remark.* Let $Q = (Q^1,\ldots, Q^n) \in U(n)$, $A \in M(n)$, and ${}^t\bar{Q}AQ = B = (b_{ij})$; we have $b_{ij} = {}^t(\bar{Q}^i)AQ^j$. Fix natural numbers $1 \leqslant r_0 \leqslant r_1 \leqslant n$, $1 \leqslant s_0 \leqslant s_1 \leqslant n$. The linear function from the space having $Q^{s_0},\ldots, Q^{s_1}$ as a basis to the space whose basis is $Q^{r_0},\ldots, Q^{r_1}$, obtained by the composition of the restriction of A to the first one with the orthogonal projection to the second, is represented with respect to the above bases by the minor of B (b_{ij}), $r_0 \leqslant i \leqslant r_1$, $s_0 \leqslant j \leqslant s_1$.

(IV) Suppose $m = n$. We need the following:

1.12. LEMMA. *In situation* IV *there exists a unique $N \in N_n$ such that ${}^t\bar{Q}AQ = B$ is upper triangular of type N for some $({}^t\bar{Q}, Q) \in H$. If $Q \in U(n)$, ${}^t\bar{Q}BQ$ is upper triangular of type N iff Q is of type $\tilde{N}$. Moreover, we can choose $({}^t\bar{Q}, Q) \in H$ in such a way that ${}^t\bar{Q}AQ$ is upper triangular of distinguished type N.*

Proof. To prove the first assertion we use induction on n. The case $n = 1$ is obvious; suppose the assertion is true for $k \leqslant n - 1$. Let $\lambda_1,\ldots, \lambda_h$ be the distinct eigenvalues of A and $n^1,\ldots, n^h$ be their multiplicities. Let V_1^1 be the eigenspace of λ_1, and n_1^1 be its dimension.

Let $P = (P^1,\ldots, P^n) \in U(n)$ such that $P^1,\ldots, P^{n_1^1}$ form a basis of V_1^1. Then

$$ {}^t\bar{P}AP = \left[\begin{array}{c|c} A(n_1^1) & * \\ \hline 0 & C(n_1^1) \end{array}\right], $$

where $A(n_1^1) = \lambda_1 I_{n_1^1}$. In the sense of Remark 1.11, $C(n_1^1)$ represents a function from $(V_1^1)^\perp$ into itself, whose eigenvalues are λ_1 (eventually), $\lambda_2,\ldots, \lambda_n$, with multiplicities $n^1 - n_1^1$, $n^2,\ldots, n^h$. By the inductive hypothesis we find $Q_0 \in U(n - n_1^1)$ such that $A_0 = {}^t\bar{Q}_0 C(n_1^1) Q_0$ is upper triangular of type $N_0 \in N_{n-n_1^1}$, where

$$ N_0 = (m_1^1,\ldots, m_{t_1}^1;\ldots;\ldots; m_1^h,\ldots, m_{t_h}^h) \qquad \text{if} \quad n_1^1 < n^1, $$
$$ N_0 = (m_1^2,\ldots, m_{t_2}^2;\ldots;\ldots; m_1^h,\ldots, m_{t_h}^h) \qquad \text{if} \quad n_1^1 = n^1. $$

Let $N = (n_1^1, m_1^1,\ldots, m_{t_1}^1;\ldots;\ldots; m_1^h,\ldots, m_{t_h}^h)$ in the first case and $N = (n_1^1; m_1^2,\ldots, m_{t_2}^2;\ldots;\ldots; m_1^h,\ldots, m_{t_h}^h)$ in the second.

We remark that in the first case $n_1^1 \geqslant m_1^1$, since otherwise, looking at

$m(1,2,{}^t\bar{P}AP)_{\hat{N}}$, there would be a nonzero kernel and we could choose $P^{n_1^1+1} \in V_1^1$, contradiction.

Thus $N \in N_n$. Let $Q = (P^1, \ldots, P^{n_1^1}, \tilde{P}^{n_1^1+1}, \ldots, \hat{P}^n)$, where $(\tilde{P}^{n_1^1+1}, \ldots, \tilde{P}^n) = (P^{n_1^1+1}, \ldots, P^n) \cdot Q_0$. Then $B = {}^t\bar{Q}AQ$ is upper triangular of type N (in the first case $m(1,2,B)_{\hat{N}}$ must have maximal rank for the same reason as before).

The unicity of N follows by a similar inductive argument.

The proof of the second assertion is straightforward; to see the third, by induction it suffices to consider $m(1,2,B)_{\hat{N}}$. If its cokernel dimension is >0, perform an elementary operation of type I on this minor, choosing a suitable basis in V_1^1, to get the required distinguished form. Q.E.D.

If $({}^t\bar{Q}, Q) \in H$ is chosen as in Lemma 1.12, we say that $({}^t\bar{Q}, Q)$ performs on A an elementary operation of type IV. With such an operation we get a matrix $B = {}^t\bar{Q}AQ$, of distinguished type $N \in N_n$, which we call in normal form of type IV. We want to determine the subgroup $\mathrm{St}_{\mathrm{IV},N}$ of H which preserves this normal form; the second assertion of Lemma 1.2 is not enough.

Given $N = (n_i^j) \in N_n$, $1 \leqslant j \leqslant h$, $1 \leqslant i \leqslant k_j$, for any such i, j define $N_i^j \in Z_n^0$ as follows:

(i) if $i = k_j$ then $N_i^j = \hat{N}$;

(ii) if $1 \leqslant i < k_j$ and $n_i^j = n_{i+1}^j$, then $N_i^j = \hat{N}$;

(iii) if $1 \leqslant i < k_j$ and $n_i^j > n_{i+1}^j$, then

$$N_i^j = (n_1^1, \ldots, n_{k_1}^1, \ldots, n_1^j, \ldots, n_i^j - n_{i+1}^j, n_{i+1}^j, n_{i+1}^j, \ldots, n_{k_j}^j, \ldots, n_1^h, \ldots, n_{k_h}^h) \in Z_n^0.$$

In case (iii), N_i^j has one element more than $\hat{N}$. Consider $D = \{S \in Z_0^n \mid S \lhd N_i^j,\ 1 \leqslant j \leqslant h,\ 1 \leqslant i \leqslant K_j\}$. D has a maximum, which we denote by N'. Then

$$\mathrm{St}_{\mathrm{IV},N} = \{({}^t\bar{Q}, Q) \in H \mid Q \text{ is of type } N' \in Z_n^0\}.$$

C. *The Normal Form*

1.13. Definition. Let $A \in M(n)$ be upper triangular of type $N \in N_n$ (N is unique by Lemma 1.12). If $S \in Z_n^0$, $S \lhd \hat{N}$, then we call (A, S)-special minors those minors $m(i,j,A)_S$ which are contained in some $m(s,t,A)_{\hat{N}}$ with $s < t$ (hence $i < j$).

