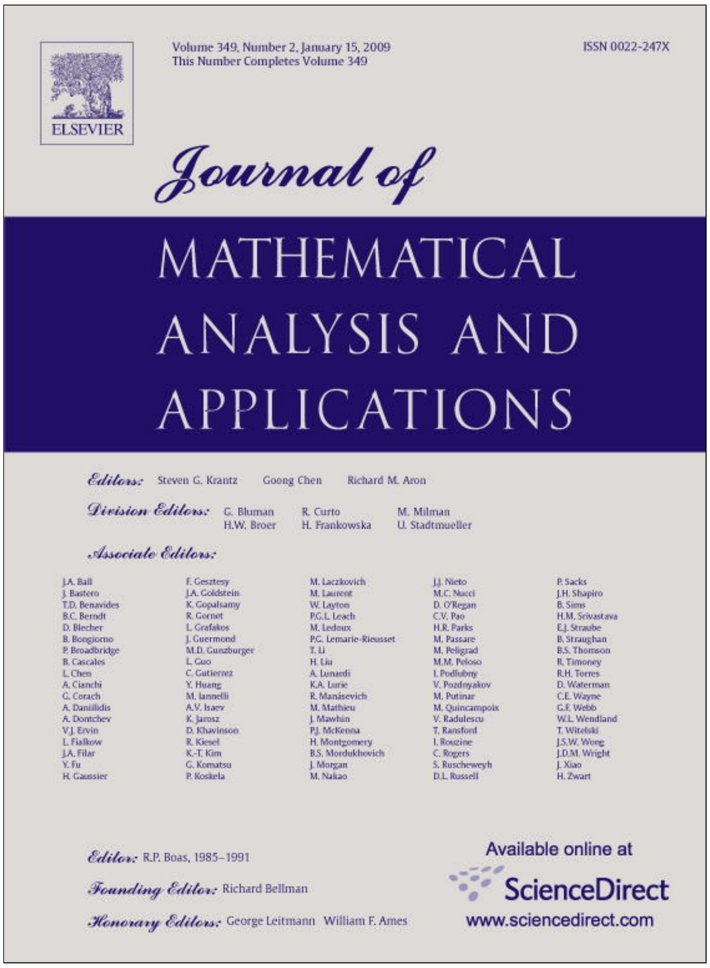
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On the global regularity of shear thinning flows in smooth domains

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ABSTRACT

In some recent papers we have been pursuing regularity results *up to the boundary*, in $W^{2,l}(\Omega)$ spaces for the velocity, and in $W^{1,l}(\Omega)$ spaces for the pressure, for fluid flows with shear dependent viscosity. To fix ideas, we assume the classical non-slip boundary condition. From the mathematical point of view it is appropriate to distinguish between the shear thickening case, $p > 2$, and the shear thinning case, $p < 2$, and between flat-boundaries and smooth, arbitrary, boundaries. The $p < 2$ non-flat boundary case is still open. The aim of this work is to extend to smooth boundaries the results proved in reference [H. Beirão da Veiga, On non-Newtonian p-fluids. The pseudo-plastic case, J. Math. Anal. Appl. 344 (1) (2008) 175–185]. This is done here by appealing to a quite general method, introduced in reference [H. Beirão da Veiga, On the Ladyzhenskaya–Smagorinsky turbulence model of the Navier–Stokes equations in smooth domains. The regularity problem, J. Eur. Math. Soc., in press], suitable for considering non-flat boundaries.

1. Introduction

The Navier–Stokes system of equations with shear dependent viscosity has been studied in the last forty years by a great number of researchers, not only in pure and applied mathematics, but also in engineering, physics and biology. A typical model of generalized stationary Navier–Stokes system of equations with shear dependent viscosity is the well-known model

$$\begin{cases} -\nabla \cdot T(u,\pi) + u \cdot \nabla u = f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where T denotes the stress tensor

$$T = -\pi I + \nu_T(u)\mathcal{D}u \tag{1.2}$$

and

$$\mathcal{D}u = \frac{1}{2}\left(\nabla u + \nabla u^T\right).$$

The first mathematical studies on the above class of equations go back to O.A. Ladyzhenskaya in a series of remarkable contributions. See [34–37]. In references [39] and [40, Chapter 2, n. 5] J.-L. Lions considers the case in which $\mathcal{D}u$ is replaced by ∇u. However in this case the Stokes principle, see [57] and [55, p. 231], is not satisfied. Such models, an instance of which is (1.1), were intensively studied in the eighties and nineties by J.Nečas and his school.

Non-linear shear dependent viscosities are used, in particular, to model properties of materials. The cases $p > 2$ and $p < 2$, see Eq. (2.1), capture shear thickening and shear thinning phenomena, respectively. The case $p = 3$ was introduced

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by Smagorinsky, see [56], as a turbulence model. In the sequel, we concentrate on, and assume that, $1 < p \leqslant 2$ and (for convenience) $n = 3$.

For comments and references, both to modeling and theory, we refer the reader to [30,43,44,49]. Let us give some references (far from being complete) on papers by other authors, concerning existence of solutions, interior regularity results, and mathematical literature on related physical problems. The first paper treating the unsteady case, for $p < 2$, is [41]. In [51,52] regularity results with periodic boundary conditions are proved. Interior regularity results are proved, for instance, in [27,28,45,54,59]. In references [1,24,50–53] the authors consider electrorheological fluids. Fluids with energy transfer, thermal viscous dependence, and related topics are treated in [14–19]. For particularly interesting results in two dimensions see [31–33,38] and references therein. Numerical results may be found, for instance, in [12,25,26,48]. For anisotropic problems see, for instance, [2,13].

In a series of recent papers, see also [42], we introduced a general scheme suitable to solve the problem of the $W^{2,q}(\Omega)$-regularity *up to the boundary*, for p-fluid flows, under typical boundary conditions. We began this series of papers by considering the half space case, see [4], and the case of a cubic domain, see [5,6]. In this last case the interesting boundary condition is given on two opposite faces, and space-periodicity is assumed in the other two directions. These two frameworks avoid the need of appealing to localization techniques and to changes of variables (in order to flatten the boundary). It is worth noting that when $p \neq 2$ the extension of regularity results from flat boundaries to arbitrary, regular, boundaries presents many new unusual obstacles, compared to the (still non-trivial) classical case $p = 2$.

Since we have dedicated a certain number of papers to the above subject, the following overview could help the interested reader. In [4], we establish the main lines to treat problems in the flat boundary case. More precisely, we consider the slip and the non-slip boundary value problem in the half space $\mathbb{R}^n_+$, and $p > 2$. In reference [5] we replace, for simplicity, the half-space $\mathbb{R}^n_+$ by the above three-dimensional cube and consider the non-slip boundary condition. Further, we introduce the convective term and the evolution problem.

In reference [6] we consider the $p < 2$ case. Here, an idea borrowed from Lemma 6 in reference [24] is crucial (see [6, Lemma 3.2]). Further, by introducing a new device (see [5, Remark 5.1]), we drop the $-\Delta u$ term from the equations.

