On Simultaneous Diagonalization of One Hermitian and One Symmetric Form

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ABSTRACT

It is remarked that if $A, B \in M_n(C)$, $A = A^t$, and $\bar{B} = B^t$, B positive definite, there exists a nonsingular matrix U such that (1) $\bar{U}^t B U = I$ and (2) $U^t A U$ is a diagonal matrix with nonnegative entries. Some related actions of the real orthogonal group and equations involving the unitary group are studied.

It is a basic result that two Hermitian (symmetric) bilinear forms on a finite-dimensional complex (real) vector space can be simultaneously diagonalized via a congruence transform, provided that at least one is positive definite. In this note we remark that the same is true for every pair consisting of a positive definite Hermitian form and a symmetric form on a finite dimensional complex vector space.

We did not find this result in several current textbooks of linear algebra and matrix theory. We were originally motivated to study this subject because it is one of the main tools in adapting the construction of [2] to the real orthogonal case (for a rough explanation see the final remark). For simplicity we shall express our statements in matrix language.

Let us fix the notation. $M_{nm}(K)$, $K = \mathbb{R}, \mathbb{C}$, is the space of real or complex $n \times m$ matrices; $M_n(K) = M_{nn}(K)$; $I = I_n \in M_n(K)$ denotes the identity; and we identify $M_{n1}(K)$ with K^n.

For every matrix A, $\bar{A}$ is its conjugate and A^t its transposte.

For $n \geqslant m$, we set $\mathrm{U}_{nm} = \{Q \in M_{nm}(\mathbb{C}) : \bar{Q}^t Q = I\}$; $\mathrm{U}_n = \mathrm{U}_{nn}$ is the unitary group, and $\mathrm{O}_n = \mathrm{U}_n \cap M_n(\mathbb{R})$ is the real orthogonal group. $\mathrm{S}_n = \{A \in M_n(\mathbb{C}) : A^t = A\}$ denotes the complex symmetric matrices.

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If X is a subset of $\mathbb{C}^n$, then $X^\perp = \{z \in \mathbb{C}^n : \bar{z}^t w = 0\ \forall w \in X\}$. We denote by G_{nk} the Grassmannian manifold of complex linear k-dimensional subspaces of $\mathbb{C}^n$.

Consider:

(A) The action on G_{nk} of O_n considered as a set of linear maps:

$$\mathrm{O}_n \times G_{nk} \ni (P, V) \to PV \in G_{nk}.$$

(B) The congruence action of U_n on S_n:

$$\mathrm{U}_n \times \mathrm{S}_n \ni (Q, A) \to Q^t A Q \in \mathrm{S}_n.$$

(C) For $A \in \mathrm{S}_m$, the equation $X^t X = A$, $X \in \mathrm{U}_{nm}$.

First we shall classify the orbits of the action A by completely elementary tools. This turns out to be equivalent to the classification of the orbits of B, when the action is restricted to the matrices A for which the equation C has solutions. Finally we shall complete the classification of B by a slightly more subtle topological argument, obtaining as a corollary a full discussion of the existence of solutions for C.

Let us denote by $G^s = G^s_{nk}$, $s = 0, \ldots, k$, the subset of G_{nk} defined by

$$G^s = \left\{ V : \dim_{\mathbb{C}} V \cap \overline{V} = s \right\}.$$

It is clear that G_{nk} consists of the union of the G^s and that each G^s is invariant under the action of O_n. Moreover:

(a) if $V, W \in G^s$, there exists $Q \in \mathrm{O}_n$ such that $Q(V \cap \overline{V}) = W \cap \overline{W}$;

(b) if $Q(V \cap \overline{V}) = W \cap \overline{W}$ for some $Q \in \mathrm{O}_n$, then $Q(V \cap \overline{V})^\perp = (W \cap \overline{W})^\perp$;

(c) $QV = W$ iff $QV^\perp = W^\perp$.

The above remarks show that it is enough to classify the orbits of O_{2p} in $G^0 \subset G_{2pp}$.

Fix a complex number $\lambda = a + ib$ ($a, b \in \mathbb{R}$, $i^2 = -1$, $b \neq 0$), and define M_p to be the set of matrices $A \in M_{2p}(\mathbb{R})$ such that:

(1) $\mathrm{Det}(A - tI) = (t - \lambda)^p (t - \overline{\lambda})^p$,

(2) $V_A = \mathrm{Ker}_{\mathbb{C}}(A - \lambda I) \in G_{2pp}$ (in fact $\in G^0$).