1.14. Definition. Let A be as in Definition 1.13 and $(S, f) \in Z_n$, $S \lhd N$. We say that the (A, S)-special minor $m(i,j,A)_S$ is (S,f)-stable iff for any $Q \in U(n)$ of type (S,f) we have: $m(i,j,{}^t\bar{Q}AQ)_S = m(i,j,A)_S$.

1.15. *Remark.* If $m(i,j,A)_S$ is (S,f)-stable and $(T,g) \lhd (S,f)$, then any $m(s,t,A)_T$ contained in $m(i,j,A)_S$ is (T,g)-stable (see Definition 1.8).

1.16. *Remark.* Let A be upper triangular of type $N \in N_n$, $S \in Z_N^0$, $S \lhd \hat{N}$, and $Q \in U(n)$ of type S; then $m(i, j, {}^t\bar{Q}AQ)_S = {}^t\overline{m(i, Q)}_S m(i, j, A)_S m(j, Q)_S$.

We shall now define inductively the algorithm which provides the normal form. Fix a total order relation α on $\mathbb{N} \times \mathbb{N}$. Let $A \in M(n)$.

Step 0. Perform an elementary operation of type IV on A and get $A(0) = {}^t\bar{Q}AQ$ upper triangular of distinguished type $N \in N_n$. Set $(S(0), f_0) = (N', \Theta) \in Z_n$, where Θ is identically zero.

Step $d+1$, $d \geqslant 0$. Suppose to have defined $A(d) \in O(A)$ upper triangular of distinguished type N and $(S(d), f_d) \in Z_n$ such that:

($\mathrm{A_d}$) $(S(d), f_d) \lhd (S(d-1), f_{d-1}) \lhd (S(0), f_0)$;

($\mathrm{B_d}$) at least d $(A(d), S(d))$-special minors are $(S(d), f_d)$-stable.

We want to construct $A(d+1) \in O(A)$ upper triangular of distinguished type N and $(S(d+1), f_{d+1}) \in Z_n$ such that ($\mathrm{A_{d+1}}$) and ($\mathrm{B_{d+1}}$) hold.

Let $m(i, j, A(d))_{S(d)}$ be the first (with respect to α) (A(d), $S(d)$)-special minor which is not $(S(d), f_d)$-stable. Let $k(i, j, A(d))_{S(d)}$ and $c(i, j, A(d))_{S(d)}$ be, respectively, its kernel and cokernel dimensions. In the sense of Remark 1.16 we shall perform on $m(i, j, A(d))_{S(d)}$ an elementary operation with a matrix $Q(d) \in U(n)$ of type $(S(d), f_d)$. This matrix is defined as follows:

$m(h, Q(d))_{S(d)} = m(i, Q(d))_{S(d)}$	iff h is coupled with i by f_d;
$m(h, Q(d))_{S(d)} = m(j, Q(d))_{S(d)}$	iff h is coupled with j by f_d;
$m(h, Q(d))_{S(d)} = \text{identity}$	otherwise.

The minors $m(i, Q(d))_{S(d)}$ and $m(j, Q(d))_{S(d)}$ are chosen as below. Two situations may arise.

(1) Suppose i and j are not coupled by f_d. Then exactly one of the cases I, II, III may occur. Perform the corresponding elementary operation yielding $m(i, Q(d))_{S(d)}$ and $m(j, Q(d))_{S(d)}$.

(2) Suppose i and j are coupled by f_d. Perform on $m(i, j, A(d)_{S(d)}$ an elementary operation of type IV, yielding $m(i, Q(d))_{S(d)} = m(j, Q(d))_{S(d)}$.

In both cases we have defined $Q(d) \in U(n)$ of type $(S(d), f_d)$. Let $A(d+1) = {}^t\overline{Q(d)}\, A(d)\, Q(d)$; then $A(d+1)$ is upper triangular of distinguished type N, since $S(d) \lhd S(0) = N'$, and at least d $(A(d+1), S(d))$-special minors are $(S(d), f_d)$-stable.

From Remark 1.15 it follows that for any $(T, g) \lhd (S(d), f_d)$ there exists at least d $((A(d+1)), T)$-special minors which are (T, g)-stable and are not contained in $m(i, j, A(d+1))_{S(d)}$. Hence, to conclude step $d+1$, we must construct $(S(d+1), f_{d+1}) \in Z_n$, $(S(d+1), f_{d+1}) \lhd (S(d), f_d)$, such that there is at least one $(A(d+1), S(d+1))$-special minor which is $(S(d+1), f_{d+1})$-stable and is contained in $m(i, j, A(d+1))_{S(d)}$. We do this in the sequel, supposing $S(d) = (..., s_i, ..., s_j, ...)$.

If the performed elementary operation was of type I, then the new stable special minor correspond to the first $c(i, j) = c(i, j, A(d+1))_{S(d)}$ rows of $m(i, j, A(d+1))_{S(d)}$ which are zero.

We define $S(d+1)$ thinking to $\mathrm{St}_{1,c(i,j)}$ and hence substituting in $S(d)$ to all the s_k which are coupled to s_i by f_d the pair $c(i, j)$, $s_k - c(i, j)$. We define f_{d+1} in order to couple all the first terms of these pairs, to couple all the second terms of these pairs, and to retain all the other couplings defined by f_d not involving s_i.

If the elementary operation was of type II, the new stable special minor corresponds to the first $k(i, j, A(d+1))_{S(d)}$ columns of $m(i, j, A(d+1))_{S(d)}$ which are zero; $S(d+1)$ and f_{d+1} are defined in a similar way.

If the elementary operation was of type III and $n_1, ..., n_p$ are the related multiplicities, $n_1 + \cdots + n_p = s_i = s_j$, then the new stable special minors are all the special minors of $A(d+1)$ which are contained in $m(i, j, A(d+1))_{S(d)}$. We define $S(d+1)$ substituting in $S(d)$ to any s_k coupled to s_i or s_j by f_d the p-tuple $n_1, ..., n_p$. We define f_{d+1} in order to couple all the corresponding terms of these p-tuples, and to retain all the other couplings defined by f_d not involving those s_k.

If the elementary operation was of type IV, let $m(i, j, A(d+1))_{S(d)}$ be upper triangular of distinguished type $M \in N_{s_i}$. The new stable special minors are the main diagonal blocks of $m(i, j, A(d+1))_{S(d)}$, the zero minors which lie below them and the zero rows of its maximal rank minors, in the sense of Definition 1.9. Thinking of $\mathrm{St}_{\mathrm{IV},M}$ define $S(d+1)$ substituting in $S(d)$ in any s_k coupled with s_i or s_j by f_d the element $M' \in Z^0_{s_i}$. Define f_{d+1} in order to couple all the corresponding terms of these M' and to retain all the other couplings defined by f_d and not involving those s_k. This concludes step $d+1$.