It is worth noting that the addition of a $-\Delta u$ term on the left-hand side of the equations simplify the proofs. Actually, it allows much stronger regularity results, specially in the $p < 2$. This case, much easier to handle, is more in accordance with the physical problems. Actually, in [8], it is shown that weak solutions belong to $W^{1,q}(\Omega)$, for any finite q, provided that an x-dependent growth condition holds, $p = p(x) \leqslant 2$. Convexity-type assumptions are not assumed. Under more classic assumptions, see for instance [23], one shows that $u \in W^{2,2}(\Omega)$.

To finish this overview on our recent contributions, we refer to [10], where the previous results on the shear thickening case are improved.

A main open problem remains the extension of the above types of results to non-flat boundaries (in this context, see also the pioneering paper [42]). This requires really new ideas, since the presence of the $\mathcal{D}u$ term together with $p \neq 2$ makes the boundary value problem particularly difficult. We solve this problem in reference [7], where $p > 2$.

In the mean-time, in references [21,22], F. Crispo has extended the $p < 2$ results in [6] to cylindrical domains, by appealing to cylindrical coordinates. This change of coordinates requires particular care, due to the non-linear p-term. Further, L.C. Berselli, see [11], improves the argument followed in [5], by replacing the classical (isotropic) Sobolev embedding theorems by anisotropic embedding theorems. This very fruitful idea is used by us below (in Section 7). Next, in reference [9], we improve previous results shown for the shear thinning case. In particular, we obtain a better value for the parameter p_0 in the Navier–Stokes problem (see below) by replacing the device borrowed from [24], by a different idea. It remains the open problem of the extension, from flat to regular boundaries, of the sharp results proved in [9]. This is the aim of this paper.

For convenience, we call "the Stokes problem" the problem without the convective term $(u \cdot \nabla)u$, and "the Navier–Stokes problem" the problem with the above term included. Concerning our approach to $W^{2,l}(\Omega)$ regularity results up to the boundary, the really new points mostly concern the stationary Stokes problem. In fact, in our proofs, the inclusion of the convective term, and the consideration of the evolution problem, are reduced in a very simple way to the stationary Stokes problem. Obviously, we do not claim that it is not possible to obtain better results by different methods. In our approach, (a) The Stokes evolution problem can be easily reduced to the stationary Stokes problem, with the same range of admissible values of p. (b) In the stationary case, the presence of the convective term requires an assumption of the type $p > p_0$ for some $p_0 < 2$, see Theorem 2.3. Under this assumption the regularity results for the Stokes and the Navier–Stokes stationary problems, coincide. (c) For the Navier–Stokes evolution problem we need a condition $p > p_1$, for some $p_1 > 2$. Hence the shear thinning case is excluded, except for sufficiently small initial data. In this last case we believe that it should be not difficult to prove the existence of a global, regular, solution.

2. Main results

In the sequel we consider the following very basic model of generalized Stokes stationary problem, where $\nu_T(u) = (1 + |\mathcal{D}u|)^{p-2}$:

$$\begin{cases} -\nabla \cdot \left((1 + |\mathcal{D}u|)^{p-2} \mathcal{D}u \right) + \nabla \pi = f, \\ \nabla \cdot u = 0, \end{cases} \tag{2.1}$$

under the non-slip boundary condition

$$u_{|\Gamma} = 0. \tag{2.2}$$

The domain Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of class C^2.

In the sequel we use the following exponents:

$$r(q) = \frac{2q}{2(2-p)+q}, \qquad \lambda(q) = \frac{2q}{2-p+q}, \qquad \mathcal{Q}(q) = \frac{6q}{8-4p+q}, \tag{2.3}$$

and also

$$\bar{q} = 4p - 2, \qquad l = \frac{4p-2}{p+1}. \tag{2.4}$$

Theorem 2.1. *Assume that $f \in L^{p'}(\Omega)$ and let $u \in V_p$ be a solution to the problem* (2.1), (2.2)*, where $\frac{3}{2} < p < 2$. Assume that*

$$\mathcal{D}u \in L^q(\Omega), \tag{2.5}$$

for some q satisfying

$$p \leqslant q \leqslant 6.$$

Then (*see* (2.3))

$$u \in W^{1,\mathcal{Q}(r)}(\Omega) \cap W^{2,r(q)}(\Omega), \qquad \nabla\pi \in L^{r(q)}(\Omega). \tag{2.6}$$

Further,

$$\|\nabla u\|_{\mathcal{Q}(q),\Omega} \leqslant C\big(1 + \|f\|_{p'}\big)\big(1 + \|\nabla u\|_{q,\Omega}^{\frac{2(2-p)}{3}}\big) \tag{2.7}$$

and

$$\big\|D^2 u\big\|_{r(q),\Omega} + \|\nabla\pi\|_{r(q),\Omega} \leqslant C\big(1 + \|f\|_{p'}\big)\big(1 + \|\nabla u\|_{q,\Omega}^{\frac{2-p}{2}}\big). \tag{2.8}$$

Note that the assumption (2.5) holds for $q = p$. This furnishes a first regularity theorem (statement left to the reader). Furthermore, Theorem 2.1 allows a bootstrap argument, similar to that introduced in references [4,5]. This leads to the following improvement, and extension to general boundaries, of Theorem 1.4 in [6].

Theorem 2.2. *Assume that $f \in L^{p'}(\Omega)$ and let $u \in V_p$, see* (3.2)*, be a solution to the problem* (2.1), (2.2)*, where $\frac{3}{2} < p < 2$. Then* (*see* (2.4))

$$u \in W^{2,l}(\Omega) \cap W^{1,\bar{q}}(\Omega), \qquad \nabla\pi \in L^l(\Omega). \tag{2.9}$$

Moreover,

$$\|u\|_{1,\bar{q}} \leqslant C\big(1 + \|f\|_{p'}^{\frac{3}{2p-1}}\big) \tag{2.10}$$

and

$$\|u\|_{2,l} \leqslant C\big(\|f\|_{p'} + \|f\|_{p'}^{\frac{5-p}{2p-1}}\big). \tag{2.11}$$

Concerning the full Navier–Stokes system

$$\begin{cases} -\nabla \cdot \big((\mu + |\mathcal{D}u|)^{p-2}\mathcal{D}u\big) + (u \cdot \nabla)u + \nabla\pi = f, \\ \nabla \cdot u = 0, \end{cases} \tag{2.12}$$

one has the following result.

Theorem 2.3. *Let u be a solution to the full Navier–Stokes equations* (2.12) *under the boundary condition* (2.2)*. Set*

$$p_0 = \frac{20}{11}. \tag{2.13}$$

Then, under the assumption $p > p_0$, (2.9) *holds.*

Besides the extension to general boundaries, the above result improves the lower bound $p_0 = \frac{15}{8}$ obtained in [6] and the lower bound $p_0 = \frac{7+\sqrt{35}}{7}$, obtained in [11]. It coincides with the value p_0 that was reached in reference [9].

It is worth noting that the boundedness of Ω is not essential here. In fact, our proof is done locally, i.e., in "small" neighborhoods of each point $x_0 \in \Gamma$. Consequently, the results hold, in particular, in any bounded subset of Ω, since boundedness is used only in order to work with a compact boundary (just to guarantee that local parameters associated with the boundary Γ have uniform bounds).

