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Effectiveness—non effectiveness in semialgebraic and PL geometry

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0. Introduction and statement of the main results

Working in real semialgebraic geometry one could easily have the feeling that the main facts can be realized by algorithms (e.g. path connecting points, triangulations, good stratifications ...) and the main trouble is to construct reasonably fast algorithms: in fact it is this (in principle) constructive nature of semialgebraic geometry and the existence of powerful computers which make it interesting in view of concrete applications.

On the other hand, Nabutovsky's examples of non recursive functions associated to natural semialgebraic constructions (see [N]) tell us that there are some conceptual limitations to the above attitude.

In this paper we consider some further examples of effectiveness-non effectiveness results in semialgebraic and PL geometry: our starting point was the study of Shiota-Yokoi solution of the subanalytic (hence semialgebraic) *Hauptvermutung* ([SY]) and the attempt to make it effective.

Before starting we need to introduce some notations.

We shall use systematically the notation:

$$[A_1, \dots, A_h] \to [B_1, \dots, B_k]$$

to mean that there exists an algorithm accepting the "objects" $A_1, \dots, A_h$ as input and producing $B_1, \dots, B_k$ as output.

With the same meaning we shall say also that "$B_1, \dots, B_k$ *can be effectively constructed, starting from* $A_1, \dots, A_h$".

Similarly we shall write:

$$[A_1, \dots, A_h] \not\to [B_1, \dots, B_k]$$

to mean that "$B_1, \dots, B_k$ *cannot be effectively constructed starting from* $A_1, \dots, A_h$", that is there doesn't exist any algorithm producing $B_1, \dots, B_k$ from $A_1, \dots, A_h$.

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Some time we shall consider also systems:

$$\begin{cases} [A_1, \ldots, A_h] \to [B_1, \ldots, B_k] \\ [C_1, \ldots, C_s] \to [D_1, \ldots, D_r] \end{cases}$$

with the obvious meaning.

A *semialgebraic set* is any subset $X \subseteq \mathbb{R}^n$ (some n) with a *presentation* of the form:

$$X = \bigcup_{i=1}^{h} \bigcap_{j=1}^{k_i} \{f_{ij} *_{ij} 0\}$$

where each $f_{ij} \in \mathbb{R}[X_1, \ldots, X_n]$, $*_{ij} \in \{ =, > \}$.

For any such presentation set:

$$r = \sum k_i \qquad d = \sup\{\text{degree of } f_{ij}\} .$$

By the notation $X \in \mathscr{A}(n, p, q)$, $n, p, q \in \mathbb{N}$, *we mean that* X *is a semialgebraic set in* $\mathbb{R}^n$ *with a given presentation such that* $r \leqq p$, $d \leqq q$.

A semialgebraic map $f\colon X \to Y$ between semialgebraic sets is a continuous map with semialgebraic graph Γ_f.

If K *is a simplicial complex in* $\mathbb{R}^n$ *(some* n*),* $P = |K|$ *is the polyhedron in* $\mathbb{R}^n$ *triangulated by* K*;* $\# K$ *denotes the number of simplexes in* K*. A PL* m*-ball is any polyhedron* $B \subseteq \mathbb{R}^n$ *PL isomorphic to the standard* m*-simplex* Δ_m*. By* K_Δ *we denote the standard triangulation of* Δ_m*.*

Let us state now some of the results of the paper.

Theorem 1 (strong effective semialgebraic Hauptvermutung for dimension $\leqq 3$; shortly: SEH_m $m \leqq 3$). *Let* K, L *be finite simplicial complexes in* $\mathbb{R}^n$. $\# K$, $\# L \leqq k$. *Let* $f\colon |K| \to |L|$ *be a semialgebraic homeomorphism with* $\Gamma_f \in \mathscr{A}(n^2, p, q)$. *Assume* $m = \dim |K| = \dim |L| \leqq 3$. *Then*

$$\begin{cases} [K, L, f] \to [g\colon K' \to L'] \\ [n, k, p, q] \to [D] \end{cases}$$

where K' *is a simplicial subdivision of* K, L' *of* L, g *is a simplicial isomorphism and* $\# K'$, $\# L' \leqq D \in \mathbb{N}$ *(see Theorem* 3.5*).*

Theorem 2 (weak effective semialgebraic Hauptvermutung for any dimension; shortly: WEH_m $\forall m$). *With the same notation as in Theorem* 1 *but with arbitrary* $m = \dim|K| = \dim|L|$, *one has*

$$[K, L, f] \to [g\colon K' \to L']$$

(see 3.3*).*

In the PL setting we have the following *effective trivialization of PL ball:*

Theorem 3. *Let* $B = |K|$ *be a triangulated PL* m*-ball,* $|K| \subseteq \mathbb{R}^m$, $\# K \leqq d$, $m \leqq 3$. *Then*

$$\begin{cases} [K] \to [g\colon K' \to L] \\ [d] \to [D] \end{cases}$$

where K' *is a simplicial subdivision of* K, L *is a subdivision of the standard triangulation of* Δ_m, g *is a simplicial isomorphism,* $D \in \mathbb{N}$ *and* $\# K' \leqq D$ *(see Problem* 2.2 *and Proposition* 2.23 *and* 2.24*).*

Remark. If $m \leqq 2$ we are able to omit the hypothesis $|K| \subseteq \mathbb{R}^m$, that is we may assume $|K| \subseteq \mathbb{R}^n$, $n \neq m$ in general.

Theorem 4. *With the same notation as in Theorem 3, if m is arbitrary and $|K| \subseteq \mathbb{R}^n$, with $m \neq n$ in general, then*

$$[K] \to [g: K' \to L]$$

(*see Problem* 2.1 *and Corollary* 2.17).

Remark. Theorems 3 and 4 are, in some sense, PL counterparts of Theorems 1 and 2; in fact they shall be employed to prove the first ones.

The $\mathrm{SEH_m}$ for $m \geqq 4$ is still open (to us). *However the "PL counterpart" is false*, that is we have the following *non effectiveness* result for trivialization of PL-balls of higher dimensions.