Conclusion of the algorithm. After at most $\frac{1}{2}n(n-1)$ steps, we get $A(k_0) \in O(A)$, upper triangular of distinguished type N, and $(S(k_0), f_{k_0}) \in Z_n$ such that all the $(A(k_0), S(k_0))$-special minors are $(S(k_0), f_{k_0})$-stable. This means ${}^t\bar{Q}A(k_0)Q = A(k_0)$ for any $Q \in U(n)$ of type $(S(k_0), f_{k_0})$.

1.17. DEFINITION. We define the map $\Delta_\alpha : M(n) \to M(n)$: $\Delta_\alpha(A) = A(k_0)$ (which depends upon the given order relation α).

We can summarize the above results in the following:

1.18. THEOREM. (i) *$\Delta_\alpha(A) \in O(A)$ and $\Delta_\alpha(A) = \Delta_\alpha(A')$ iff $O(A) = O(A')$;*

(ii) $\mathrm{St}(\Delta_\alpha(A)) = \{Q \in U(n) \mid Q$ *is of type* $(S(k_0), f_{k_0})\}$;

(iii) *if* $S(k_0) = (s_1, ..., s_t)$ *and* $f_{k_0}\colon \{1, ..., t\} \to \{0, ..., p\}$ *then*

$$\dim \mathrm{St}(\Delta_\alpha(A)) = \sum_{i \in f_{k_0}^{-1}(0)} s_i^2 - \sum_{\substack{j = \min f^{-1}(n) \\ n = 1, \ldots, p}} s_j^2$$

and $\dim O(A) = n^2 - \dim \mathrm{St}(\Delta_\alpha(A))$.

Proof. Assume A and B lie in the same orbit V. Let $A(d)$, $(S^A(d), f_d^A)$ and $B(d)$, $(S^B(d), f_d^B)$, $d \geqslant 0$, be the sequences defined performing the normal form algorithm on A and B, respectively. By induction on d, the following assertion $p(d)$ is true:

> $(S^A(d), f_d^A) = (S^B(d), f_d^B)$; $A(d)$ and $B(d)$ are conjugated by a matrix $Q(d) \in U(n)$ of type $(S^A(d), f_d^A)$; all the $(A(d), S^A(d))$-special minors which are $(S^A(d), f_d^A)$-stable are equal to the corresponding $(B(d), S^B(d))$-special minors.

Then (i) follows from $p(k_0)$.

Next, let $\mathrm{St}(A, d)$ be the subgroup of those $Q \in U(n)$ which satisfy:

(i) ${}^t\bar{Q} A(d) Q$ is upper triangular of distinguished type N;

(ii) if $m(r, s, A(d))_{S(d)}$ is a $(A(d), S(d))$-special minor $(S(d), f_d)$-stable then

$$m(r, s, {}^t\bar{Q} A(d) Q)_{S(d)} = m(r, s, A(d))_{S(d)}.$$

By induction it follows that:

$$\mathrm{St}(A, d) = \{Q \in U(n) \mid Q \text{ is of type } (S(d), f_d)\}.$$

One inclusion is immediate; the other follows by the inductive hypothesis and the form of the stabilizer of the performed elementary operation. Since $\mathrm{St}(A^0) = \mathrm{St}(A, k_0)$, (ii) is proved; (iii) is a consequence. Q.E.D.

1.19. *Remark.* Δ_α depend on the order relation α. There should not be canonical choices for α. In the sequel we choose the following: $(i, j)\, \alpha (s, t)$, that means (i, j) precedes (s, t), iff $j > t$ or $j = t$ and $i < s$. The reason will be clear in Remark 3.3. In the sequel we shall omit to mention α, so we shall write Δ instead of Δ_α.

2. Versal Deformations

In the sequel, regular means infinitely differentiable or real analytic.

2.1. DEFINITION. If $A \in M(n)$, we call a k-parameter deformation of A the germ g of regular function $g\colon (\mathbb{R}^k, 0) \to (M(n), A)$. $M(n)$ has the real $2n^2$ dimensional structure. Equality of deformations means equality of germs.

2.2. DEFINITION. We say that the k-parameter deformations g and h of A are equivalent if there exists a germ of regular function $a\colon (\mathbb{R}^k, 0) \to (U(n), I)$ such that the k-parameter deformations g and $a^{-1} \cdot h \cdot a$ of A are equal. $U(n)$ has the structure of n^2 dimensional real analytic manifold.

2.3. DEFINITIONS. Let $\varphi\colon (\mathbb{R}^s, 0) \to (\mathbb{R}^k, 0)$ be a germ of regular function, and let g be a k-parameter deformation of A. Then we define the s-parameter deformation of A φ^*g as the germ $g \circ \varphi$.

2.4. DEFINITION. We say that the k-parameter deformation g of A is versal if for any s-parameter deformation h of A there exists a germ of regular function $\varphi\colon (\mathbb{R}^s, 0) \to (\mathbb{R}^k, 0)$ such that the s-deformations h and φ^*g are equivalent.

The trivial versal deformation of A is the $2n^2$-parameter deformation by $t(X) = X + A$, $X \in M(n) = \mathbb{R}^{2n^2}$. We want to construct a versal deformation for a given $A \in M(n)$ with minimum number of parameters. The basic tool we use is the following:

2.5. THEOREM. *The k-parameter deformation g of A is versal iff g is transversal to $O(A)$ in $O \in \mathbb{R}^k$.*

The proof is similar to that of Lemma 2.2 in [1].

If g is a versal deformation of A and $Q \in U(n)$, then $Q^{-1} \cdot g \cdot Q$ is a versal deformation of $Q^{-1} \cdot A \cdot Q$. So it is enough to construct a versal deformation with minimum number of parameters of the normal form $\Delta(A) = A^0$ of A.

Let $J\colon U(n) \to M(n)$ be defined by $J(Q) = Q^{-1}A^0Q$, so that $O(A) = O(A^0) = J(U(n))$. Let $\Gamma = dJ_I\colon TU(n)_I \to TM(n)_{A^0}$; after the canonic identification $TM(n)_{A^0} = M(n)$, we have: $TU(n)_I = \{C \in M(n) \mid {}^tC = -\bar{C}\}$. We have $-\Gamma(C) = [C, A^0] = CA^0 - A^0C$, so that

$$\operatorname{Ker} \Gamma = T\,\mathrm{St}(A^0)_I, \qquad \operatorname{Im} \Gamma = TO(A^0)_{A^0}.$$

2.6. *Remark.* The minimum number of parameters of a versal deformation of A^0 is: $\dim_{\mathbb{R}} M(n) - \dim O(A^0) = n^2 + \dim \mathrm{St}(A^0)$.