The function $M_p \ni A \to V_A \in G^0$ is a natural bijection, and it is immediate that:

$M_p \ni A = P^t B P$ for some $P \in \mathrm{O}_{2p}$ iff $PV_B = V_a$.

Thus it is sufficient to classify the orbits of the conjugacy action of O_{2p} on M_p.

As before, let $\lambda = a + ib$, $b \neq 0$, be a fixed complex number. Let us define for every $c \in (0,1]$

$$A(c) = A(\lambda, c) = \begin{pmatrix} a & bc^{-1} \\ -bc & a \end{pmatrix} \in M_1,$$

and for every $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_p \in (0,1]$ define

$$A(c_1,\ldots,c_p) = \left[A(c_1),\ldots,A(c_p)\right] \in M_p,$$

that is, the block-diagonal matrix having the $A(c_j)$ along the main diagonal. Our aim is to show that in fact the $A(c_1,\ldots,c_p)$ are the "normal form representatives" for the conjugacy action of O_{2p} on M_p.

Let $(e_1,\ldots,e_{2p})$ be the natural basis of $\mathbf{C}^{2p}$. Set

$$z_j = e_{2j-1} + ic_je_{2j} \text{ and } w_j = e_{2j} + ic_je_{2j-1}, \qquad j = 1,\ldots,p.$$

Note that

$$\left(\bar{z}_j^tz_j\right)^{-1/2} = \left(\bar{w}_j^tw_j\right)^{-1/2} = \left(1+c_j\right)^{-1/2} \stackrel{\text{def}}{=} P(c_j)$$

and that

$$\left(P(c_1)z_1,\ldots,P(c_p)z_p, P(c_1)w_1,\ldots,P(c_p)w_p\right)$$

is a unitary basis of $\mathbf{C}^{2p}$. Let $F = F_p$ be the unitary matrix representing this change of basis; by a short computation we have:

Lemma 1.

$$F^{-1}A(c_1,\ldots,c_p)F = \hat{A}(c_1,\ldots,c_p)$$

where

$$\hat{A}(c_1,\ldots,c_p) = \begin{pmatrix} \lambda I_p & bS(c_1,\ldots,c_p) \\ 0 & \bar{\lambda} I_p \end{pmatrix}$$

and

$$S(c_1,\dots,c_p) = \left[S(c_1),\dots,S(c_p)\right],$$
$$S(c_j) = \left(1 - c_j^2\right)c_j^{-1}.$$

We can now state our first result.

PROPOSITION 2. *For every $A \in M_p$ there exist $Q \in \mathrm{O}_{2p}$ and a uniquely determined $A(c_1,\dots,c_p)$ such that $Q^{-1}AQ = A(c_1,\dots,c_p)$.*

The proof splits into the following two lemmas.

LEMMA 3. *Proposition 2 still holds if one replaces $Q \in \mathrm{O}_{2p}$ with $P \in \mathrm{U}_{2p}$.*

Proof. By the previous lemma, in order to prove the existence part it is enough to show that $Z^{-1}AZ = \hat{A}(c_1,\dots,c_p)$ for some $Z \in \mathrm{U}_{2p}$. After choosing any unitary basis of V_A and of $V_A^{\perp}$ respectively, we may assume that

$$A_1 = W^{-1}AW = \begin{pmatrix} \lambda I_p & D \\ 0 & \bar{\lambda} I_p \end{pmatrix}, \qquad W \in U_{2p}.$$

Let $D = TH$ be the polar decomposition of D ($T \in U_p$, $\bar{H} = H^t$, H positive semidefinite and uniquely determined). Let $N \in U_p$ be such that $N^{-1}HN$ is equal to the diagonal matrix $[t_1,\dots,t_p]$, $t_j \geqslant t_{j+1} \geqslant 0$. Setting $R = [T^{-1}N, N]$, clearly $R^{-1}A_1R = \hat{A}(c_1,\dots,c_p)$, where $t_j = S(c_j)$ (note that $S(c)$ is a decreasing homeomorphism between $(0,1]$ and $[0,+\infty)$). The uniqueness of $(c_1,\dots,c_p)$ follows from the uniqueness of the Hermitian part of the polar decomposition and from the fact that if $D' = BDC$, $B, C \in U_p$, then $D' = T'H'$ and $H' = C^{-1}HC$. ■