Theorem 5. (*a*) *For every $m, n, d \in \mathbb{N}$ there exists $s = s(m, n, d) \in \mathbb{N}$ such that for every PL m-ball $B = |K| \subseteq \mathbb{R}^n$, triangulated by K with $\# K \leqq d$, there exists a simplicial isomorphism $g: K' \to L$ where K' is a subdivision of K, L of K_Δ, and $\# K' \leqq s$.*

(*b*) *For every $m \geqq 6$*

$$[m, d] \not\to [D]$$

such that $D \in \mathbb{N}$ and $s(m, d) = s(m, m, d) \leqq D$ (i.e. $s(m, d)$ cannot be bounded by any recursive function of m, d) (see Corollary 2.18 *and Remark* 2.21).

In the paper we are concerned with the opposition 'effectiveness-non effectiveness"; we shall not consider the problem of the complexity of the algorithms (when they exist).

We shall assume the existential solution of the (semialgebraic) Hauptvermutung by Shiota-Yokoi; however the eventual effectiveness (strong or weak) is not a consequence of their proof, not even in low dimensions: we shall discuss shortly the reason of it in (3.10).

For basic facts about semialgebraic geometry we refer to [BCR] or to [BR] (in the second one some effectiveness questions are explicitly pointed out).

For basic facts about PL-geometry we refer to [RS] and [Hu].

1. Recall on a result of Novikov

Integrating Smale's h-cobordism theorem (and its consequences) with the results of Adyan, Markov, on the "non-decidability of the triviality for finitely presented groups", Novikov (see [VKF]) proved the following result (in a formulation which is more convenient to our aim).

Theorem (Novikov). *For every $m \geqq 6$ one can construct a sequence $\mathcal{M} = \{M_S = |K_S|\}_{S \in \mathbb{N}}$ of triangulated PL submanifolds of $\mathbb{R}^m$, $\dim M_S = m$, $\partial M_S \neq \varnothing$ such that*

(*i*) $$[|K_S| \in \mathcal{M}] \not\to [t]$$

where $t \in \{yes, no\}$ and t is "yes" if M_S is a PL m-ball, t is "no" otherwise.

(*ii*) $$[|K_S| \in \mathcal{M}] \not\to [t]$$

where $t \in \{yes, no\}$ and t is "yes" if ∂M_S is a PL $(m-1)$-sphere, i.e. PL isomorphic to $\partial \Delta_m$), "no" otherwise.

Novikov's theorem shall be one of the main tools in order to prove Theorem 5 of the Introduction.

The idea to use it (in fact a smooth analogous of it) in order to produce examples of non recursive functions in semialgebraic geometry is due to Nabutovsky (see [N]).

2. On the effective trivialization of PL-balls

We shall adopt the notation of [RS], in particular: we distinguish between m-cells and *PL* m-balls; if A, B are cells $A \leqq B$ means that A is a *face* of B; if K and L are (cell or simplicial) complexes, $L \lhd K$ means that L is a *subdivision* of K; $\mathrm{St}(A, K)$ denotes the *star* of A in K.

By definition a PL m-ball is PL isomorphic to the standard m-simplex Δ_m. We denote by K_Δ the standard triangulation of $\Delta = \Delta_m$.

By a *trivialization* of B we mean a simplicial isomorphism $g: K \to L$ where $B = |K|$ and $L \lhd K_\Delta$.

Problem 2.1 (WT_m: weak effective trivialization problem in dim m). *It asks for*

$$[K] \to [g: K' \to L]$$

where $B = |K|$ is any triangulated PL m-ball, $K' \lhd K$, $L \lhd K_\Delta$, g is a simplicial isomorphism.

Problem 2.2 (ST_m: strong effective trivialization problem in dim m). *It asks for*

$$\begin{cases} [K] \to [g: K' \to L] \\ [d] \to [D] \end{cases}$$

where $B = |K|$ is any triangulated PL m-ball, with $\# K \leqq d$, $K' \lhd K$, $L \lhd K_\Delta$, g is a simplicial isomorphism $D \in \mathbb{N}$ and $\# K' \leqq D$.

We shall see, in particular, that WT_m has solution for every m; on the contrary ST_m has not solution for $m \geqq 5$. In low dimension we have some stronger results.

(*A*) *Recall of some simple facts of PL-geometry*

We shall use systematically the following well-known facts which we state with emphasis on effectiveness.

Lemma 2.3. *For any cell-complex K with a number of cells $\# K \leqq d$ one has*

$$\begin{cases} [K] \to [H] \\ [d] \to [D] \end{cases}$$

where H is a simplicial complex such that $H \lhd K$, $\# H \leqq D \in \mathbb{N}$, the 0-skeleton H^0 and K^0 coincide.

Proof. See [RS], Proposition 2.9. □

Lemma 2.4. *If K, M are two simplicial complexes such that $|K| = |M|$, $\# K$, $\# M \leqq d$ then one has:*

$$\begin{cases} [K, M] \to [H] \\ [d] \to [D] \end{cases}$$

where H is a simplicial complex such that $H \lhd K, M$, $\# H \leqq D \in \mathbb{N}$.

Proof. Apply (2.3) to the cell complex $K \cap M$. □

Two cell-complexes K, L are said *formally isomorphic* if there exists a bijection $q: K \to L$ such that $\forall \sigma, \tau \in K$ $\sigma \leqq \tau \Rightarrow q(\sigma) \leqq q(\tau)$.

Lemma 2.5. *If K, L are cell-complexes, $q: K \to L$ is a formal isomorphism, $\# K$, $\# L \leqq d$ then one has:*

$$\begin{cases} [q: K \to L] \to [f: H \to M] \\ [d] \to [D] \end{cases}$$

where H, M are simplicial complexes, $H \lhd K$, $M \lhd L$, f is a simplicial isomorphism, $\# H = \# M \leqq D \in \mathbb{N}$.

Proof. The construction of H in Lemma 2.3 uses a suitable order on the cells of K. Induce the order on L via q. Construct M as in 2.3 using this order. One gets two formally isomorphic simplicial complexes, hence simplicially isomorphic. □

(B) Effective shelling and effective trivialization of PL balls

Let $B = |K|$ be a triangulated PL m-ball.

Definition 2.6. A shelling of K is an ordering of the m-simplexes of K:

$$\sigma_1, \ldots, \sigma_h$$

such that for every $i = 1, \ldots, h-1$

$$B_i := \overline{B \setminus \bigcup_{1 \leqq j \leqq i} \sigma_j} \qquad (B_0 = B)$$

is a PL m-ball (triangulated by the restriction of K).

Remark 2.7. If $m \leqq 2$ every triangulation K of a PL m-ball B carries some shelling (for $m = 1$ is trivial, for $m = 2$ see [Mo]).