We want to construct a linear versal deformation, so we need a linear real subspace W of $M(n)$ such that $\operatorname{Im} \Gamma \oplus W = M(n)$. We shall do this inductively. Let $A(d)$, $(S(d), f_d)$, $0 \leqslant d \leqslant k_0$, be the sequence defined by the algorithm which yields the normal form A^0. For any d, $0 \leqslant d \leqslant k_0$, we want to define $p(d) \in \mathbb{N}$ and a linear function $\pi(d)\colon M(n) \to \mathbb{R}^{p(d)}$ which satisfy the following conditions:

($\mathrm{i_d}$) $\pi(d) \circ \Gamma\colon TU(n)_I \to \mathbb{R}^{p(d)}$ is surjective:

($\mathrm{ii_d}$) $\ker \pi(d) \circ \Gamma = \{C \in TU(n)_I \mid C \text{ is of type } S(d), f_d)\}$.

Suppose we have got such $p(d)$, $\pi(d)$. Let $p = p(k_0)$ and $\pi = \pi(k_0)$. Then

$$\ker \pi \circ \Gamma = \{C \in TU(n)_I \mid C \text{ is of type } (S(k_0), f_{k_0})\} = T\,\mathrm{St}(A^0)_I = \operatorname{Ker} \Gamma.$$

It follows easily $\operatorname{Im} \Gamma \cap \operatorname{Ker} \pi = \{0\}$ and $\dim \operatorname{Im} \Gamma + \dim \operatorname{Ker} \pi = \dim M(n)$, so that $\operatorname{Im} \Gamma \oplus \ker \pi = M(n)$, and the required W is the solution space of the linear system with maximal rank $\pi(Z) = 0$, $Z \in M(n)$.

We shall now construct the sequence $p(d)$, $\pi(d)$ for $0 \leqslant d \leqslant k_0$. In order to simplify the versal deformation, we shall define $\pi(d)$ as a canonic projection, obtained by taking suitable real and imaginary parts of the entries z_{ij} of $Z \in M(n)$. Instead of defining $p(d)$ and $\pi(d)$, we shall write the linear system $\pi(d)(\Gamma(C)) = 0$, $C \in TU(n)_I$, which we call $L(d)$, which defines $p(d)$ and $\pi(d)$ in the natural way.

Step 0. If $N \in N_n$ is the type of A^0, $N = (n_j^i)$, $1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant k_i$, $\sum_{j=1}^{k_i} n_j^i = n^i$, we set $N'' = (n^1, \dots, n^h) \in Z_n^0$. We define $L(0)$ to be:

(a) for any $s > t$: $m(s, t, \Gamma(C))_{N''} = 0$;

(b) for any i, j such that $1 \leqslant i \leqslant h$, $1 \leqslant j < k_i$, $N_j^i = \hat{N}$:

$$m\left(j + \sum_{r=1}^{i-1} k_r, s + \sum_{r=1}^{i-1} k_r, \Gamma(C)\right)_{N_j^i} = 0 \qquad \text{for} \quad 1 \leqslant s \leqslant j;$$

(c) for any i, j such that $1 \leqslant i \leqslant h$, $1 \leqslant j < k_i$, $N_j^i \neq \hat{N}$:

$$m\left(j + 1 + \sum_{r=1}^{i-1} k_r, s + \sum_{r=1}^{i-1} k_r, \Gamma(C)\right)_{N_j^i} = 0 \qquad \text{for} \quad 1 \leqslant s \leqslant j + 1,$$

$$m\left(j + \sum_{r=1}^{i-1} k_r, j + 2 + \sum_{r=1}^{i-1} k_r, \Gamma(C)\right)_{N_j^i} = 0.$$

We claim that ($\mathrm{ii_0}$) holds, that is, given $C \in TU(n)_I$, we have $\pi(0)(\Gamma(C)) = 0$ iff $m(s, t, C)_{N'} = 0$ for any $s \neq t$. Observe that ${}^tC = -\bar{C}$, so $m(s, t, C)_{N'} = 0$ iff $m(t, s, C)_{N'} = 0$.

First we remark that Eqs. (a) are equivalent to $m(s, t, C)_{N''} = 0$ for $s \neq t$. To see this, consider the following order relation on the minors of $\Gamma(C)$: $m \ll m'$ iff m lies strictly on the left of m', or, if this does not happen, m is below m'. Then solve Eq. (a) in increasing order, starting from the minimum.

It remains to show that Eqs. (b) and (c) are equivalent to $m(s, t, C)_{N'} = 0$

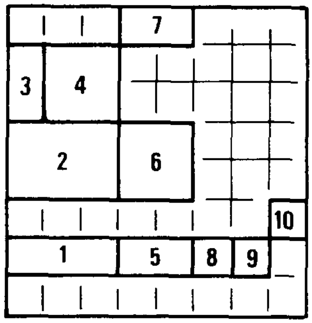

FIGURE 1

for any $s \neq t$ such that $m(s, t, c)_{N'}$ is contained in some $m(i, i, C)_{N''}$. We may assume $h = 1$, which means there is only one eigenvalue. Then it suffices to solve Eqs. (b) and (c) in increasing order, with respect to the above order relation on the minors of $\Gamma(C)$, starting from the minimum. In this computation it is essential that A^0 is in *distinguished* upper triangular form.

As an example, if $h = 1$, $N = (3, 2, 2, 1)$, the order of solving Eqs. (b) and (c) is showed by Fig. 1.

Property (i_0) follows by computation of dimensions: if $N' = (s_1, \ldots, s_t)$, then

$$2\,(\text{number of equations}) + \sum s_j^2 = n^2.$$

This can be seen in the following way: moving opportunely the minors involved in the equations, adding the minors which are symmetric with respect to the main diagonal, and adding the minors $m(i, i)_{N'}$, we cover the whole $n \times n$ matrix.

Step $d + 1$. Suppose we have defined $L(d)$, $d \geqslant 0$, such that (i_d) and (ii_d) are satisfied. We want to construct $L(d+1)$, satisfying (i_{d+1}) and (ii_{d+1}), taking the system $L(d)$ and adding to it further equations which depend on the elementary operation performed on the minor $m(i, j, A(d))_{S(d)}$ in Step $d + 1$ of the normal form algorithm.