LEMMA 4. *Let $A, B \in M_n(\mathbb{R})$; then the following facts are equivalent:*

(a) *there exists $U \in \mathrm{U}_n$ such that $U^{-1}AU = B$;*

(b) *there exists $P \in \mathrm{GL}(n,\mathbb{R})$ such that $P^{-1}AP = B$ and $P^{-1}A^tP = B^t$;*

(c) *there exists $P \in \mathrm{GL}(n,\mathbb{R})$ such that $P^{-1}A_0P = B_0$ and $P^{-1}A_1P = B_1$, where A_0, A_1 (B_0, B_1) are respectively the symmetric and the skew-symmetric parts of A (B);*

(d) *there exists $Q \in \mathrm{O}_n$ such that $Q^tAQ = B$.*

Proof. That (d) implies (a) and that (b) is equivalent to (c) are trivial.

(a) implies (b): let $U = V + iW$, V and W real. Since A and B are real matrices, we get immediately from

$$\bar{U}^t A U = B,$$

$$U^t A^t \bar{U} = B^t$$

the relations (0)

$$A(V+iW) = (V+iW)B \qquad \text{and hence} \quad AV = VB \text{ and } AW = WB,$$

$$A^t(V-iW) = (V-iW)B^t \qquad \text{and hence} \quad A^tV = VB^t \text{ and } A^tW = WB^t.$$

On the other hand, there exists $t \in \mathbf{R}$ such that $P = V + tW$ is nonsingular. Then it follows immediately from (0) that $P^{-1}AP = B$ and $P^{-1}A^tP = B^t$.

(c) implies (d): Let $P = HR$ be the polar decomposition of P (H symmetric and positive definite). It is easy to prove in succession the equations

$$RB_0R^t = H^{-1}A_0H = A_0 \quad \text{and} \quad RB_1R^t = H^{-1}A_1H = A_1.$$

Thus the proof of the lemma and also of Proposition 2 is complete. ■

We can call each $A(c_1,\dots,c_p)$ the normal-form representative for the orbit. By translating everything into terms of the action on G^0 we can say also that the eigenspace V_λ of $A(c_1,\dots,c_p)$—that is, the subspace having as a unitary basis $(P(c_1)z_1,\dots,P(c_p)z_p)$ defined before Lemma 1—is the normal representative of the corresponding orbit. Notice that the restriction to V of the canonical symmetric bilinear form $[(x,y) = x^t y]$ is represented with respect to that basis by the diagonal matrix $T = [t_1,\dots,t_p]$, $t_j = P(c_j)^2(1-c_j^2) \in [0,1)$. Summarizing:

Proposition 5. *For each $V \in G_{nk}$ there exist a unitary basis of V and a uniquely determined diagonal matrix $T = [t_1,\dots,t_k]$ $t_j \in [0,1]$, $t_j \geqslant t_{j+1}$, which represents the canonical symmetric bilinear form with respect to that basis. V and W admit the same T iff there exists $Q \in \mathrm{O}_n$ such that $QV = W$. The first $t_1,\dots,t_s$ are equal to 1 iff $\dim_{\mathbf{C}} V \cap \bar{V} = s$.*

In this way we have obtained the claimed simultaneous diagonalization theorem only for those symmetric matrices A for which the equation in (C) above has a solution. We want now to generalize this result to any symmetric matrix, that is, we shall prove the

PROPOSITION 6. *Let $A = A^t \in M_n(\mathbb{C})$. Then there exist $U \in \mathrm{U}_n$ and $T = [t_1, \dots, t_n]$, $t_j \in \mathbb{R}$, $t_j \geq t_{j+1} \geq 0$, such that $U^t A U = T$. The t_j are uniquely determined to be the eigenvalues of the Hermitian part H of the polar decomposition $A = ZH$ of A.*

An equivalent (perhaps more precise) statement is the following

THEOREM 7. *Let $A, B \in M_n(\mathbb{C})$ be such that $B = \bar{B}^t$ and is positive definite, and $A = A^t$. Then there exists a basis $(b_1, \dots, b_n)$ of $\mathbb{C}^n$ such that:*

(a) $b_i^t A b_j = \bar{b}_i^t B b_j = 0$ *for every* $i \neq j$;
(b) $\bar{b}_i^t B b_i = 1$ *for every* i;
(c) $b_i^t A b_i = t_i$, $t_i \in \mathbb{R}$, $t_i \geq 0$ *for every* i.