On the contrary, if $m \geqq 3$ there are triangulations of PL m-balls (in fact of Δ_m) without any shelling (see [R]). It is known (see [BM; S], and also [Ac]) that for any triangulation of any PL m-ball there exists a subdivision admitting some shelling.

The proof (see [BM]) in the general case is not effective and uses the existence of a PL isomorphism with Δ_m. In fact we want to prove that the effectiveness of shellings (up to subdivision) is *equivalent* to the effectiveness of trivializations.

In analogy with the trivialization problems we can state:

Problem 2.8 (WS$_m$: weak effective shelling problem in dim m). *It asks for*

$$[K] \to [K', \mathscr{S}]$$

where $B = |K|$ is any triangulated PL m-ball, $K' \lhd K$, and $\mathscr{S}$ is a shelling of K'.

Problem 2.9 (SS_m: strong effective shelling problem in dim m). *It asks for*

$$\begin{cases}[K] \to [K', \mathscr{S}]\\ [d] \to [D]\end{cases}$$

where $B = |K|$, $K' \lhd K$, $\mathscr{S}$ *are as above*, $\# K \leqq d$, $D \in \mathbb{N}$ *and* $\# K' \leqq D$.

We want to prove:

Proposition 2.10. *For every m, the strong (weak) trivialization problem for PL m-balls has solutions if and only if the strong (weak) shelling problem has solutions. Shortly:* $WT_m \Leftrightarrow WS_m$ *and* $ST_m \Leftrightarrow SS_m$.

Proof. We shall prove $ST_m \Leftrightarrow SS_m$. The proof works also in the weak case.

(*a*) $ST_m \Rightarrow SS_m$

It is enough to find solutions of SS_m for any $K \lhd K_\Delta$ with $\# K \leqq d$. This is a particular case of a more general statement.

Lemma 2.11. *If* $C = |K|$ *is any triangulated m-cell in* $\mathbb{R}^m$ *(i.e. C is a convex m-ball),* $\# K \leqq d$ *then one has*

$$\begin{cases}[K] \to [K', \mathscr{S}]\\ [d] \to [D]\end{cases}$$

where $K' \lhd K$, $\# K' \leqq D \in \mathbb{N}$, *and* $\mathscr{S}$ *is a shelling of* K'.

We postpone the proof.

(*b*) $ST_m \Leftarrow SS_m$

Assume, for the moment, the following lemma:

Lemma 2.12 (effective trivialization of collars of $\partial \Delta_m$). *If* $L \lhd K_\Delta$, $\# L \leqq p$ *then one has*

$$\begin{cases}[L] \to [L', h\colon C \to M]\\ [p] \to [q]\end{cases}$$

where $L' \lhd L$, $\# L' \leqq q \in \mathbb{N}$, $|M| = \partial \Delta_m \times [0, 1]$, *C is a subcomplex of L' triangulating a collar of* $\partial \Delta_m$ *and* $h\colon C \to M$, *is an effective trivialization such that* $h_{|\partial \Delta_m} = id \times \{0\}$.

We shall prove Lemma 2.12 later. It is enough to find solutions of ST_m for any triangulation K of $B = |K|$, with $\# K \leqq d$ and a given shelling $\sigma_1, \sigma_2, \ldots, \sigma_r$, where r is the number of the m-simplexes of K. We do it by induction on r. For $r = 1$ it is trivial. Assume that we have done for $(r - 1)$ and try to do for r. Set $|K_1| = \overline{K \setminus \sigma_1}$: it is (by hypothesis) a PL m-ball triangulated by the restriction of K, say K_1. $\mathscr{S}$ restricts to a shelling $\mathscr{S}_1 = (\sigma_2, \ldots, \sigma_r)$ and K_1 has $(r - 1)$ m-simplexes; $\# K_1 \leqq d$.

By induction we have a solution of ST_m for K_1, that is we construct effectively $K'_1 \lhd K_1$, $L_1 \lhd K_\Delta$, $g_1\colon K'_1 \to L_1$ a simplicial isomorphism, $D_1 \in \mathbb{N}$ (depending on d) such that $\# K'_1 \leqq D_1$.

By pulling-back via g_1 a collar of $\partial \Delta_m$ in Δ_m given by Lemma 2.12 we may assume also that there exists a subcomplex T of K'_1 such that $|T|$ is an effectively trivialized collar of $\partial |K'_1|$ in $|K'_1|$, by $h\colon T \to M$ where $|M| = \partial |K'_1| \times [0, 1]$, $h_{|\partial |K'_1|}$:

$\partial|K_1'| \to \partial|K_1'| \times \{0\}$ is $id \times \{0\}$. We want to construct a simplicial isomorphism

$$f: \hat{K} \to K_1''$$

where $\hat{K} \lhd K$, $K_1'' \lhd K_1'$ and $\#\hat{K}$ is effectively controlled by d. Thus we shall complete the proof by taking the simplicial map $g = g_1 \circ f: |K| \to \Delta_m$.

σ_1 intersect $|K_1|$ along $\partial|K_1|$ and the union of the $(m-1)$-faces of K_1 in $\sigma_1 \cap \partial|K_1|$ is an $(m-1)$-ball $B' = \sigma_1 \cap \partial|K_1|$. Using the trivialized collar we have that $B'' = h^{-1}(B' \times [0,1])$ is isomorphic to $B' \times [0,1]$ with an effectively given simplicial isomorphism (for a suitable subdivision). To conclude it is enough to show that $id: \overline{\partial B'' \setminus B'} \to \overline{\partial B'' \setminus B'}$ extends to a PL isomorphism $q: \sigma_1 \bigcup_{B'} B'' \to B''$ which becomes simplicial for suitable subdivisions effectively constructed with bounds on the number of simplexes.

Set $B''' = \sigma_1 \bigcup_{B'} B''$. First remark that the complement in $\partial B'''$ of the interior of $\overline{\partial B'' \setminus B'}$ coincides with the complement in $\partial\sigma_1$ of the interior of B' and also that $\partial B'' = B' \cup \overline{\partial B'' \setminus B'}$.

Both B' and $\overline{\partial\sigma_1 \setminus B'}$ are cones over their intersection $B' \cap \overline{\partial\sigma_1 \setminus B'}$, hence we can extend q to an isomorphism $q': \partial B''' \to \partial B''$.