If the elementary operation was of type I, then $m(i, j, A(d+1))_{S(d)}$ is the union of certain $m(u, v, A(d+1))_{S(d+1)} = 0$ and $(u+1, v, A(d+1))_{S(d+1)}$ of maximal rank. The equations to be added are $m(u, v, \Gamma(C))_{S(d+1)} = 0$. Likewise in case II.

If the operation was of type III, $m(i, j, A(d+1))_{S(d)}$ is the union of $m(s_p, t_q, A(d+1))_{S(d+1)}$, $p, q = 1, \ldots, r$, which are 0 if $p \neq q$ and are positive real scalar multiples of the identity if $p = q$.

We introduce the following notation: if $B = (b_{ij}) \in M(k)$ then

$HB = (\operatorname{Im} b_{ii}, b_{st})$, $1 \leqslant i \leqslant k$; $s < t$. Note that HB depends on k^2 real parameters and if ${}^t\bar{B} = -B$, $HB = 0$, then $B = 0$.

The equations to be added are:

$$m(s_p, t_q, \Gamma(C))_{S(d+1)} = 0 \qquad \text{for} \quad p \neq q;$$
$$Hm(s_p, t_p, \Gamma(C))_{S(d+1)} = 0 \qquad \text{for} \quad 1 \leqslant p \leqslant r.$$

Let v_p be the order of $m(s_p, t_p, \Gamma(C))_{S(d+1)}$. Then (i_{d+1}) holds since $p(d+1) - p(d)$ and the decrement of the kernel dimension are both equal to $\sum_1^r v_p^2 + 4\sum_{p<q} v_p v_q$; (ii_{d+1}) follows from direct computation, using (ii_d).

If the operation was of type IV, then the minors corresponding to the equations to be added are all contained in $m(i, j, \Gamma(C))_{S(d)}$ and are defined like in Step 0; (i_{d+1}) and (ii_{d+1}) hold for the same reason as (i_0) and (ii_0).

Let $p = p(k_0)$ and $\pi = \pi(k_0)$ be defined by the above procedure. If $Z = (z_{jk}) \in M(n)$, $z_{jk} = x_{jk} + iy_{jk}$, let

$$\pi(Z) = (x_{j(1)k(1)}, y_{j(1)k(1)}, \ldots, x_{j(r)k(r)}, y_{j(r)k(r)}, y_{h(1)t(1)}, \ldots, y_{h(s)t(s)}),$$

where $p = 2r + s$. Let $x = (x_l) \in \mathbb{R}^{2n^2-p}$ and $G(x) \in M(n)$ be the matrix whose (j, k) entry is 0 if $(j, k) \in \{(j(1), k(1)), \ldots, (j(r), k(r))\}$; is x_l if $(j, k) \in \{(h(1), t(1)), \ldots, (h(s), t(s))\}$; and $x_u + ix_v$ otherwise, so that $G\colon \mathbb{R}^{2n^2-p} \to M(n)$ is linear and injective. Then we can conclude that a versal deformation with minimum number of parameters of A^0 is the germ defined by $A(x) = A^0 + G(x)$.

3. Remarks and Examples

A. *Connection with Jordan Normal Form*

Let $N = (n_j^i) \in N_n$, $1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant k_i$, $n^i = \sum_j n_j^i$, and $\alpha = (\alpha_1, \ldots, \alpha_h) \in \mathbb{C}^h$, $\alpha_1 > \alpha_2 > \cdots > \alpha_h$. We denote by $\alpha_s^p \in M(p)$ the $p \times p$ Jordan block with eigenvalue α_s. We define the following two matrices:

$J(N, \alpha) \in M(n)$ is the matrix in Jordan normal form whose diagonal blocks are, respectively,

$$\alpha_1^{n_1^1}, \ldots, \alpha_1^{n_{k_1}^1}; \ldots; \alpha_h^{n_1^h}, \ldots, \alpha_h^{n_{k_h}^h}.$$

$K(N, \alpha) \in M(n)$ is the upper triangular of distinguished type N matrix whose eigenvalues are $\alpha_1, \ldots, \alpha_h$, respectively, with multiplicities $n^1, \ldots, n^h$, whose maximal rank square minors are identity matrices, and whose special minors are all zero.

For example, if $h = 1$, $N = (3, 2, 1)$ as in Fig. 2. If the Jordan normal form of $A \in M(n)$ is $J(N, \alpha)$, then we call $N \in N_n$ the Jordan type of A; if the

$$J(N,\alpha_1)=\begin{pmatrix} \alpha_1 & 1 & 0 & & & \\ 0 & \alpha_1 & 1 & & 0 & \\ 0 & 0 & \alpha_1 & & & \\ & & & \alpha_1 & 1 & \\ & 0 & & 0 & \alpha_1 & \\ & & & & & \alpha_1 \end{pmatrix} \qquad K(N,\alpha_1)=\begin{pmatrix} \alpha_1 & 0 & & & & 0 \\ & \alpha_1 & & 1 & 0 & \\ 0 & & \alpha_1 & 0 & 1 & \\ & & & \alpha_1 & 0 & \\ & 0 & & 0 & \alpha_1 & 1 \\ & & & & & \alpha_1 \end{pmatrix}$$

FIGURE 2

normal form A^0 of A is upper triangular of distinguished type $S \in N_n$, we call S the upper triangular type of A. We want to find the relation between the Jordan type and the upper triangular type of a given matrix $A \in M(n)$.

We denote by τ_p the set of tables of triangular form as shown in Fig. 3, having p boxes and such that if q_j is the number of boxes of the jth row, then $q_1 \geqslant q_2 \geqslant q_3 \geqslant \cdots$. If $T \in \tau_p$, then the transpose ${}^tT$ is naturally defined and ${}^tT \in \tau_p$.

To a given $N \in N_n$ we associate the element $(T_N^1,..., T_N^h) \in \tau_{n^1} \times \cdots \times \tau_{n^h}$ in the following way: $T_N^i \in \tau_{n^i}$ is the table having n_j^i boxes in the jth row. Now define $\tilde{N} \in N_n$ to the unique element of N_n corresponding to $({}^tT_N^1,..., {}^tT_N^h)$. We remark that the so-defined function $\sim: N_n \to N_n$ is a bijection and that $\tilde{\tilde{N}} = N$ for any $N \in N_n$.

3.1. *Remark.* (a) If $B \in M(n)$ is upper triangular of type $N \in N_n$ and has $\alpha = (\alpha_1,..., \alpha_h)$ as ordered eigenvalues with multiplicities $n^1,..., n^h$, then B is similar to $K(N, \alpha)$;

(b) $K(N, \alpha)$ is similar to $J(\tilde{N}, \alpha)$.