Moreover the t_i are uniquely determined up to permutation.

Proof. After choosing any orthonormal basis for B, we may assume that $B = I$. Furthermore note that it is enough to prove the theorem when A is nonsingular: in fact, let A be any symmetric matrix, and set $V_0 = \{v \in \mathbb{C}^n;\ v^t A w = 0 \text{ for each } w \in \mathbb{C}^n\}$. Of course A (as a bilinear form) is not degenerate on $V_0^\perp$. Let $(b_1, \dots, b_s)$ be a basis of $V_0^\perp$ satisfying the required properties, and fix any unitary basis $(b_{s+1}, \dots, b_n)$ of V_0. Then the basis $(b_1, \dots, b_n)$ of $\mathbb{C}^n$ works. Thus we may assume A to be nonsingular. We now use induction on n; that is, if $p(k)$ is the statement that part (a) of the theorem holds for all $k \times k$ matrices A_0, B_0 satisfying the hypotheses, we shall prove:

(i) $p(2)$ [$p(1)$ is trivial].
(ii) $p(2k)$ implies $p(2k+2)$, $k \geq 1$.
(iii) $p(2k)$ implies $p(2k+1)$, $k \geq 1$.

Step 1: Proof of $p(2)$. Let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

We want to find a nontrivial solution $z, w \in \mathbb{C}^2$ for the problem

$$\begin{aligned} &\text{(a)} \qquad \bar{z}^t w = 0, \\ &\text{(b)} \qquad z^t A w = 0. \end{aligned} \tag{0}$$

Set $\alpha = w_1/w_2$, where $w^t = (w_1, w_2)$, and consider α as an element of $\mathbb{P}_1\mathbb{C}$. It follows from (a) that $\bar{\alpha} = -z_2/z_1 \in \mathbb{P}_1\mathbb{C}$, $z^t = (z_1, z_2)$, and from $(z_1 w_2)^{-1} z^t A w = 0$ that

$$\bar{\alpha} = \frac{a\alpha + b}{b\alpha + c}. \tag{00}$$

It is enough to find a solution for (00). Setting $\alpha = x + iy$, $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, $B = (b_1, b_2)^t$, we get from (00)

$$\left(\|\alpha\|^2 - 1\right)B = MD, \tag{000}$$

where

$$D = (x, y)^t, \qquad M = \begin{pmatrix} a_1 - c_1 & -a_2 - c_2 \\ a_2 - c_2 & a_1 + c_1 \end{pmatrix}.$$

In the solution of (000) there are three possibilities:

(1) $\det M = 0$; then there exists $D \in \mathbb{R}^2$ such that $MD = 0$ and $\|D\| = 1$. D is a solution.

(2) $B = 0$; then $D = (1, 0)^t$ is a solution.

(3) $\det M \neq 0$, $B \neq 0$. In this case there exists $H \in \mathbb{R}^2$ such that $MH = B$. We look for a solution of the form tH, $t \in \mathbb{R}$, $t \neq 0$. It is enough to solve the equation $q(t) = \|H\|^2 t^2 - t - 1 = 0$, which is always possible.

Step 2. Denote by $f^{n,k}$ the involution defined on G_{nk} by $f^{n,k}(V) = \bar{V}$, and by $L^{n,k}$ its Lefschetz number (see for instance [3]).