Finally, using again the effective trivialization of the collar, we can realize both B''' and B'' as cones over their boundary; hence we can extend to $q'': B''' \to B''$. Remark that this is the proof of 3.25 of [RS], which becomes effective because of the effective trivialization of the collar. The proposition is proved (assuming Lemma 2.12). □

Remark 2.13. The same proof shows also that ST_m and SS_m are equivalent to the following

$$\begin{cases} [K] \to [K', \mathscr{S}] \\ [d] \to [D] \end{cases}$$

where $B = |K|$, is any triangulated PL m-ball, $|K'| = B$ and $\mathscr{S}$ is a shelling of K'; $\#K \leqq d$, $\#K' \leqq D \in \mathbb{N}$ (that is we do not require that $K' \lhd K$). Similarly for the weak statements.

Proof of Lemma 2.11. With a slightly different statement the proof is contained in [Ac]. For the sake of completeness we sketch the construction with emphasis on effectiveness. The proof is by induction on n.

Let B be a convex triangulated n-polyhedron, $B = |K|$. We shall describe an explicit subdivision which is shellable.

Consider a family $E_0, \ldots, E_S$ of parallel hyperplanes with the following properties:

i) $E_0 \cap K = \{v^0\}$; v^0 is a vertex.
ii) each vertex v of K is in some E_j.
iii) In a suitable coordinate system

$$E_j = \{x_n = \alpha_j\} \quad \text{with} \quad \alpha_0 < \alpha_1 < \ldots < \alpha_S$$

It is easy to see that one can do this effectively.

$E_j \cap B$ is a convex $(n-1)$-polyhedron (or possibly a single vertex). By Lemma 2.4 and the induction hypothesis we can subdivide explicitly the cellularization induced by K on $E_j \cap B$ in a shellable triangulation K_j'. Let

$$F_i = B \cap \{\alpha_{i-1} \leqq x_n \leqq \alpha_i\} \quad i = 1, \ldots, s$$

F_i is a convex polyhedron with a cellularization induced by K, K'_{i-1} and K'_i. We can find a triangulation of F_i, $i = 1, \ldots, s$ without introducing new vertices. So we obtain a linear simplicial subdivision K' of K such that the vertices of K' lie in $E_0 \cup \ldots \cup E_S$. Define:

$$V_{i,k} = \{\sigma \in K' | \dim \sigma = m, \sigma \subset F_i, \dim(\sigma \cap E_{i-1}) = k - 1\} .$$

If $\sigma \in V_{i,k}$ and $\sigma' \subset F_i$ is such that σ and σ' have a common $(m-1)$-face, then $\sigma' \in V_{i,k-1} \cup V_{i,k} \cup V_{i,k+1}$. So we can define the *worst* type simplices in $V_{i,k}$ as the simplices which have no common $(m-1)$-face with any simplex in $V_{i,k+1}$.

It is rather easy to prove, as in [Ac], that the following rules define a shelling $\mathscr{S}$ for K'.

(*1*) decreasing i.
(*2*) increasing k.
(*3*) for fixed i and k, first the worst type simplices with more free faces.

So the lemma is proved. □

Proof of Lemma 2.12. By means of Lemma 2.4 it is enough to construct an effectively trivialized collar of $\partial\Delta_m$ for some triangulation of Δ_m. Consider Δ_m as the cone over $\partial\Delta_m$ with center in the barycenter a of Δ_m.

Every $v \in \Delta_m$ is uniquely expressed as $v = ta + (1-t)v'$ where $t \in [0, 1]$ and $v' \in \partial\Delta_m$. Set:

$$\Delta(s) = \{v \in \Delta_m | v = sa + (1-s)v' \quad \text{for some } v' \in \partial\Delta_m\} \quad 0 \leqq s \leqq 1$$

Note that $\Delta(0) = \partial\Delta_m$. Consider $\Delta(\frac{1}{2}) \subseteq \Delta_m$. Using the cone structure we find a natural cell-decomposition of Δ_m having as vertices the union of the vertices of $\Delta(0)$ and $\Delta(\frac{1}{2})$ and such that the induced cell-decomposition on $\bigcup_{0 \leqq s \leqq \frac{1}{2}} \Delta(s)$ is formally isomorphic to the standard cellular decomposition of $\partial\Delta_m \times I$ (i.e. the product $K_{\Delta|\partial\Delta_m} \times I$). Thus we can conclude using Lemma 2.4 and Lemma 2.5. □

Remarks 2.14. (*i*) The proof of 2.13 works for any m-cell in $\mathbb{R}^n$.
(*ii*) Consider $K = K_\Delta$ and let $K^{(r)}$ denote the rth-barycentric subdivision of K. One can explicitly construct (it is elementary but not completely trivial) a simplicial isomorphism between the subcomplex of $K^{(2)}$, $\mathrm{St}(\partial\Delta_m, K^{(2)})$ with a suitable subdivision of $\partial\Delta_m \times I$, (obtaining in this way another proof of 2.12): PL topology tells us (see [RS]) that if $M = |K|$ is a triangulated manifold with boundary, $\mathrm{St}(\partial M, K^{(2)})$ is a collar of ∂M in M. The problem to us is to make the trivialization effective. We have not been able to do it in general, not even under the hypothesis that M is a PL ball (but K is arbitrary); in fact there are some evidence that it should not be possible in higher dimensions see (Remark 2.19 and Corollary 2.20).

(*C*) *Listing bounded subdivisions and effective trivialization of PL balls*

The aim of this section is to prove the following:

Proposition 2.15 (listing bounded subdivisions). *Let $X = |K|$ be any triangulated polyhedron in $\mathbb{R}^n$ with $\# K \leq d$ and fix a positive integer $D \geqq d$ then we have*

$$\begin{cases} [X = |K|] \to [\{H_1, \ldots, H_s\}] \\ [d, n, D] \to [S] \end{cases}$$

where $S \in \mathbb{N}$ and $s \leqq S$, for every $i = 1, \ldots, s$, $H_i \lhd K$, $\# H_i \leqq D$, and for every $H \lhd K$ such that $\# H \leqq D$ there exists i and a simplicial isomorphism ϕ: $H \to H_i$.

Before proving the proposition we deduce some corollaries. In particular we prove Theorem 4 and Theorem 5 of the Introduction.