3. Notation. Weak solutions

In general we set

$$T_{\mathrm{sym}} = \frac{1}{2}\big(T + T^T\big), \tag{3.1}$$

where T is a generic tensor field and T^T is it transpose. In particular, $\mathcal{D}u = (\nabla u)_{\mathrm{sym}}$.

The symbol $\|.\|_p$ denotes the canonical norm in $L^p(\Omega)$. Further, $\|.\| = \|.\|_2$. We denote by $W^{k,p}(\Omega)$, k a positive integer and $1 < p < \infty$, the usual Sobolev space of order k, by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ and by $W^{-1,p'}(\Omega)$ the strong dual of $W_0^{1,p}(\Omega)$, where $p' = p/(p-1)$. The canonical norms in these spaces are denoted by $\|.\|_{k,p}$. $L^p_\#(\Omega)$ denotes the subspace of L^p consisting of functions with vanishing mean value.

In notation concerning duality pairings and norms, we will not distinguish between scalar and vector fields. Very often we also omit from the notation the symbols indicating the domains Ω or Γ, provided that the meaning remains clear.

We set

$$\mathbb{L}^p(\Omega_0) = \big[L^p(\Omega_0)\big]^3, \qquad \mathbb{W}^{k,p}(\Omega_0) = \big[W^{k,p}(\Omega_0)\big]^3, \qquad \mathbb{W}_0^{1,p}(\Omega_0) = \big[W_0^{1,p}(\Omega_0)\big]^3,$$

for any open subset Ω_0 of $\mathbb{R}^3$.

We set

$$V_p = \big\{v \in \mathbb{W}^{1,p}(\Omega)\colon\ (\nabla \cdot v)_{|\Omega} = 0;\ \ v_{|\Gamma} = 0\big\}. \tag{3.2}$$

Note that, by appealing to inequalities of Korn's type, one shows that there is a positive constant c such that

$$\|\nabla v\|_p + \|v\|_p \leqslant c\|\mathcal{D}v\|_p, \tag{3.3}$$

for each $v \in V_p$. Hence the two quantities above are equivalent norms in V_p. Actually, $\|\mathcal{D}v\|_p$ is a norm in $\mathbb{W}_0^{1,p}$.

We denote by $c, \bar{c}, c_1, c_2$, etc., positive constants that depend, at most, on Ω and p. The dependence of the constants c on p is not crucial provided that $1 < p_0 \leqslant p \leqslant p_1 < \infty$. The same symbol c may denote different constants, even in the same equation.

Definition. We say that a pair (u, π) is a *weak solution* of problem (2.1), (2.2) if it belongs to $\mathbb{W}_0^{1,p}(\Omega) \times L_\#^{p'}(\Omega)$, and if it satisfies

$$\int_\Omega \big(1 + |\mathcal{D}u|\big)^{p-2}\mathcal{D}u \cdot \mathcal{D}\phi\, dx - \int_\Omega \pi(\nabla \cdot \phi)\, dx + \int_\Omega (\nabla \cdot u)\psi\, dx = \int_\Omega f \cdot \phi\, dx, \tag{3.4}$$

for each $(\phi, \psi) \in \mathbb{W}_0^{1,p}(\Omega) \times L_\#^{p'}(\Omega)$.

Note that $(1+b)^{p-2}b \leqslant 2^{p-1}(1+b^{p-1})$, for $b \geqslant 0$.

Since a solution u of (3.4) necessarily satisfies

$$\int_\Omega \nabla \cdot u\, dx = 0,$$

it readily follows that (3.4) holds for each $(\phi, \psi) \in \mathbb{W}_0^{1,p}(\Omega) \times L^{p'}(\Omega)$.

Existence and uniqueness of the above solution is well known.

By replacing v by u and ψ by π in Eq. (3.4) one gets

$$\int_\Omega \big(1 + |\mathcal{D}u|\big)^{p-2}|\mathcal{D}u|^2\, dx \leqslant \langle f, u\rangle, \tag{3.5}$$

where the symbols $\langle\cdot,\cdot\rangle$ denote a duality pairing. Hence, by setting $A = \{x\colon\ |\mathcal{D}u| \leqslant 1, \}$, $B = \{x\colon\ |\mathcal{D}u| > 1\}$, one shows that

$$\int_\Omega \big(1+|\mathcal{D}u|\big)^{p-2}|\mathcal{D}u|^2\,dx \geqslant 2^{p-2}\int_A |\mathcal{D}u|^2\,dx + 2^{p-2}\int_B |\mathcal{D}u|^p\,dx.$$

By appealing to the obvious inequality $|\mathcal{D}u|^p \leqslant 1+|\mathcal{D}u|^2$, one shows that

$$\int_\Omega \big(1+|\mathcal{D}u|\big)^{p-2}|\mathcal{D}u|^2\,dx \geqslant 2^{p-2}\bigg(\int_\Omega |\mathcal{D}u|^p\,dx - |\Omega|\bigg).$$

It follows that

$$\|\mathcal{D}u\|_p^p \leqslant 2^{2-p}\langle f,u\rangle + |\Omega|. \tag{3.6}$$

Hence

$$\|\nabla u\|_p^{p-1} \leqslant c\big(\|f\|_{-1,p'}+1\big), \tag{3.7}$$

where, in general, q' denotes the dual exponent of q, namely

$$q' = \frac{q}{q-1}. \tag{3.8}$$

Remark 3.1. Since

$$\int_\Omega (u\cdot\nabla)u\cdot u\,dx = 0,$$

it readily follows that all the above estimates hold for weak solutions u to the complete Navier–Stokes equations

$$\begin{cases} -\nabla\cdot\big(\big(1+|\mathcal{D}u|\big)^{p-2}\mathcal{D}u\big)+\nabla\pi = F,\\ \nabla\cdot u = 0, \end{cases} \tag{3.9}$$

where

$$F = f - (u\cdot\nabla)u.$$

This means, in particular, that (3.7) holds with the external force f *not replaced* by F.

The following result, basically due to Nečas (see [46]), is well known.

Lemma 3.1. *If a distribution g is such that $\nabla g \in \mathbb{W}^{-1,\alpha}(\Omega)$ then $g \in L^\alpha(\Omega)$ and*

$$\|g\|_{L^\alpha_\#} \leqslant c\|\nabla g\|_{-1,\alpha}, \tag{3.10}$$

where $L^\alpha_\# = L^\alpha/\mathbb{R}$.

By setting in (3.4) $\psi = 0$ and by using test-functions $\phi \in C_0^\infty(\Omega)$ one gets

$$\nabla\pi = -\nabla\cdot\big[\big(1+|\mathcal{D}u|\big)^{p-2}\mathcal{D}u\big] + f. \tag{3.11}$$

By appealing to (3.10) we prove that

$$\|\pi\|_{p'} \leqslant c\big(\|f\|_{-1,p'}+1\big). \tag{3.12}$$

For convenience we fix π by assuming that its mean value in Ω vanishes.

4. The change of variables

In order to reduce our problem, by a suitable change of variables, to a problem involving a flat boundary, we need to consider functions with a sufficiently small support.