In (a) we may assume $h = 1$; then a suitable coordinate change provides the result. In (b) the similarity is given by a permutation of coordinates $(z_1,..., z_n) \to (z_{\sigma(1)},..., s_{\sigma(n)})$, where σ is defined by N and is easy to describe. Suppose for simplicity $h = 1$, $N = (3, 2, 4)$; then the situation is illustrated by

$$T_N = \boxed{\begin{array}{lll} \sigma(1)=3 & \sigma(4)=2 & \sigma(7)=1 \\ \sigma(2)=5 & \sigma(5)=4 & \\ \sigma(3)=7 & \sigma(6)=6 & \end{array}}\,.$$

3.2. COROLLARY. *Let $A \in M(n)$. Then the upper triangular type of A is $N \in N_n$ iff the Jordan type of A is $\tilde{N} \in N_n$.*

3.3. *Remark.* For any $N \in N_n$, $\beta = (\beta_1,..., \beta_h) \in \mathbb{C}^h$, $\beta_1 > \beta_2 > \cdots > \beta_h$,

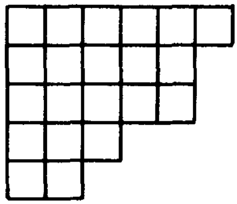

FIGURE 3

we have $\Delta(O(K(N,\beta)))=K(N,\beta)$, that is, $K(N,\beta)$ is in normal form. This fact motivates the choice of α in Remark 1.19.

B. *Stratification of* $M(n)$

We shall see that the action of $U(n)$ defines a natural stratification of $M(n)$ based on the type of the normal form, which refines the Jordan bundle stratification [1]. Denote $O_n=\{O(A) \mid A \in M(n)\}$.

3.4. DEFINITION. We denote by Z_n^* the set of all elements of the type $(s_{1,f(1)},..., s^*_{i,f(i)},..., s^*_{j,f(j)},..., s_{k,f(k)})$, where $(s_{1,f(1)},..., s_{k,f(k)}) \in Z_n$, $i \neq j$, and exactly two elements are starred. We define $\tilde{Z}_n = Z_n \cup Z_n^*$.

For any $k \in \mathbb{N}$ a natural forgetting function $p_k\colon (\tilde{Z}_n)^k \to (Z_n)^k$ is defined.

3.5. DEFINITION. We denote by P_n the set consisting of all elements of the type $(\varepsilon_1,..., \varepsilon_t)$, where $1 \leqslant t \leqslant n$ and $\varepsilon_j \in \{0, 1\}$, and of the element $\varnothing$.

To any orbit $V \in O_n$ we associate $N_V \in N_n$, $C_V=(C_{V_j}) \in (\tilde{Z}_n)^{k_V+1}$, $C_V^0=(C_{V_j}^0)=p_{k_V+1}(C_V)$, $\varepsilon_V=(\varepsilon V_j) \in (P_n)^{k_V+1}$, in the following way:

(1) N_V is the upper triangular type of $\Delta(V)$.

(2) Let $C^0_{V_j}=(S(j), f_j) \in Z_n$, $0 \leqslant j \leqslant k_V$, be the sequence obtained performing the normal form algorithm on $\Delta(V)$. Then C_{V_j} is obtained starring the rth and the sth components of $C^0_{V_j}$ if at step $j+1$ we operate on $m(r_j, s_j, \Delta(V))_{C^0_{V_j}}$.

(3) ε_V is defined as follows. If at step j we perform an operation of type I, II, or III, $\varepsilon_{V_j}=\varnothing$. If the operation is of type IV, let $\alpha_1^j > \alpha_2^j > \cdots > \alpha_h^j$ be the distinct eigenvalues of $\Delta(V)$ if $j=0$ and of $m(r_{j-1}, s_{j-1}, \Delta(V))_{C^0_{V,j-1}}$ if $1 \leqslant j$. Then $\varepsilon_{V_j}=(\varepsilon^1_{V_j},..., \varepsilon^{h-1}_{V_j})$, where $\varepsilon^i_{V_j}=0$ if $\operatorname{Re} \alpha_i^j > \operatorname{Re} \alpha_{i+1}^j$, and $\varepsilon^i_{V_j}=1$ otherwise.

3.6. *Remark.* It follows from proposition $p(j)$ of the proof of Theorem 1.18 that the N_V, C_V, ε_V are well defined.

3.7. DEFINITION. If $V \in O_n$ we set $\chi(V)=(N_V, C_V, \varepsilon_V)$. We say that V,

$W \in O_n$ are equivalent iff $\chi(V) = \chi(W)$, and we call a bundle of orbits any equivalence class of O_n.

3.8. DEFINITION. For any j, $0 \leqslant j \leqslant k_V$, let $H_j^V \in \{\mathrm{I, II, III, IV}\}$ be the type of operation we perform at step j of the algorithm. We set $\eta_j^V = 0$ iff $H_j^V \in \{\mathrm{I, II}\}$ and otherwise $\eta_j^V =$ number of distinct eigenvalues of $\Delta(V)$ if $j = 0$ and of $m(r_{j-1}, s_{j-1}, \Delta(V))_{C^0_{V,j-1}}$ if $j > 0$.

3.9. *Remark.* C_V^0 determines H_j^V, η_j^V, $0 \leqslant j \leqslant k_V$; but C_V^0 does not determine C_V, as the following example shows.

$$V \ni \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad C_V = ((1^*, 1, 1^*), (1_1, 1^*, 1_1^*), (1_1, 1_1, 1_1));$$

$$V \ni \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C_V = ((1^*, 1, 1^*), (1_1^*, 1^*, 1_1), (1_1, 1_1, 1_1)).$$

3.10. DEFINITION. We denote by M_j^V, $0 \leqslant j \leqslant k_V$, then union of the following minors of $\Delta(V)$:

(a) if $j = 0$, the minors $m(r, s, \Delta(V))_{C^0_{V,0}}$ with $r \geqslant s$ and the zero minors described in Definition 1.9;

(b) if $j > 0$ and $H_j^V \in \{\mathrm{I, II}\}$, the new stable zero minor contained in $m(r_{j-1}, s_{j-1}, \Delta(V))_{C^0_{V,j-1}}$;

(c) if $j > 0$ and $H_j^V = \mathrm{III}$, the minor $m(r_{j-1}, s_{j-1}, \Delta(V))_{C^0_{V,j-1}}$;

(d) if $j > 0$ and $H_j^V = \mathrm{IV}$, the minors contained in $m(r_{j-1}, s_{j-1}, \Delta(V))_{C^0_{V,j-1}}$ corresponding to those described in (a).