CLAIM. *Assume $L^{n-k,k} \neq 0$, $p(k)$, and $p(n)$. Then $p(n+k)$ is true.*

Proof of the claim. For every subset W of $\mathbb{C}^{n+k}$ define

$$W^T = \{ v \in \mathbb{C}^{n+k} : v^t A w = 0 \text{ for each } w \in W \}.$$

Define also $f: G_{n+k\,k} \to G_{n+k\,k}$ by $f(V) = V^{\perp T}$. Suppose that f has a fixed point V_0. If $\mathscr{B}$ is a good basis of V_0 [in the sense that it satisfies $p(k)$ for the restriction to V_0 of the bilinear form $v^t A w$] and if $\mathscr{B}'$ is such a basis for $V_0^{\perp} = V_0^T$, then $\mathscr{B} \cup \mathscr{B}'$ is a good basis of $\mathbb{C}^{n+k}$. On the other hand, denote by $u_1, \dots, u_{n+k}$ the roots of the polynomial $\det(uA - (1-u)I)$, and choose a continuous arc $q: [0,1] \to \mathbb{C} - \{u_1, \dots, u_{n+k}\}$ such that $q(0) = 0$, $q(1) = 1$. Set

$A(s) = q(s)A + [1 - q(s)]I$. For $s \in [0,1]$, $A(s)$ is symmetric nonsingular, and $A(0) = I$, $A(1) = A$.

As before, set $W^{T(s)} = \{v \in \mathbf{C}^{n+k} : v^t A(s) w = 0 \ \forall w \in W\}$, and define $f_s(V) = V^{\perp T(s)}$. Then f_s gives a homotopy between $f_1 = f$ and f_0, where $f_0(V) = V^{\perp T(0)} = f^{n+k,k}(V)$. By hypothesis the Lefschetz number of f is nonzero, and hence a fixed point V_0 actually exists. The claim is proved. ■

Step 3. *For $n \geq 1$, $L^{2n+2,2} = n+1$ and hence $p(2n) \Rightarrow p(2n+2)$.*

Proof of the assertion. Write G for G_{2n+22} and let $e_1, \ldots, e_{2n+2}$ be a basis of $\mathbf{C}^{2n+2}$. Every $W \in G$ can be represented by a matrix $(v_{ij}) \in M_{2n+2,2}(\mathbf{C})$ having rank 2. Let

$$p_{b_0 b_1}(W) = \det \begin{pmatrix} v_{b_0 1} & v_{b_0 2} \\ v_{b_1 1} & v_{b_1 2} \end{pmatrix}, \qquad 1 \leq b_0 < b_1 \leq 2n+2$$

be the Plückerian coordinates on G. Setting $(b_1, b_2) \not< (a_1, a_2)$ whenever $1 \leq a_1 < a_2 \leq 2n+2$, $1 \leq b_1 < b_2 \leq 2n+2$, and $b_1 > a_1$ or $b_2 > a_2$, let

$$[a_1, a_2] = \left\{ W \in G : p_{b_1 b_2}(W) = 0 \text{ for each } (b_1, b_2) \not< (a_1, a_2) \right\}.$$

It is well known (see [4]) that:

(1) $[a_1, a_2] = \{W \in G : W \cap A_1 \neq \varnothing \text{ and } W \subset A_2\}$, where A_1 (A_2) is the subspace of $\mathbf{C}^{2n+2}$ generated by $e_1, \ldots, e_{a_1}$ ($e_1, \ldots, e_{a_2}$);

(2) the interior of $[a_1, a_2]$ is homeomorphic to $\mathbf{C}^d$, $d = a_1 + a_2 - 3$;

(3) the homology of G is given by

$$H_{2j+1}(G, \mathbf{Z}) = 0, \qquad H_{2j}(G, \mathbf{Z}) = \bigoplus_{a_1 + a_2 - 3 = j} \mathbf{Z}[a_1, a_2].$$

Since $p_{b_1 b_2}(\overline{W}) = \overline{p_{b_1 b_2}(W)}$, then $f^{2n+2,2}([a_1, a_2]) = [a_1, a_2]$. Moreover, via the identification of the interior of $[a_1, a_2]$ with $\mathbf{C}^d$, we see that $f^{2n+2,2}$ actually acts as complex conjugation. Thus the map preserves the orientation of the interior of $[a_1, a_2]$ iff d is an even number. It follows that the matrix representing

$$\left(f_{2j}^{2n+2,2} \right)_* : H_{2j}(G, \mathbf{Z}) \to H_{2j}(G, \mathbf{Z})$$

is equal to $I_{q(j)}$ if j is even or to $-I_{q(j)}$ if j is odd, where $q(j)$ is the number of cells of dimension $2j$.