Let $x_0 \in \Gamma$ be given and let Π be the tangent plane to Γ at x_0. We assume that the axes of x_i, $i = 1,2,3$, are such that the origin coincides with x_0 and the x_3 axis has the direction of the inward normal to Γ at x_0. Hence the axes of x_i, $i = 1,2$, lie in the plane Π. We may use this particular system of coordinates since the analytical expressions that appear on the left-hand side of (3.4) are invariant under orthogonal transformations.

We assume that Γ is a manifold of class C^2. Let $x_0 \in \Gamma$ be given and let $(x', x_3) = (x_1, x_2, x_3)$, be the above system of coordinates. We assume that there is a positive real constant $\bar{a}$ and a real function $x_3 = \eta(x')$, of class C^2 defined on the sphere $\{x' \colon |x'| < \bar{a}\}$, such that: the points x for which $x_3 = \eta(x')$ belong to Γ; and the points x for which $\eta(x') < x_3 < \bar{a} + \eta(x')$ belong to Ω; the points x for which $-\bar{a} + \eta(x') < x_3 < \eta(x')$ belong to $\mathbb{R}^3 - \Omega$. Without loss of generality, we assume that $\bar{a} \leqslant 1$. In principle $\bar{a}$ may depend on the point $x_0 \in \Gamma$. However, since Γ is regular and bounded, the greatest lower bound $\bar{a}$ of the values $\bar{a}(x_0)$ is positive. Note that if we do not assume that Ω is bounded then the above greatest lower bound could be equal to zero. In this case our results hold on any bounded subset of $\overline{\Omega}$.

The positive values of a for which $a \leqslant \bar{a}$ are called admissible. For each admissible a define

$$\begin{aligned} I_a &= \bigl\{x \colon |x'| < a,\ -a + \eta(x') < x_3 < a + \eta(x')\bigr\}, \\ \Omega_a &= \bigl\{x \in I_a \colon \eta(x') < x_3\bigr\}, \\ \Gamma_a &= \bigl\{x \in I_a \colon x_3 = \eta(x')\bigr\}. \end{aligned} \tag{4.1}$$

Clearly $\Omega_a = \Omega \cap I_a$ and $\Gamma_a = \Gamma \cap I_a$.

Actually, we extend the function $\eta(x')$ to the whole of Ω_a by setting $\eta(x', x_3) = \eta(x')$. Nevertheless, since η is independent of x_3, we use the notation $\eta(x')$.

It is worth noting that along the course of our proof (more than once, but a finite number of times) we need to impose additional smallness assumptions on the positive parameter $\bar{a}$, i.e., on the admissible values of a. Actually, each time we appeal to (4.8) we are just imposing a smaller positive upper bound to the admissible values of a.

Next we introduce the change of variables $y = Tx$ given by

$$(y_1, y_2, y_3) = \bigl(x_1, x_2, x_3 - \eta(x')\bigr), \qquad (x_1, x_2, x_3) = \bigl(y_1, y_2, y_3 + \eta(y')\bigr), \tag{4.2}$$

and set

$$\begin{aligned} J_a &= \bigl\{y \colon |y'| < a,\ -a < y_3 < a\bigr\}, \\ Q_a &= \{y \in J_a \colon 0 < y_3\}, \\ \Lambda_a &= \{y \in J_a \colon y_3 = 0\}. \end{aligned} \tag{4.3}$$

The map T is a C^2 diffeomorphism of I_a onto J_a, that maps Ω_a onto Q_a and Γ_a onto Λ_a. Note that the Jacobian determinant of the map T is equal to 1.

We define functions $\widetilde{g}$ by setting $\widetilde{g}(y) = g(x)$ or, more precisely, by

$$\widetilde{g}(y) = g\bigl(T^{-1}(y)\bigr), \tag{4.4}$$

where g denotes an arbitrary scalar or vector field. As a notation rule, $g = g(x)$ and $\widetilde{g} = \widetilde{g}(y)$. Moreover, partial derivatives and differential operators when applied to functions g concern the x variables and when applied to functions $\widetilde{g}$ concern the y variables. We primarily use the notation $\partial_k g$ instead of $\frac{\partial g}{\partial x_k}$. Hence

$$\partial_k \widetilde{g} = \frac{\partial \widetilde{g}(y)}{\partial y_k}$$

and

$$\partial_k g = \frac{\partial g(x)}{\partial x_k}.$$

Note the distinction between $\widetilde{\nabla f}$ and $\nabla \widetilde{f}$. Actually, $\widetilde{\nabla f}(y) = (\nabla_x f)(T^{-1}(y))$ and $(\nabla \widetilde{f})(y) = \nabla_y[f(T^{-1}(y))]$.

Since some expressions are quite long, in addition to the "tilde" notation we also use the symbol $\mathcal{T}$ to denote the map $f \to \widetilde{f}$. In other words,

$$(\mathcal{T} f)(y) = \widetilde{f}(y).$$

Vector fields are transformed here coordinate by coordinate (as independent scalars). More precisely

$$\widetilde{v}_j(y) = v_j(x) = v_j\bigl(y', y_3 + \eta(y')\bigr), \tag{4.5}$$

where $j = 1, 2, 3$. Conversely,

$$v_j(x) = \widetilde{v}_j(y) = v_j\bigl(x', x_3 - \eta(x')\bigr). \tag{4.6}$$

Given x, if $y = Tx$ then $y' = x'$. Hence $\widetilde{\eta}(y) = \eta(x) = \eta(x') = \eta(y')$, moreover $\frac{\partial \eta(x')}{\partial x_j} = \frac{\partial \eta(y')}{\partial y_j}$, and so on. In the sequel we identify the above functions and use the sole notation $\eta(y')$.

We set

$$\mathbb{V}(\Omega_a) = \{v\colon\ v \in \mathbb{W}_0^{1,p}(\Omega_a),\ \operatorname{supp} v \subset I_a\},$$
$$\mathbb{V}(Q_a) = \{\widetilde{v}\colon\ \widetilde{v} \in \mathbb{W}_0^{1,p}(Q_a),\ \operatorname{supp} \widetilde{v} \subset J_a\}.$$

Clearly, if a test-function $\phi(x)$ belongs to $\mathbb{V}(\Omega_a)$ the transformed function $\widetilde{\phi}(y)$ belongs to $\mathbb{V}(Q_a)$.

A main point in the sequel is that

$$\partial_j \eta(0) = 0, \quad j = 1, 2, \tag{4.7}$$

which holds since Π is tangential to Γ at x_0. The following trivial, but fundamental, result is a consequence of (4.7) together with the continuity of $\nabla\eta$ over Γ.

Lemma 4.1. *Given a positive ϵ_0 there is an $a(\epsilon_0) > 0$ such that*

$$|\nabla\eta(y')| < \epsilon_0, \quad \text{for each } y' \text{ such that } |y'| < a(\epsilon_0). \tag{4.8}$$

Moreover, $a(\epsilon_0)$ is independent of the point x_0.

Note that $a(\epsilon_0)$ depends on the $C^1(J_a)$ norm of η. Since Γ is compact the desired independence holds.

In the sequel we express the derivatives with respect to the y variables of functions $\widetilde{\phi}(y)$ in terms of the transformations of the derivatives of the original functions $\phi(x)$.