M_j^V is the union of the new stable minors which are described at step j of the algorithm. We remark that $\bigcup_{0 \leqslant s \leqslant j} M_s^V$ is contained but not necessarily equal to the union of minors of $\Delta(V)$ which are $C^0_{V_j}$-stable.

3.11. DEFINITION. We denote by $\gamma_{j,i}^V$, $1 \leqslant j \leqslant k_V$, the ith among the $(\Delta(V), C^0_{V,j-1})$-special minors which precede $m(r_{j-1}, s_{j-1}, \Delta(V))_{C^0_{V,j-1}}$ and are not contained in $\bigcup_{0 \leqslant s < j} M_s^V \cup \bigcup_{0 \leqslant s < j, t} \gamma_{s,t}^V$. We denote by $\gamma_{k_V+1,i}^V$ the ith among the $(\Delta(V), C^0_{V,k_V})$-special minors which are not contained in $\bigcup_{0 \leqslant s \leqslant k_V} M_s^V \cup \bigcup_{0 \leqslant s \leqslant k_V, t} \gamma_{s,t}^V$. We denote by $H_{j,i}^V \in \{\mathrm{I, II, III, IV}\}$ the type of operation which corresponds to $\gamma_{j,i}^V$.

3.12. LEMMA. (a) *Every $\gamma_{j,i}^V$ is $C^0_{V,j-1}$-stable.*

(b) *C_V determines $\gamma_{j,i}^V$ and $H_{j,i}^V$.*

(c) $H^V_{j,i} \in \{\mathrm{I}, \mathrm{II}, \mathrm{IV}\}$; *if* $H^V_{j,i} \in \{\mathrm{I}, \mathrm{II}\}$, *then* $\gamma^V_{j,i} = 0$; *if* $H^V_{j,i} = \mathrm{IV}$, *then* $\gamma^V_{j,i} = \lambda I_p$, $\lambda \in \mathbb{C}$.

Proof. (a) and (b) are clear; (c) holds since every other hypothesis would contradict (a). Q.E.D.

3.13 Definition. For any $p \in \mathbb{N}$ we set

$$\Gamma^p_{\mathbb{R}} = \{(x_1, ..., x_p) \in \mathbb{R}^p \mid x_1 > x_2 > \cdots > x_p > 0\}.$$

For any $\varepsilon = (\varepsilon_1, ..., \varepsilon_t) \in P_n$ we set

$$\Gamma^\varepsilon_{\mathbb{C}} = \{(z_1, ..., z_t) \in \mathbb{C}^t \mid z_1 > \cdots > z_t, \operatorname{Re} z_i > \operatorname{Re} z_{i+1} \text{ if } \varepsilon_i = 0,$$
$$\operatorname{Re} z_i = \operatorname{Re} z_{i+1} \text{ if } \varepsilon_i = 1\}.$$

We remark that $\Gamma^p_{\mathbb{R}}$ is open in $\mathbb{R}^p$; $\Gamma^\varepsilon_{\mathbb{C}}$ is naturally homeomorphic to some open set in $\mathbb{R}^q$, $q \leqslant 2t$. We are now able to describe the stratification.

3.14. Proposition. *Let T be a bundle of orbits and fix $V \subset T$ an orbit. Then T is the total space of a smooth trivial fiber bundle, whose base space is*

$$B_T = \Gamma^{\varepsilon_{V,0}}_{\mathbb{C}} \times \prod_{H^V_j = \mathrm{III}} \Gamma_{\mathbb{R}}^{\eta^V_j} \times \prod_{H^V_j = \mathrm{IV}} \Gamma^{\varepsilon_{V,j}}_{\mathbb{C}} \times \prod_{H^V_{i,j} = \mathrm{IV}} \mathbb{C}.$$

The fibre is $F_T = V$ and $\zeta_T = \{\Delta(W) \mid W \in O_n, W \subset T\}$ defines a global section.

The partition in bundles of orbits gives a good real semialgebraic stratification of $M(n)$, which is a refinement of the Jordan bundles stratification (see [1]*).*

Proof. Let $\varphi \colon T \to B_T \times F_T$ be defined as follows: For any $A \in T$, let $Q_A \in U(n)$ such that $A^0 = \Delta(O(A)) = {}^t\bar{Q}_A A Q_A$. We remark that A^0 defines in a natural way a unique element $\pi(A^0) \in B_T$, since every matrix of T has the "same type" of normal form (see Remark 3.9 and Lemma 3.12). Set $\varphi(A) = (\pi(A^0), Q_A \Delta(V) {}^t\bar{Q}_A)$. Then φ is well defined and a real analytic isomorphism, which gives a banalization of T.

Next, if $A^0 \in \zeta_T$ consider the versal deformation $A^0 + G_{A^0}(x)$ defined in Section 2. It follows from the construction that G_{A^0} does not depend on $A^0 \in \zeta_T$ and we can denote it by G_T. It is not difficult to define $\mathbb{R}$-linear functions G^1_T and G^2_T such that $\operatorname{Im} G^1_T = T\zeta_{T,A_0}$, $\operatorname{Im} G_T = \operatorname{Im} G^1_T \oplus \operatorname{Im} G^2_T$, so that $A^0 + G^2_T(x)$ is transversal to T in A^0 for any $A^0 \in \zeta_T$. Using this deformation it follows by straightforward computation that the partition in bundles is a good stratification of $M(n)$. The last assertion follows from Section 3,A. Q.E.D.

C. *Examples and Final Remarks*

1. We give an example. Let $n=8$, $\alpha=(2,1)$, $N=(3,2,1;2)$, $A^0=K(N,\alpha)=\Delta(V)$, $V=O(A^0)$. Then we have:

$$C_V=((1,2,1,1^*,1^*,2),(1,2^*,1,1_1^*,1_1,2),(1,1,1^*,1,1_1^*,1_1,2),$$
$$(1,1^*,1_1,1^*,1_1,1_1,2),(1,1_1,1_2,1_1,1_2,1_2,2));$$
$$\varepsilon_V=((1,1)),\varnothing,\varnothing,\varnothing,\varnothing); \qquad \dim V=57;$$
$$(H_j^V)=(\mathrm{IV},\mathrm{III},\mathrm{I},\mathrm{III},\mathrm{III}).$$

The versal deformation of A^0 is $A^0+G(x)$, where G is shown in Fig. 4. The white entries represent complex independent parameters; the black entries represent zero; α_j, $j=1,2,3$ represent real independent parameters.

Let T be the bundle to which V belongs. Then $\dim T=62$, $\dim B_T=5$, and G_T^2 is represented in Fig. 5, where β_1 and β_2 are real independent parameters and the white and black entries have the above meaning.