It is easy to verify that $q(2j)=q(2j+1)$ if $0 \leqslant j \leqslant n-1$ and $q(2j-1)=q(2j)$ if $n+1 \leqslant j \leqslant 2n$. So we get

$$L^{2n+2,2} = \sum_{i=0}^{8n} (-1)^i \operatorname{tr}\left(f_i^{2n+2,2}\right)_* = n+1. \qquad \blacksquare$$

Step 4. $L^{2n+1,1}=1$ *for* $n \geqslant 1$, *and hence* $p(2n)$ *implies* $p(2n+1)$.

In the same way as before, we get

$$G_{2n+11} = \mathbf{P}_{2n}\mathbf{C}, \qquad q(j)=1 \quad \text{for } 0 \leqslant j \leqslant 2n,$$

and finally $L^{2n+1,1}=1$.

Thus we have shown that a basis certainly exists such that parts (a) and (b) of the statement of the theorem is satisfied. To achieve also part (c), it is now sufficient to multiply each element of such a basis, v_j say, by a suitable $\exp(is_j)$. The uniqueness of $(t_1,\dots,t_n)$ follows immediately from the same argument of the end of Lemma 3. $\blacksquare$

The following corollary is a consequence of Proposition 5 and the previous theorem:

COROLLARY 8. *The equation* $U^tU=A$, $U \in \mathrm{U}_{nm}$, $A=A^t$, *always has a solution, provided that the eigenvalues of the Hermitian part of the polar decomposition of* A *belong to* $[0,1]$.

REMARK 9. The stabilizers of the normal-form representatives for the above actions and hence the "number of solutions" in the last corollary can be easily computed. Let $T=[t_1I_{s_1},\dots,t_kI_{s_k}]$, as in Proposition 6. Clearly,

$$\begin{aligned} st(T) &= \left\{ U \in \mathrm{U}_n : U^tTU = T \right\} \\ &= \left\{ [Q_1,\dots,Q_k] : Q_i \in \mathrm{O}_{s_i} \text{ if } t_i \neq 0,\ Q_k \in \mathrm{U}_{s_k} \text{ iff } t_k = 0 \right\}. \end{aligned}$$

Set (with the notation of Lemma 1)

$$\mathrm{St}_R(\hat{A}) = \mathrm{St}_R\left(\hat{A}(c_1,\dots,c_p)\right) = \left\{ Q \in \mathrm{U}_{2p} : Q^{-1}\hat{A}Q = \hat{A} \text{ and } FQF^{-1} \in \mathrm{O}_{2p} \right\}.$$

It is almost immediate that:

(a) $\mathrm{St}(A) = \mathrm{St}(A(c_1,\ldots,c_p)) = \{Q \in \mathrm{O}_{2p} : Q^t A Q = A\} = F\,\mathrm{St}_R(\hat{A})F^{-1}$,
(b) $\mathrm{St}_R(\hat{A}) = \{[Q_1,\ldots,Q_k,\bar{Q}_1,\ldots,\bar{Q}_k] : [Q_1,\ldots,Q_k] \in \mathrm{St}(T)$ and

$$T = \left[P(c_1)^2(1-c_1)^2,\ldots,P(c_p)^2(1-c_p)^2\right]\}.$$

Finally, a less immediate but straightforward computation shows that for each $R = [Q_1,\ldots,Q_k,\bar{Q}_1,\ldots,\bar{Q}_k] \in \mathrm{St}_R(\hat{A})$, we have $FRF^{-1} = [f(Q_1),\ldots,f(Q_k)]$, where if $C = (c_{ij})_{i,j=1,\ldots,s} \in M_s(\mathbb{C})$, then $f(C) = B = (B_{ij})_{i,j=1,\ldots,s} \in M_{2s}(\mathbb{R})$ and

$$B_{ij} = \begin{pmatrix} \operatorname{Re} c_{ij} & \operatorname{Im} c_{ij} \\ -\operatorname{Im} c_{ij} & \operatorname{Re} c_{ij} \end{pmatrix}, \qquad c_{ij} = \operatorname{Re} c_{ij} + i \operatorname{Im} c_{ij}.$$

FINAL REMARK. In [2] we developed the following program:

(a) To construct an algorithm which yields:

(1) a map $J : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $J(B) = J(C)$ iff $B = P^{-1}CP$ for some $P \in U_n$ and $J \circ J = J$;
(2) the stabilizer of each $J(B)$.