Lemma 4.2. *One has the following formulas*:

$$(\partial_k\widetilde{\phi})(y) = (\widetilde{\partial_k\phi})(y) + (\partial_k\eta)(y')(\widetilde{\partial_3\phi})(y) \tag{4.9}$$

and

$$(\widetilde{\partial_k\phi})(y) = (\partial_k\widetilde{\phi})(y) - (\partial_k\eta)(y')(\partial_3\widetilde{\phi})(y). \tag{4.10}$$

If $k = 3$ the second terms on the above right-hand sides vanish identically.

Proof. Since

$$\widetilde{\phi}(y) = \phi(T^{-1}y)$$

it follows that

$$(\partial_k\widetilde{\phi})(y) = (\partial_k\phi)(T^{-1}y) + (\partial_3\phi)(T^{-1}y)(\partial_k\eta)(y') = (\widetilde{\partial_k\phi})(y) + (\partial_k\eta)(y')(\widetilde{\partial_3\phi})(y).$$

Note that $\partial_3\widetilde{\phi} = \widetilde{\partial_3\phi}$. □

From the above lemma it follows that

$$(\widetilde{\nabla\phi})(y) = (\nabla\widetilde{\phi})(y) - (\nabla\eta)(y') \otimes (\partial_3\widetilde{\phi})(y) \tag{4.11}$$

and that

$$(\widetilde{\nabla\cdot\phi})(y) = (\nabla\cdot\widetilde{\phi})(y) - (\nabla\eta)(y')\cdot(\partial_3\widetilde{\phi})(y). \tag{4.12}$$

Lemma 4.3. *Given an $\epsilon_0 \in]0,1[$ there is an $a(\epsilon_0) > 0$ such that if $a \leqslant a(\epsilon_0)$ then*

$$|(\nabla\widetilde{\phi})(y) - (\widetilde{\nabla\phi})(y)| \leqslant \epsilon_0 |(\widetilde{\partial_3\phi})(y)|, \quad \forall y \in Q_a. \tag{4.13}$$

The same result holds if we replace y by $y - h$ (a tangential translation, see the next section). Clearly we may replace ∇ by $\mathcal{D}$.

Proof. From (4.9) one shows that the left-hand side of (4.13) is bounded by $|\nabla\eta(y')||(\widetilde{\partial_3\phi})(y)|$. Since $\nabla\eta(0) = 0$ it follows that $|\nabla\eta(y')| \leqslant \epsilon_0$ in a sufficiently small neighborhood of x_0. □

Remark. Note that the identity $\nabla\phi(x) = (\widetilde{\nabla\phi})(y)$ together with (4.13) leads to a "point wise equivalence" between $|\nabla\phi(x)|$, $|(\widetilde{\nabla\phi})(y)|$ and $|(\nabla\widetilde{\phi})(y)|$. In particular, the L^q-norms of these quantities are equivalent.

Next, from (4.9) one gets (point wisely in y) that

$$\partial_s(\partial_k\widetilde{\phi}) - \partial_s(\widetilde{\partial_k\phi}) = (\partial_s\partial_k\eta)(\partial_3\widetilde{\phi}) + (\partial_k\eta)(\partial_s\partial_3\widetilde{\phi}).$$

In particular, by appealing to (4.8), one shows that

$$\left|\left|\partial_s(\partial_k\widetilde{\phi})\right| - \left|\partial_s(\widetilde{\partial_k\phi})\right|\right| \leqslant c\left|D^2\eta\right||\partial_3\widetilde{\phi}| + \epsilon_0\left|\partial_s(\widetilde{\partial_3\phi})\right|,$$

point wisely in the y variables. By appealing to the above estimate for $k = 1, 2, 3$, and to (4.8) with (for instance) $\epsilon_0 = 1$, one proves the following result.

Lemma 4.4. *Let $s = 1, 2, 3$ be fixed. For a sufficiently small $a(\epsilon_0) > 0$ one has*

$$\left|\partial_s(\nabla\widetilde{\phi})\right| \leqslant 2\left|\partial_s(\widetilde{\nabla\phi})\right| + c\left|D^2\eta\right||\partial_3\widetilde{\phi}|, \quad \forall y \in Q_a. \tag{4.14}$$

The left-hand side and the first term on the right-hand side may be switched.

Clearly ϕ may be a vector field since the above result holds for each component. In particular, by considering $s = 1, 2$, one proves the first estimate in the following lemma.

Lemma 4.5. *For a sufficiently small $a(\epsilon_0) > 0$ one has*

$$\left|\nabla_*(\nabla\widetilde{u})\right| \leqslant c\left|\nabla_*(\widetilde{\nabla u})\right| + c\left|D^2\eta\right||\partial_3\widetilde{u}|, \quad \forall y \in Q_a. \tag{4.15}$$

Moreover,

$$\left|\nabla_*(\mathcal{D}\widetilde{u})\right| \leqslant \left|\nabla_*(\widetilde{\mathcal{D}u})\right| + c|\nabla\eta||\nabla_*\partial_3\widetilde{u}| + c\left|D^2\eta\right||\partial_3\widetilde{u}|, \quad \forall y \in Q_a. \tag{4.16}$$

The left-hand side and the first term on the right-hand side in both estimates may be switched.

The estimate (4.16) follows since, if we apply the above transformation formulae to $\partial_s(\mathcal{D}\widetilde{u})$, we get

$$\partial_s(\mathcal{D}\widetilde{u})_{i,j} = \partial_s(\widetilde{\mathcal{D}u})_{i,j} + \frac{1}{2}\left[(\partial_j\eta)(\partial_s\partial_3\widetilde{u}_i) + (\partial_i\eta)(\partial_s\partial_3\widetilde{u}_j)\right] + \frac{1}{2}\left[(\partial_s\partial_j\eta)(\partial_3\widetilde{u}_i) + (\partial_s\partial_i\eta)(\partial_3\widetilde{u}_j)\right].$$

By iteration, (4.10) may be extended to higher order derivatives (not used in the sequel):

$$\mathcal{T}\left(\partial^2_{jk}\phi\right) = \partial^2_{jk}\widetilde{\phi} - (\partial_k\eta)\partial^2_{j3}\widetilde{\phi} - (\partial_j\eta)\partial^2_{k3}\widetilde{\phi} + (\partial_j\eta)(\partial_k\eta)\partial^2_3\widetilde{\phi} - \left(\partial^2_{jk}\eta\right)\partial_3\widetilde{\phi}.$$

Remark. We want to emphasize that, basically, our regularity results will be proved in the following local form. Let x_0 and Ω_a be as above. If $(u, \pi) \in \mathbb{W}^{1,p}(\Omega_a) \times L^{p'}(\Omega_a)$ satisfies (2.1) in the weak sense in Ω_a and satisfies (2.2) in Γ_a, then the regularity results hold in Ω_r for $r < a$ (for instance, for $r = \frac{a}{2}$). We prove this local result by assuming that $a > 0$ is sufficiently small. Our final value of a is not necessarily equal to the initial one. As we proceed through the proof we may need to consider smaller values of a. However we will show explicitly that each new (smaller) value of a depends only on an upper bound of the $C^2(J_a)$ norm of η. In particular, a positive lower bound for a, independent of the point x_0, exists since Γ is compact. This leads to the global result in the whole of Ω.