2. A Jordan bundle of orbits (see [1]) is the union of the elements of an equivalence class of O_n with respect to the equivalence relation: $V\sim W$ iff $N_V=N_W$. We denote by $S_{\tilde N}$ the Jordan bundle defined by $N\in N_n$; $S_{\tilde N}$ is a disjoint union of bundles. We remark that in the Jordan bundle $\varepsilon_{V,0}$ is not considered.

There exists a unique bundle of minimal dimension among those contained in $S_{\tilde N}$, precisely that containing $K(N,(h,h-1,\ldots,1))$, if N is as in Section 3,A. $S_{\tilde N}$ contains a unique bundle T_N of maximal dimension; $\dim T_N=\dim S_{\tilde N}$, and T_N is defined by the following properties:

(i) for any orbit $V\subset T_N$ and for any i,j we have $\{\gamma_{i,j}^V\}=\varnothing$ or $\gamma_{i,j}^V$ is a 1×1 minor and $H_{i,j}^V=\mathrm{IV}$;

(ii) $\varepsilon_{V,0}=(0,0,\ldots,0)$;

G =

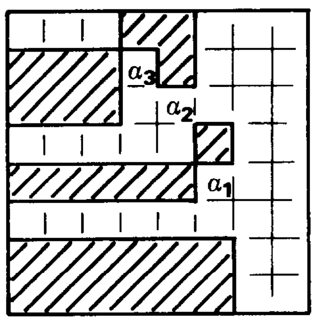

FIGURE 4

$G_T^2 =$

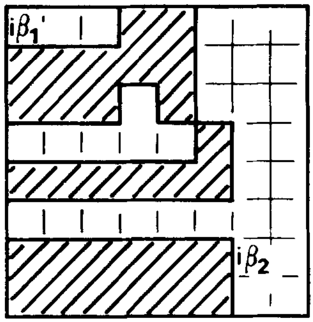

FIGURE 5

(iii) if $H_j^V \in \{\mathrm{I}, \mathrm{II}\}$ then the rank ρ_j of $m(r_{j-1}, s_{j-1}, \Delta(V))_{C^0_{V,j-1}}$ is maximal;

(iv) if $H_j^V = \mathrm{III}$ then $\eta_j^V = \rho_j$ and if $H_j^V = \mathrm{IV}$ then $\varepsilon_{V,j} = (0, \ldots, 0)$.

3. In $M(n)$ there exists a unique bundle of codimension 0, defined by:

$$N = (1; 1; \ldots; 1).$$

$$C = ((1^*, 1, \ldots, 1^*), (1_1, 1^*, \ldots, 1_1^*), (1_1, 1_1, 1^*, \ldots, 1_1^*), \ldots,$$
$$(1_1, 1_1, \ldots, 1_1, 1^*, 1_1^*), (1_1, 1_1, \ldots, 1_1)).$$

$$\varepsilon = ((0, \ldots, 0), \varnothing, \ldots, \varnothing).$$

There exist $n-1$ bundles of codimension 1, obtained taking N, C as above and

$$\varepsilon_1 = ((1, 0, \ldots, 0), \varnothing, \ldots, \varnothing)$$
$$\varepsilon_2 = ((0, 1, 0, \ldots, 0), \varnothing, \ldots, \varnothing)$$
$$\vdots$$
$$\varepsilon_{n-1} = ((0, \ldots, 0, 1), \varnothing, \ldots, \varnothing).$$

The bundles of codimension 2 are of two types:

(a) There are $n-1$ with $N = (1; \ldots; 1)$, obtained as follows

$$C_1 = ((1, 1^*, \ldots, 1^*), (1, 1_1, 1^*, \ldots, 1_1^*), \ldots, (1, 1_1, \ldots, 1^*, 1_1^*),$$
$$(1^*, 1_1, 1_1, \ldots, 1_1^*, 1_1), (1_1, 1_1, \ldots, 1_1)).$$

$$\varepsilon_1 = ((0, \ldots, 0)\, \varnothing, \ldots, \varnothing) = \varepsilon.$$

$$C_2 = ((1^*, 1, \ldots, 1^*), (1_1, 1, 1^*, \ldots, 1_1^*), (1_1, 1, 1_1, 1^*, \ldots, 1_1^*), \ldots,$$
$$(1_1, 1^*, 1_1, \ldots, 1_1^*, 1_1), (1_1, 1_1, \ldots, 1_1)).$$
$$\varepsilon_2 = \varepsilon_1.$$
$$\vdots$$
$$C_{n-1} = ((1^*, \ldots, 1^*), (1_1, 1^*, \ldots, 1_1^*), \ldots, (1_1^*, \ldots, 1_1, 1^*, 1_1), (1_1, \ldots, 1_1)).$$
$$\varepsilon_{n-1} = \varepsilon_1.$$

(b) There are $n-1$ with

$$N_1 = (1, 1; 1; \ldots; 1), \qquad N_2 = (1; 1, 1; 1; \ldots; 1), \ldots, \qquad N_{n-1} = (1; \ldots; 1; 1, 1).$$
$$C_i = C, \qquad \varepsilon_i = \varepsilon, \qquad 1 \leqslant i \leqslant n-1.$$

4. The preceding examples suggest an inductive way to order the bundles contained in $S_{\tilde{N}}$ with respect to increasing codimension.

Consider the conditions (i), (ii), (iii), (iv) of **2** which define the maximal dimension bundle T_N of $S_{\tilde{N}}$; if exactly one of these fails, you define a family $\mathscr{F}^1$ of bundles contained in $S_{\tilde{N}}$. For the family $\mathscr{F}^j$ you define in a natural way $(\mathrm{i})^j$, $(\mathrm{ii})^j$, $(\mathrm{iii})^j$, $(\mathrm{iv})^j$; if exactly one of these fails, you get the family $\mathscr{F}^{j+1}$.

5. A necessary condition that one bundle contains in its closure another is that the same holds for the Jordan bundles to which they belong. The problem to determine the bundles which are contained in the closure of a given one is not easy to solve in general, but the following principle should be true: among the bundles such that $\varepsilon_{\nu,i} = (0, \ldots, 0)$ or $\varepsilon_{\nu,i} = \varnothing$, every bundle T contains in its closure all the bundles of higher codimension belonging to a Jordan bundle which is contained in the closure of the Jordan bundle which contains T.

6. As we remarked, the normal form algorithm and hence the bundle stratification depend on the order relation α. The examples of this paragraph are based on the order relation defined in Remark 1.19, but many of the remarks we made above do not depend on this choice. The following principle should be true: the bundle stratifications of $M(n)$ defined by different order relations are isomorphic, so the singularities arising from the action of $U(n)$ on $M(n)$ are smoothly, uniquely determined.

On the questions related to **5** and **6** we want to return.

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