Lemma 2.16. *Given $X = |K|$, $Y = |H|$ triangulated polyhedra, we can effectively decide if there exists a simplicial isomorphism $f: K \to H$.*

Proof. In the worse case (that is if there are not evident distinctions like $\dim X \neq \dim Y$ or $\# K \neq \# H$) we have to look for formal isomorphism between K and H, thus we have to analyze a finite number of possibilities. □

Corollary 2.17 (solution of the weak trivialization and shelling problems). *For every m WT_m and WS_m have solution.*

Proof. Since $\mathrm{WT}_m \Leftrightarrow \mathrm{WS}_m$ (Proposition 2.10) it is enough to show that WT_m (for example) has solutions. Take a triangulated m-ball $B = |K|$. By hypothesis there exists some $K' \lhd K$, $\# K' = R_0$, $L \lhd K_\Delta$ such that $g: K' \to L$ is a simplicial isomorphism. If $r = \# K$, for every $R \geqq r$ we can list (Proposition 2.15) all the subdivisions of K with no more than R simplexes and all the analogous subdivisions of K_Δ. Using Lemma 2.16, for every R we can check if there are any such subdivisions which are isomorphic. By hypothesis there exists some R_0 such that the algorithm stops. □

Corollary 2.18 (non effectiveness for the strong trivialization and shelling problems). *For $m \geqq 5$ ST_m and SS_m do not have solution. More precisely: if $m \geqq 6$ they do not have solution even under the stronger hypothesis that the m-balls are embedded in $\mathbb{R}^m$; if $m = 5$, ST_5 and SS_5 fail even for codimension one 5-balls in $\mathbb{R}^6$.*

Proof. Assume first $m \geqq 6$ and ST_m has a solution. Take any triangulated manifold $M_S = |K_S| \subseteq \mathbb{R}^n$ given by Novikov theorem recalled in Sect. 1. If $\# K_S \leqq d$ and if M_S should be a PL m-ball one could effectively determine $D \in \mathbb{N}$ such that there exists a simplicial isomorphism $g: K'_S \to L_S$ where $K'_S \lhd K_S$ and $\# K'_S \leqq D$, $L_S \lhd K_\Delta$. Using the listing of bounded subdivisions as in the previous proof one could decide if M_S is isomorphic to Δ_m (by actually finding isomorphic subdivisions with no more than D simplexes or concluding that they are not PL isomorphic otherwise). This contradicts Novikov's theorem.

Let be $m = 5$ and suppose that ST_5 has a solution. Take again $M_S = |K_S| \subseteq \mathbb{R}^6$, take off a simplex σ from the boundary ∂M_S and consider $N_S = \overline{\partial M_S \backslash \sigma}$. N_S is a triangulated 5-manifold and by ST_5 and Lemma 2.16 we can decide whether N_S is a 5-ball or not. So we can decide whether ∂M_S is a 5-sphere or not and finally, by h-cobordism in dimension 5, we can decide whether M_S is a 6-ball or not, contradicting Novikov's theorem. □

Remark 2.19. As a consequence of Corollary 2.18 we cannot have a strongly effective proof of h-cobordism theorem: indeed, if so, taking off a simplex τ from the interior of a triangulated m-ball B, one could explicitly triangulate $\overline{B \backslash \tau}$ as the product $\partial B \times I$; this would be enough to have ST_m. A rough analysis of the proof of h-cobordism theorem shows that there are two a priori non effective steps: the first is the trivialization of the handle decomposition, where trivialization of collars is an essential tool; the second is the choice of a disk bounded by the curve of "Whitney trick".

Actually 2.18 has also a rather surprising corollary: collars cannot be strongly effectively trivialized.

We thank A. Marin who remarked it. The argument is the following.

Recall that, if not specified otherwise, by PL m-balls we mean balls generally embedded in some $\mathbb{R}^n$, n and m not necessarily equal. Given a PL m-ball B with boundary S, a *trivialized collar* of S in B is a PL homeomorphism between a regular neighbourhood N of S in B and $S \times [0, 1]$, identifying identically S and $S \times \{0\}$. We have, with the obvious meaning, a *strong effectiveness problem for construction of such trivialized collars.* In case of positive answer, we say shortly that *collars can be trivialized.*

Corollary 2.20 (non effectiveness for the trivialization of the collars). *There exists m, $3 < m < 7$, such that collars in m-balls cannot be trivialized.*

Proof. Let m_0 the minimal number m such that ST_m (extended to all balls) has negative answer. By Corollary 2.18 m_0 exists and in fact $2 < m_0 < 6$. Then one deduces that $(m_0 - 1)$-spheres can be strongly effectively trivialized on the boundary of the $(m_0 - 1)$-simplex (otherwise we contradict the minimality of m_0). For every m_0-ball B consider the $(m_0 + 1)$-ball B' obtained by coning over B. If collars of the boundary S' of B' can be strongly effectively trivialized, the above remark on the trivialization of the boundary of B together with the *unicity of links* (which is a strongly effective result) imply that B can be strongly effectively trivialized. This is a contradiction. □

Remark 2.21. Since Novikov's manifolds M_S are of codimension 0 in $\mathbb{R}^m$, we have proved the point (*b*) of Theorem 5 stated in the Introduction. It remains to prove the point (*a*). It is a consequence of a more general fact:

Lemma 2.22. *For every $n, d \in \mathbb{N}$ there exists $D = h(n, d)$ such that for every couple of triangulated polyhedra $X = |K|$, $Y = |H| \subseteq \mathbb{R}^n$ with $\# K$, $\# H \leqq d$, if X, Y are PL isomorphic then there exist subdivisions $K' \lhd K$, $H' \lhd H$ with $\# K'$, $\# H' \leqq D$ which are simplicially isomorphic.*

Proof. It is a consequence of (the proof of) the theorem of local triviality of semialgebraic maps as stated and proved, for example, in [BCR] Theorem 9.3.1, following the proof of Hardt ([Ha]) (see also [BR], 2.8.3.). Note that in our case the situation is much simpler because we are working with PL objects.

Every triangulated polyhedron in $\mathbb{R}^n$ with no more than d simplexes can be presented using no more than $\bar{d}$ (effectively computed by means of n and d) polynomial of degree 1, is such a way that it is given as the union of its simplexes.