5. Translations and related properties

In the sequel we deal with translations of h_j in the y_j-direction, $j = 1, 2$. For notational convenience we consider the case $j = 1$ and set $h = h_1$. We use the following convention:

$$y + h = (y_1 + h, y_2, y_3), \qquad y' + h = (y_1 + h, y_2).$$

The amplitude $|h|$ of the translations is always assumed to be smaller than the distance from the support of $\widetilde{\phi}$ to the set $(\partial Q_a)\backslash\Lambda_a$.

A test-function $\phi(x)$ is transformed into a function $\widetilde{\phi}(y)$. Since in the following we make translations in the y variables we need to determine (and study) the differential properties of the test-function $\phi_h(x)$ such that $\widetilde{(\phi_h)}(y) = \widetilde{\phi}(y + h)$. This is the aim of this section.

Lemma 5.1. *Let $\phi(x) \in \mathbb{V}(\Omega_a)$. Define ϕ_h by*

$$\phi_h(x) = \phi\left(x_1 + h, x_2, x_3 - \eta(x') + \eta(x' + h)\right). \tag{5.1}$$

Then

$$\widetilde{\phi}_h(y) = \widetilde{\phi}(y + h). \tag{5.2}$$

The proof is left to the reader.

Next we want to establish the transformation law for derivatives of the "pseudo-translations" $\phi_h(x)$. One has the following result.

Lemma 5.2. *Let $\phi(x) \in \mathbb{V}(\Omega_a)$, let $\phi_h(x)$ be as in the previous lemma, and let $k \leqslant 3$ be fixed. Then*

$$(\widetilde{\partial_k \phi_h})(y) = (\widetilde{\partial_k \phi})(y+h) + (\widetilde{\partial_3 \phi})(y+h)\big[(\partial_k \eta)(y'+h) - (\partial_k \eta)(y')\big]. \tag{5.3}$$

If $k = 3$ the second term on the right-hand side vanishes identically.

Proof. From (5.1) it readily follows that

$$\begin{aligned}(\partial_k \phi_h)(x) &= (\partial_k \phi)\big(x'+h, x_3 + \eta(x'+h) - \eta(x')\big)\\ &\quad + (\partial_n \phi)\big(x'+h, x_3 + \eta(x'+h) - \eta(x')\big)\big[(\partial_k \eta)(x'+h) - (\partial_k \eta)(x')\big].\end{aligned} \tag{5.4}$$

Note that the last term is not taken into account if $k = 3$. By the definition of the "tilde" functions

$$(\widetilde{\partial_k \phi_h})(y) = (\partial_k \phi_h)\big(T^{-1}y\big) = (\partial_k \phi_h)(\bar{x}),$$

where

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = \big(y', y_3 + \eta(y')\big).$$

Hence from (5.4) with x replaced by $\bar{x}$ we get an expression for $(\widetilde{D_k \phi_h})(y)$ in terms of $\bar{x}$. By taking into account the definition of $\bar{x}$ we obtain

$$(\widetilde{\partial_k \phi_h})(y) = (\partial_k \phi)\big(y'+h, y_3 + \eta(y'+h)\big) + (\partial_3 \phi)\big(y'+h, y_3 + \eta(y'+h)\big)\big[(\partial_k \eta)(y'+h) - (\partial_k \eta)(y')\big]. \tag{5.5}$$

Since $(y'+h, y_3 + \eta(y'+h)) = T^{-1}(y+h)$ it follows that

$$(\partial_k \phi)\big(y'+h, y_3 + \eta(y'+h)\big) = (\widetilde{\partial_k \phi})(y+h).$$

Consequently (5.3) follows from (5.5). □

By setting in general

$$(\nabla \phi)_{ik} = \partial_k \phi_i$$

it follows from (5.3) that

$$(\widetilde{\nabla \phi_h})(y) = (\widetilde{\nabla \phi})(y+h) + (\widetilde{\partial_3 \phi})(y+h) \otimes \big[(\nabla \eta)(y'+h) - (\nabla \eta)(y')\big], \tag{5.6}$$

where, since η does not depend on the 3rd variable, we set

$$\nabla \eta = (\partial_1 \eta, \partial_2 \eta).$$

In particular, since $\mathcal{D}u = (\nabla u)_{\mathrm{sym}}$,

$$(\widetilde{\mathcal{D} \phi_h})(y) = (\widetilde{\mathcal{D} \phi})(y+h) + \big\{(\widetilde{\partial_3 \phi})(y+h) \otimes \big[(\nabla \eta)(y'+h) - (\nabla \eta)(y')\big]\big\}_{\mathrm{sym}}. \tag{5.7}$$

Moreover,

$$(\widetilde{\nabla \cdot \phi_h})(y) = (\widetilde{\nabla \cdot \phi})(y+h) + (\widetilde{\partial_3 \phi})(y+h) \cdot \big[(\nabla \eta)(y'+h) - (\nabla \eta)(y')\big]. \tag{5.8}$$

Lemma 5.3. *Given an $\epsilon_0 \in]0,1[$ there is an $a(\epsilon_0) > 0$ such that if $a \leqslant a(\epsilon_0)$ then*

$$\begin{aligned}&\big|\big((\widetilde{\nabla} \phi)(y) - (\widetilde{\nabla} \phi)(y-h)\big) - \big((\nabla \widetilde{\phi})(y) - (\nabla \widetilde{\phi})(y-h)\big)\big|\\ &\quad \leqslant \epsilon_0 \big|(\partial_3 \widetilde{\phi})(y) - (\partial_3 \widetilde{\phi})(y-h)\big| + |h| \|\eta\|_{C^2(Q_a)} \big|(\partial_3 \widetilde{\phi})(y-h)\big|.\end{aligned} \tag{5.9}$$

Proof. From (4.10) one has

$$\begin{aligned}&\big((\widetilde{\nabla} \phi)(y) - (\widetilde{\nabla} \phi)(y-h)\big) - \big((\nabla \widetilde{\phi})(y) - (\nabla \widetilde{\phi})(y-h)\big)\\ &\quad = -\nabla \eta(y') \otimes \big((\partial_3 \widetilde{\phi})(y) - (\partial_3 \widetilde{\phi})(y-h)\big) - \big(\nabla \eta(y') - \nabla \eta(y'-h)\big) \otimes (\partial_3 \widetilde{\phi})(y-h).\end{aligned}$$

Hence, in a sufficiently small neighborhood of x_0, (5.9) holds. □

5.1. Estimates for some second-order derivatives of the velocity in terms of the pressure

For convenience, in the sequel C denotes positive constants which are bounded from above provided that the quantities $\|\nabla\eta\|_{C^1(\Lambda_a)}$ and $\|\nabla\theta\|_{C^1(\Omega_a)}$ are bounded from above. These constants may also depend on the bounded quantities $\|\nabla u\|_p, \|\pi\|_{p'}, \|f\|_{p'}$ (recall (3.7) and (3.12)). In short,

$$C = C\big(\|\nabla u\|_p, \|\pi\|_{p'}, \|\nabla\eta\|_{C^1(\Lambda_a)}, \|\nabla\theta\|_{C^1(\Omega_a)}\big). \tag{5.10}$$

Explicit expressions for these quantities follow easily from our calculations. For the reader's convenience (and for completeness) we often write the explicit dependence on the above quantities before including them in a constant of type C. Multiplicative constants of type c will be incorporated in C.