(b) To construct a second algorithm (in fact parallel to the first one) which yields a *versal deformation of every $J(B)$ with the minimum number of parameters* (in the sense of [5]) of the form

$$J(B) + L(a_1,\ldots,a_r), \qquad a_j \in \mathbb{R}, \quad L \text{ real linear}.$$

Moreover, by collecting in the same *bundle* all matrices B which reach $J(B)$ by the same formal steps of the algorithm, we obtain a "good" stratification of $M_n(\mathbb{C})$ in trivial fiber bundles, each admitting the restriction of J as a global "smooth" section. Versal deformations can be used to study the diagrams of bifurcation of this stratification.

A similar program can be developed also in the real case by replacing U_n with O_n. It is not the purpose of this note to give the details of this new construction; we limit ourself to the following remarks:

(1) We obtain the starting step of the first algorithm by means of the above proposition 2. In fact it is easy to prove by induction that for every $B \in M_n(\mathbb{R})$ there exists $P \in \mathrm{O}_n$ such that $P^tBP = B_1$ is a block upper triangular matrix of a "distinguished type." This means that B_1 has along the main

diagonal either:

a distribution $rI_{k_1},\dots,rI_{k_s}$ for every real eigenvalue r, or
a distribution $A(\lambda, c_1^1,\dots,c_{k_1}^1),\dots,A(\lambda, c_1^s,\dots,c_{k_s}^s)$ for every eigenvalue $\lambda = a+ib$ with $b>0$.

Here $k_j \geqslant k_{j+1} \geqslant 1$, and they are intrinsically determined (in fact by the similarity class of B); the c_j^i depend only on the congruence orbit of O_n containing B. Note that this is, for instance, a generalization of the classical normal form of the real orthogonal matrices up to orthogonal change of basis.

(2) The second step is obtained by acting on B_1 with $\mathrm{St}_1(B)$ (the subgroup of O_n which preserves the distinguished block-upper triangular type of B_1) and performing a certain "elementary operation" among a well-defined finite list in order to obtain a more specialized form B_2.

(3) The third step is given by operating on B_2 by $\mathrm{St}_2(B)$ [the subgroup of $\mathrm{St}_1(B)\dots$].

(4) After a finite number of steps the process stabilizes[1] and we define $J(B)=B_\infty$.

(5) As it is evident from Remark 9, $\mathrm{St}_1(B)$ [and *a fortiori* $\mathrm{St}_k(B)$] can be a sort of "twisted" subgroup of O_n. To avoid this, define $\hat{B}_1 = G^{-1}B_1G$, where G is the unitary block-diagonal matrix having along the main diagonal a block I_k, $k=k_1+\cdots+k_s$, corresponding to every real eigenvalue and a block $[F_{k_1},\dots,F_{k_s}]$ (with the notation of Lemma 1) corresponding to every eigenvalue $\lambda = a+ib$, $b>0$, of B_1. It is clear that $\mathrm{St}_R^1(B) \overset{\text{def}}{=} G^{-1}\mathrm{St}_1(B_1)G$ is now a reasonably "tame" subgroup of U_n. So it is convenient actually to perform the steps of the algorithm by acting with $\mathrm{St}_R^1(B)$, and so on.

(6) It is now evident how certain auxiliary classifications (such as that of the orbits of the congruence action of U_n on S_n, which is the topic of this note) may arise naturally as a tool to list the elementary operations of the algorithm mentioned above. We end by noting that the fixed-point argument of Theorem 8 can be applied to get similar auxiliary classifications. For instance one can easily prove the following

PROPOSITION. *Let $B \in M_n(\mathbb{C})$, $B = -B^t$. Then there exists $U \in \mathrm{U}_n$ such that $U^tBU = [r_1S_{k_1},\dots,r_sS_{k_s}]$, where $r_i \in \mathbb{R}$, $r_i \geqslant 0$, $(r_1,\dots,r_s)$ is uniquely determined up to permutation, and*

$$S_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}.$$

[1] In fact we are able to prove point (4) for most matrices, and some special cases remain to be studied.

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