We can identify the polynomial in n variables of degree 1 with $\mathbb{R}^{n+1}$ and set for every $h \leqq \bar{d}$

$$\mathbf{R} = \mathbf{R}_h = (\mathbb{R}^{n+1})^h$$

Set:

$$S = S_h = \{ >, =, < \}^h$$

Take $\mathbb{R}^n \times \mathbf{R}$ with the natural projections

$$\pi\colon \mathbb{R}^n \times \mathbf{R} \to \mathbf{R} \qquad q\colon \mathbb{R}^n \times \mathbf{R} \to \mathbb{R}^n$$

For every $t = (t_1, \ldots, t_h) \in S$ consider

$$V_t = \{(x, P_1, \ldots, P_h) \in \mathbb{R}^n \times \mathbf{R}\colon P_i(x) t_i 0\}$$

Finally set for every k:

$$\mathscr{V} = \mathscr{V}_{h,k} = \{V \mid V = V_{t_1} \cup \ldots \cup V_{t_k} \quad t_i \in S \quad 1 \leqq i \leqq k\}$$

$\mathscr{V}$ is a finite set. It is easy to find an effective upper bound B for k (in term of $\bar{d}$) such that every triangulated polyhedron with no more than d simplexes is of the form $X = q(\pi^{-1}(v^0) \cap V)$ for some $v^0 \in \mathbf{R}$, $V \in \mathscr{V}$, $h, k \leqq B$. Apply the local triviality theorem to the maps $\pi_{|V}$: $V \to \mathbf{R}$, V varying in $\mathscr{V}$. It turns out from the proof that $\pi_{|V}$ is trivialized over a finite number of semialgebraic subsets of $\mathbf{R}$ giving a partition of $\mathbf{R}$ (in fact in our case they are defined by polynomial of degree 1). More over we note that:

(*1*) For every V the number of such subsets of $\mathbf{R}$ is effectively bounded in term of n and $\bar{d}$ (hence of d).
(*2*) $\# \mathscr{V}$ is effectively bounded.
(*3*) Two fibers of the same trivialization are PL isomorphic by a simplicial isomorphism between subdivisions with bounded number of simplexes (see [BR] where all this facts are explicitly noted).

Eventually we may have fibers of different trivializations which are nevertheless PL isomorphic. Anyway there is a finite number of possibilities hence the function $D = h(n, d)$ exists. □

Proof of Proposition 2.15. It is again an application of the proof of the local triviality for semialgebraic maps ([Ha; BCR; BR]) adapted (and simplified) in our setting. Given a subdivision $K' \lhd K$, $X = |K|$, $\# K' \leqq D$ then the polyhedron $X = |K| \subseteq \mathbb{R}^n$ is presented as union of the simplexes of K' using no more than $\bar{D}$ polynomials of degree 1, and $\bar{D}$ is effectively computable using n, d and D. Let $P_1, \ldots, P_r$ be such polynomials. They produce a cell-decomposition of X by cells of the form:

$$\sigma_t = \{x \in \sigma \,|\, P_i(x) t_i 0\}$$

where σ is a simplex of K, $t = (t_1, \ldots, t_r)$ and t_i is a value in $\{>, =, <\}$ associated to P_i, $i = 1, \ldots, r$.

It is clear that such a cell-decomposition subdivides K and has a number of cells effectively bounded.

On the other hand every subdivision $K' \leqq K$ with $\# K' \leqq D$ can be obtained by some cell decomposition of the previous type (that is defined by no more than $\bar{D}$ polynomials of degree 1) by deleting some cells: since the number of the cells is effectively bounded, then there is only a finite number of possibilities to delete cells in order to get (if any) a required subdivision of K.

We work similarly to the proof of Lemma 2.22.

Let $A \subseteq \mathbb{R}^{n+1}$ be the subset of polynomials of degree 1 with zero set intersecting X. Set for $h \leqq \bar{D}$:

$$\mathbf{R} = \mathbf{R}_h = A^h \qquad S = S_h = \{>, =, <\}^h$$

$$\pi: \mathbb{R}^n \times \mathbf{R} \to \mathbf{R} \qquad q: \mathbb{R}^n \times \mathbf{R} \to \mathbb{R}^n$$

For every $t = (t_1, \ldots, t_h) \in S$ set

$$V_t = \{(x, P_1, \ldots, P_h) \in X \times \mathbf{R}: P_i(x) t_i 0\}$$

$$\mathscr{V} = \mathscr{V}_{h,k} = \{V \,|\, V = V_{t_1} \cup \ldots \cup V_{t_k}\ t_i \in S\ 1 \leqq i \leqq k\}$$

$$k \leqq B \text{ effectively bounded}$$

Apply the local trivialization theorem to $\pi_{|V}$: $V \to \mathbf{R}$, V varying in $\mathscr{V}$. For every $r \in \mathbf{R}$ $\pi_{|V}^{-1}(r)$ is a cell-subdivision of X of the type described above.

It turns out from the proof of the triviality theorem that, in particular, two fibers in the same trivialization of π over some $\mathbf{R}_i$ of the partition of $\mathbf{R}$ are formally isomorphic; hence the subdivision of K obtained by deleting faces with the same name in both such fibers are simplicially isomorphic. Since the number of $\mathbf{R}_i$ is effectively bounded the proposition is proved. Note that in this way we may obtain several representatives of simplicial isomorphism class of subdivisions of K presented by no more than $\bar{D}$ polynomials; what is important is that we get at least one representative for each class. □

(D) Strong effective trivialization of PL balls in low dimension

In this section we shall prove, in particular, Theorem 3 of the Introduction.

Proposition 2.23. $\mathrm{SS_m}$ *and* $\mathrm{ST_m}$ *have solution for* $m \leqq 2$.

Proof. Since $\mathrm{SS_m} \Leftrightarrow \mathrm{ST_m}$ (2.11) it is enough to show $\mathrm{SS_m}$ (for example). Now as already noted in 2.8 *every* triangulation of a PL m-ball with $m \leqq 2$ carries some shelling and at least one of these can be detected effectively. □

Proposition 2.24. SS_3 *and* ST_3 *have solutions for PL 3-balls embedded in* $\mathbb{R}^3$.

Proof. ST_3 is a consequence of an effective version of the proof of the PL Shönflies theorem in $\mathbb{R}^3$; we follow Moise's statement (see Theorem 17.12 in [Mo]), whose proof is very similar to the original one (see [A1]). Since what follows Lemma 1, p. 122 of [Mo] is straightforwardly effective, it is enough to prove the effectiveness of the first part. Here are the steps.