In the sequel, in the absence of an explicit indication, tilde-functions inside integrals are calculated at the generic point y. Compare Eqs. (5.11) and (5.12). Moreover, in the absence of an explicit indication, norms of functions of the x variable concern the domain Ω_a and norms of tilde-functions concern the domain Q_a.

From (3.4), by making the change of variables $x \to Tx = y$, it follows that

$$\int \big(1 + |\widetilde{\mathcal{D}u}|\big)^{p-2}\widetilde{\mathcal{D}u}\cdot\widetilde{\mathcal{D}\phi}\,dy - \int \widetilde{\pi}\,(\widetilde{\nabla\cdot\phi})\,dy + \int (\widetilde{\nabla\cdot u})\widetilde{\psi}\,dy = \int \widetilde{f}\cdot\widetilde{\phi}\,dy, \tag{5.11}$$

for each $\widetilde{\phi} \in \mathbb{V}(Q_a)$ and each $\widetilde{\psi} \in L^p(Q_a)$. Recall that the Jacobian determinant of the T-transform is equal to one.

Next we consider Eq. (5.11) with ϕ and ψ replaced by the admissible test-functions ϕ_h and ψ_h, respectively. Then by the change of variables $y \to y - h$ we show that

$$\begin{aligned}
&\int \big(1 + \big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h) : \widetilde{\mathcal{D}\phi_h}(y-h)\,dy - \int \widetilde{\pi}(y-h)(\widetilde{\nabla\cdot\phi_h})(y-h)\,dy + \int (\widetilde{\nabla\cdot u})(y-h)\widetilde{\psi_h}(y-h)\,dy \\
&\quad = \int \widetilde{f}(y-h)\cdot\widetilde{\phi}_h(y-h)\,dy,
\end{aligned} \tag{5.12}$$

for each $\widetilde{\phi} \in \mathbb{V}(Q_a)$ and each $\widetilde{\psi} \in L^p(Q_a)$.

By appealing to (5.2), (5.3), (5.6), (5.7) and (5.8) we may write Eq. (5.12) in the form

$$\begin{aligned}
&\int \big(1 + \big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h) : \widetilde{\mathcal{D}\phi}(y)\,dy \\
&\qquad + \int \big(1 + \big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h) : \big[(\widetilde{\partial_3\phi})(y)\otimes\big[(\nabla\eta)(y') - (\nabla\eta)(y'-h)\big]\big]_{\mathrm{sym}}\,dy \\
&\qquad - \int \widetilde{\pi}(y-h)(\widetilde{\nabla\cdot\phi})(y)\,dy - \int \widetilde{\pi}(y-h)(\widetilde{\partial_3\phi})(y)\cdot\big[(\nabla\eta)(y') - (\nabla\eta)(y'-h)\big]\,dy + \int (\widetilde{\nabla\cdot u})(y-h)\widetilde{\psi}(y)\,dy \\
&\quad = \int \widetilde{f}(y-h)\cdot\widetilde{\phi}(y)\,dy.
\end{aligned} \tag{5.13}$$

Finally by taking the difference, side by side, between Eqs. (5.11) and (5.13) we get

$$\begin{aligned}
&\int \big(\big(1 + \big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y) - \big(1 + \big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big) : (\widetilde{\mathcal{D}\phi})(y)\,dy \\
&\qquad - \int \big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)(\widetilde{\nabla\cdot\phi})(y)\,dy + \int \big((\widetilde{\nabla\cdot u})(y) - (\widetilde{\nabla\cdot u})(y-h)\big)\widetilde{\psi}(y)\,dy \\
&\quad = -\int \widetilde{f}(y)\cdot\big(\widetilde{\phi}(y+h) - \widetilde{\phi}(y)\big)\,dy \\
&\qquad + \int \big(1 + \big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h) : \big[(\widetilde{\partial_3\phi})(y)\otimes\big[(\nabla\eta)(y') - (\nabla\eta)(y'-h)\big]\big]_{\mathrm{sym}}\,dy \\
&\qquad - \int \widetilde{\pi}(y-h)(\widetilde{\partial_3\phi})(y)\cdot\big[(\nabla\eta)(y') - (\nabla\eta)(y'-h)\big]\,dy.
\end{aligned} \tag{5.14}$$

Remark 5.1. Now we would like to replace in (5.14) $\widetilde{\nabla\phi}(y)$ with $\widetilde{\nabla u}(y) - \widetilde{\nabla u}(y-h)$ and, by consequence, $\widetilde{\mathcal{D}\phi}(y)$ with $\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)$. Unfortunately this is not allowed since $\widetilde{\nabla u}(y-h)$ is not the transformation of the gradient of an x-test function. However our goal will be obtained "up to a perturbation term" by setting in Eq. (5.14)

$$\phi(x) = \big(u(x) - u_{-h}(x)\big)\theta^2(x), \tag{5.15}$$

where θ is an arbitrary regular real function such that

$$\operatorname{supp}\theta \subset I_a.$$

Just for the reader's convenience, assume from now on that $0 < \theta(x) \leqslant 1$. Note that $\widetilde{(\theta^2)} = (\widetilde{\theta})^2$ and $\widetilde{\nabla \theta^2} = 2\widetilde{\theta}\widetilde{\nabla}\theta$. Clearly

$$\widetilde{\phi}(y) = \big(\widetilde{u}(y) - \widetilde{u}(y-h)\big)(\widetilde{\theta})^2(y). \tag{5.16}$$

Lemma 5.4. *Let $\phi(x)$ be the admissible test-function given by* (5.15). *Then the y-transformed of $\nabla\phi(x)$, $\mathcal{D}\phi(x)$, $\partial_3\phi(x)$ and $\nabla\cdot\phi(x)$ are respectively given by* (5.17), (5.18), (5.19) *and* (5.20) *below.*

Proof. By taking the gradient of both sides of Eq. (5.15), by passing from the x to the y variables and by appealing to (5.6) it readily follows that

$$\begin{aligned}\widetilde{\nabla\phi}(y) &= \big(\widetilde{\nabla u}(y) - \widetilde{\nabla u}(y-h)\big)(\widetilde{\theta})^2(y) + (\widetilde{\partial_3 u})(y-h)\otimes\big[(\nabla\eta)(y') - (\nabla\eta)(y'-h)\big](\widetilde{\theta})^2(y)\\ &\quad + 2\widetilde{\theta}(y)\big(\widetilde{u}(y) - \widetilde{u}(y-h)\big)\otimes\widetilde{\nabla}\theta(y).\end{aligned} \tag{5.17}$$