(1) The effective push property EPP

Let C be a polyhedral 3-cell in $\mathbb{R}^3$ and $D_1 \subset \partial C^3$ be a polyhedral 2-cell. Let J be the boundary of D_1 and define $D_2 = \overline{\partial C \setminus D_1}$. *We say that C has the EPP at D_1 if for each closed neighborhood N of $C \setminus J$ one has*:

$$\begin{cases} [K, G, H] \to [f\colon H' \to L] \\ [d] \to [D] \end{cases}$$

where K, G, H are compatible linear triangulations of C, D_1, N, $\# K$, $\# G$, $\# H \leqq d$; $H' \lhd H$; L is a linear triangulation of N, $\# H'$, $\# L \leqq D$ and f is a simplicial isomorphism such that

(*i*) $f_{|\partial N} = id$

(*ii*) $f(D_1) = D_2$

C has the EPP if this is true at any polyhedral 2-cell $D \subset \partial C$.

(2) Any convex polyhedral 3-cell has the EPP

This is true since proofs of Theorems 4–9 of Chap. 17 in [Mo] are (or can be easily reduced to) effective proofs.

(3) Simply embedded spheres

We say that a polyhedral sphere S is simply embedded if it bounds a polyhedral 3-cell B and ST_3 is true for B.

(4) *Theorem 17.11 in* [Mo]

Theorem. *Let S_1, S_2 be polyhedral 2-spheres in $\mathbb{R}^3$ such that $S_1 \cap S_2$ is a plane 2-cell D. Let*

$$S = (S_1 \cup S_2)\backslash \operatorname{int} D$$

If S_1 and S_2 are simply embedded, then so also is S.

Proof. See Theorem 17.11 in [Mo], using EPP instead of the push-property.

(5) *The index of a polyhedral 2-sphere*

Define the index of a polyhedral 2-sphere as in [Mo]: first choose a system $\mathscr{R}$ of orthogonal coordinates x, y, z in $\mathbb{R}^3$ in such a way that each plane $z =$ const. does not contain more than one vertex of S: this can be done effectively. For any horizontal plane E, $E \cap S$ can be: a single point, a (disconnected) 1-manifold, the disjoint union of a 1-manifold and a single point, the union of k polygons, 2 or more through a point P. In the latter case we call P a *singular* point and define the index at P, $\operatorname{Ind}_{\mathscr{R}} P$, to be $k-1$. Then define $\operatorname{Ind}_{\mathscr{R}} S$, as the sum of the numbers $\operatorname{Ind}_{\mathscr{R}} P$ for all the singular points P in S.

(6) **Lemma.** *If S is a polyhedral sphere and $\operatorname{Ind}_{\mathscr{R}} S = n$, then one can effectively find a coordinate system $\mathscr{R}'$ and two polyhedral spheres S_1, S_2 such that S is simplicially isomorphic to their union (as in the step 4) and $\operatorname{Ind}_{\mathscr{R}'} S_i \leqq n-1$ $i = 1, 2$.*

Proof. (see Lemma 17.1 of [Mo], p. 122). Let P be a singular point of the section $E \cap S$, where $E = \{z = z_E\}$ in the system $\mathscr{R}$, and let $J \subseteq E \cap S$ be the *inmost* polygon. J bounds a 2-cell D_J in E.

The only remark to be made is that the PL homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ constructed in the proof of [Mo], which preserves horizontal planes and makes D_J convex, is simplicial on an effectively computable triangulation of S and $\mathbb{R}^3$: but this is clear because of SS_2 and Lemma 2.5. □

Remark 2.25. SS_m and ST_m for $m = 4$, are open (to us), even under the restriction to 4-balls in $\mathbb{R}^4$.

The above facts suggest the following:

Problem 2.26 (embedding problem (E_m)). *It asks for*

$$\begin{cases} [K] \to [g\colon K' \to H] \\ \ [d] \to [D] \end{cases}$$

where $B = |K|$ is a triangulated PL m-ball, $\# K \leqq d$; $K' \lhd K$, $D \in \mathbb{N}$ and $\# K' \leqq D$, g is a simplicial isomorphism, $|H|$ is a PL m-ball embedded in $\mathbb{R}^m$.

3. On the effectiveness of the solutions of the real semialgebraic Hauptvermutung

We recall some well-known facts in semialgebraic geometry (see [BCR] or [BR]). We use the notations introduced in Sect. 0.

Theorem 3.1 (semialgebraic triangulation). *Let $X \in \mathscr{A}(n, s, D)$ be a compact semialgebraic set in $\mathbb{R}^n$. Let $Y_i \in \mathscr{A}(n, s, D)$ be a closed subset of X, $i = 1, \dots, h$. Then*

$$\begin{cases} [X, Y_i] \to [K, f] \\ [n, s, D] \to [r, p, q, R] \end{cases}$$

where $|K|$ is a triangulated polyhedron of $\mathbb{R}^n$, $f\colon |K| \to X$ is a semialgebraic homeomorphism such that: $f^{-1}(Y_i) = |K_i|$ and K_i is a subcomplex of K, $i = 1, \dots, h$; $\Gamma_f \in \mathscr{A}(p, q, R)$, $\# K \leqq r$.

We say that any $f\colon |K| \to X$ as above (without the limitation $|K| \subseteq \mathbb{R}^n$) is *a semialgebraic triangulation of X relative to $\{Y_i\}$.*

The following is an almost immediate corollary of 3.1.

Corollary 3.2. *Let $f_l\colon |K_l| \to X$ $l = 1, 2$ be a semialgebraic triangulation of X (relative to $\{Y_i\}$). Let $\# K_l \leqq d$, $\Gamma_{f_l} \in \mathscr{A}(n, s, D)$. Then*

$$\begin{cases} [f_1, f_2] \to [K, h_1, h_2] \\ [n, s, d, D] \to [r, m, q, R] \end{cases}$$

where for $l = 1, 2$ $h_l\colon |K| \to |K_l|$ is a semialgebraic triangulation of $|K_l|$ relative to the family of all simplexes of K_l, $\Gamma_{h_l} \in \mathscr{A}(m, q, R)$, $\# K \leqq r$.

Thus we have a diagram

$$\begin{tikzcd} & {|K_1|} \arrow[dr, "f_1"] & \\ {|K|} \arrow[ur, "h_1"] \arrow[dr, "h_2"'] & & X \\ & {|K_2|} \arrow[ur, "f_2"'] & \end{tikzcd}$$

and $(|K|, h_1, h_2)$ is called a simultaneous semialgebraic subdivision of the triangulations f_1, f_2 of X. Clearly $(|K|, f_l \circ h_l)$ $l = 1, 2$ is a semialgebraic triangulation of X.

(A) WEH_m

3.3. Proof of the WEH_m $\forall m$ (Theorem 2 of the Introduction). Let $f\colon |K| \to |L|$ be a semialgebraic homeomorphism between two polyhedra in $\mathbb{R}^n$ as in the statement of the theorem. Using Corollary 3.2 we may assume that f satisfies the further hypothesis: *for every simplex σ of K, $f(\sigma)$ is triangulated by a subcomplex of L.*

By Shiota-Yokoi existential solution of the Hauptvermutung we know that $B = f(\sigma)$ is a triangulated PL m-ball, $m = \dim \sigma$.

By means of Theorem 4 of the Introduction (see 2.17) we can effectively construct a trivialization of B. Thus we conclude (as in [SY], p. 737) by induction on $\dim |K|$, constructing effectively a PL isomorphism on the skeletons in increasing dimension and applying several times the "Alexander trick". □

Remark 3.4. The same proof as above (in the same way of [SY], p. 737) produces in fact a more precise result: that is *we can construct effectively a semialgebraic isotopy $H\colon |K| \times [0, 1] \to |L|$ such that $H_0 = f$ and $H_1 = g$, that is the simplicial isomorphism of the statement of* WEH_m.

Theorem 3.5 (Theorem 1 of the Introduction). *SEH_m is true for $m \leqq 3$.*

Proof. For $m \leqq 2$ we can work as in the Proof 3.3 using Proposition 2.23 instead of 2.17.

For $m = 3$ we want to reduce the Proposition 2.24, that is to work with 3-balls embedded in $\mathbb{R}^3$.

Consider a semialgebraic homeomorphism $f: |K| \to |L|$ such that for every simplex $\sigma \in K$, $B = f(\sigma)$ is triangulated by a subcomplex L_1 of L (like in the proof 3.3).

Take a 3-simplex σ of K, clearly we may consider $\sigma \subseteq \mathbb{R}^3$.

$g = f^{-1}: B = |L_1| \to \sigma$ is a semialgebraic triangulation of σ. Apply the semialgebraic triangulation theorem and its proof (see for instance [BCR] or [BR]) to σ relatively to the family given by the images of the simplexes of L_1. We obtain a straight triangulation of σ, say $\sigma = |H|$ such that every simplex of L_1 goes by g' over the support of a subcomplex of H, where $g': B \to \sigma$ is a suitable semialgebraic homeomorphism of effectively bounded complexity.

Thus we can effectively construct a PL isomorphism between $|L_1|$ and σ working like in the Proof 3.3 (induction, construction on skeletons of increasing dimension, Alexander trick) applying 2.25 to each $g'(\tau) \subseteq \sigma$, τ varying among the 3-simplexes of L_1. □

Remark 3.6. (in analogy with Remark 3.4.). For $m \leqq 3$ we can construct the semialgebraic isotopy H_t in such a way that $\Gamma_H \in \mathscr{A}(k, p, q)$ where k, p, q, are effectively bounded by the data of the situation.

(B) SEH_m and semialgebraic homeomorphism of bounded complexity.

At the present we are not able to decide if SEH_m holds for $m \geqq 4$.

In this section we show simply that SEH_m and an other natural semialgebraic problem cannot be solved effectively at the same time. The spirit of the result is analogous to Nabutovsky's examples.

Lemma 3.7. *For every $m, n, d \in \mathbb{N}$ there exist $P, Q \in \mathbb{N}$ such that P, Q depend on n, m, d and for every $X, Y \in \mathscr{A}(m, n, d)$ which are semialgebraically homeomorphic, there exists a semialgebraic homeomorphism $h: X \to Y$ with $\Gamma_h \in \mathscr{A}(m^2, P, Q)$.*

Proof. It is essentially the same of 2.22, using the local triviality theorem of semialgebraic maps. □

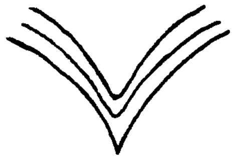
Problem 3.8. *Can P, Q as above be bounded by a recursive function of n, m, d?*

Proposition 3.9. *For $m \geqq 5$, the statement: "SEH_m has solution $\Rightarrow$ Problem 3.8 has negative answer" is true.*

Proof. Let $M = |K|$ be any compact polyhedron. If 3.8 has a positive solution we can effectively determine P, Q, depending only on m and $\# K$, such that if M is homeomorphic with Δ_m, then there exists a semialgebraic homeomorphism $h: M \to \Delta_m$ with $\Gamma_h \in \mathscr{A}(m^2, P, Q)$. If also SEH_m has solution we can deduce that one can effectively determine D depending only on $\# K$ and m such that if M is PL isomorphic with Δ_m then there exists a simplicial isomorphism $g: K' \to L$ where $K' \leqq K$, $L \lhd K_\Delta$ and $\# K' \leqq D$. This contradicts 2.18. □

Remark 3.10. The hard part in Shiota-Yokoi solution of subanalytic Hauptvermutung is in showing that if a polyhedron $P = |K|$ is subanalytically homeomorphic to Δ_m, then it is PL isomorphic to Δ_m (see also the Proof of 3.3 of the present paper). Shiota-Yokoi proof doesn't get any essential simplification in the semialgebraic setting and it is far from giving SEH_m, not even WEH_m. The main reason of it is that Shiota-Yokoi use Cairns-Whithehead C^∞-triangulations and the (already known) solution of the Hauptvermutung for this class of triangulations (see [Mu]). This produce at least two troubles of increasing difficulty:

(*a*) It is not clear if a semialgebraic manifold $M \in \mathscr{A}(n, r, p)$ could be triangulated in the C^∞-sense, $M = |K|$ say, in such a way that $\# K$ is effectively controlled by n, r, q. The usual way (secant approximation ... (see [Mu])) doesn't work: consider for example a sequence of manifolds $M_S \in \mathscr{A}(n, r, q)$ converging to some cusps as in the figure. The semialgebraic triangulation algorithm (again for the cusp presence) doesn't produce C^∞-triangulations.



(*b*) (More serious) Shiota-Yokoi apply the uniqueness of C^∞-triangulations to the diffeomorphisms obtained by *integrating vector fields.* Even if these vector fields are defined by semialgebraic functions of bounded complexity, how to bound, or even how to define, the *complexity* of the resulting diffeomorphism is a hard problem.

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