In particular, one has

$$\begin{aligned}\widetilde{\mathcal{D}\phi}(y) &= \big(\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)\big)(\widetilde{\theta})^2(y) + \big\{(\widetilde{\partial_3 u})(y-h)\otimes\big[(\nabla\eta)(y') - (\nabla\eta)(y'-h)\big](\widetilde{\theta})^2(y)\big\}_{\mathrm{sym}}\\ &\quad + 2\widetilde{\theta}(y)\big\{\big(\widetilde{u}(y) - \widetilde{u}(y-h)\big)\otimes\widetilde{\nabla}\theta(y)\big\}_{\mathrm{sym}},\end{aligned} \tag{5.18}$$

and also (since $\partial_3\eta = 0$)

$$\widetilde{\partial_3\phi}(y) = \big(\widetilde{\partial_3 u}(y) - \widetilde{\partial_3 u}(y-h)\big)(\widetilde{\theta})^2(y) + 2\widetilde{\theta}(y)\big(\widetilde{u}(y) - \widetilde{u}(y-h)\big)\widetilde{\partial_3\theta}(y). \tag{5.19}$$

Similarly, from (5.2) and (5.8) it readily follows that

$$\begin{aligned}\widetilde{\nabla\cdot\phi}(y) &= \big(\widetilde{\nabla\cdot u}(y) - \widetilde{\nabla\cdot u}(y-h)\big)(\widetilde{\theta})^2(y) + (\widetilde{\partial_3 u})(y-h)\cdot\big[(\nabla\eta)(y') - (\nabla\eta)(y'-h)\big](\widetilde{\theta})^2(y)\\ &\quad + 2\widetilde{\theta}(y)\big(\widetilde{u}(y) - \widetilde{u}(y-h)\big)\cdot\widetilde{\nabla}\theta(y). \qquad \square\end{aligned} \tag{5.20}$$

On the other hand, by setting

$$\psi(x) = \big(\pi(x) - \pi_{-h}(x)\big)\theta^2(x), \tag{5.21}$$

it follows

$$\widetilde{\psi}(y) = \big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)(\widetilde{\theta})^2(y). \tag{5.22}$$

Next we replace in Eq. (5.14) the test-functions ϕ and ψ by the expressions in Eqs. (5.15) and (5.21). We start by estimating each term which appears in Eq. (5.14). In order to treat the first integral on the left-hand side of (5.14) we appeal to the following well-known result. For the proof see, for instance, Lemma 2.19 in reference [53].

Let A, B be two symmetric matrices. Then

$$\begin{aligned}&\big((1+|A|)^{p-2}A - (1+|B|)^{p-2}B\big)\cdot(A-B) \geqslant c\big(1+|A|+|B|\big)^{p-2}|A-B|^2,\\ &\big|(1+|A|)^{p-2}A - (1+|B|)^{p-2}B\big| \leqslant c\big(1+|A|+|B|\big)^{p-2}|A-B|.\end{aligned} \tag{5.23}$$

Proposition 5.1. *Let $\widetilde{\phi}(y)$ be given by* (5.15). *Then*

$$\begin{aligned}&\int\big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y) - \big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big) : \widetilde{\mathcal{D}\phi}(y)\,dy\\ &\quad\geqslant c\int\big(1+\big|(\widetilde{\mathcal{D}u})(y)\big| + \big|(\widetilde{\mathcal{D}u})(y-h)\big|\big)^{p-2}\big|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)\big|^2(\widetilde{\theta})^2(y)\,dy - C\|\nabla\widetilde{u}\|_p^p h^2.\end{aligned} \tag{5.24}$$

Proof. For convenience, denote by S_1 the left-hand side of (5.24). By (5.18) one has

$$\begin{aligned}S_1 &= \int\big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y) - \big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big) : \big(\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)\big)(\widetilde{\theta})^2(y)\,dy\\ &\quad + \int\big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y) - \big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big)\\ &\quad \cdot\big\{(\widetilde{\partial_3 u})(y-h)\otimes\big[(\nabla h)(y') - (\nabla h)(y'-h)\big]\big\}_{\mathrm{sym}}(\widetilde{\theta})^2(y)\,dy\\ &\quad + \int\big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y) - \big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big)\\ &\quad \cdot\big\{\big(\widetilde{u}(y) - \widetilde{u}(y-h)\big)\otimes\widetilde{\nabla}\theta(y)\big\}_{\mathrm{sym}}\widetilde{\theta}(y)\,dy.\end{aligned} \tag{5.25}$$

From (5.23) it follows that

$$S_1 \geqslant c\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big|^2(\widetilde{\theta})^2(y)\,dy$$
$$-c\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big|\big|(\widetilde{\partial_3 u})(y-h)\big|\big|(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big|(\widetilde{\theta})^2(y)\,dy$$
$$-c\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big|\widetilde{\theta}(y)\big|\widetilde{u}(y)-\widetilde{u}(y-h)\big|\big|\widetilde{\nabla\theta}(y)\big|\,dy. \tag{5.26}$$

By appealing to Cauchy–Schwartz inequality one easily shows that

$$S_1 \geqslant c\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big|^2(\widetilde{\theta})^2(y)\,dy$$
$$-c\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|(\widetilde{\partial_3 u})(y-h)\big|^2\big|(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big|^2(\widetilde{\theta})^2(y)\,dy$$
$$-c\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{u}(y)-\widetilde{u}(y-h)\big|^2\big|\widetilde{\nabla\theta}(y)\big|^2\,dy. \tag{5.27}$$

The last two integrals are bounded by

$$ch^2\big(\big\|D^2\eta\big\|_\infty^2+\big\|D^2\theta\big\|_\infty^2\big)\big\|\widetilde{\nabla u}(y)\big\|_p^p. \qquad \square$$

Next we estimate the second integral on the right-hand side of (5.14).

Proposition 5.2. *Let $\widetilde{\phi}(y)$ be given by* (5.15)*. Then*

$$\left|\int \big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h):\big[(\widetilde{\partial_3\phi})(y)\otimes\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big]\big]_{\mathrm{sym}}\,dy\right|$$
$$\leqslant c\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big|^2(\widetilde{\theta})^2(y)\,dy+Ch^2\|\nabla\widetilde{u}\|_p^p. \tag{5.28}$$

Proof. Denote by S the integral on the left-hand side of (5.28). By (5.19) one has

$$S=\int \big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h):\big[\big(\widetilde{\partial_3 u}(y)-\widetilde{\partial_3 u}(y-h)\big)(\widetilde{\theta})^2(y)\otimes\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big]\big]_{\mathrm{sym}}\,dy$$
$$+2\int \big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)$$
$$\cdot\big[\big(\widetilde{u}(y)-\widetilde{u}(y-h)\big)\widetilde{\partial_3\theta}(y)\otimes\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big]\big]_{\mathrm{sym}}\widetilde{\theta}(y)\,dy. \tag{5.29}$$

It is easily seen that the second integral on the right-hand side of (5.29) is bounded by

$$c\|\nabla\theta\|_\infty\big\|D^2\eta\big\|_\infty\|\nabla\widetilde{u}\|_p^p h^2,$$

hence it is bounded by the last term in the right-hand side of Eq. (5.28).

Denote by I_1 the first integral on the right-hand side of (5.29). By splitting this integral into two integrals, the first one including the term $\widetilde{\partial_3 u}(y)$ and the second one including the term $\widetilde{\partial_3 u}(y-h)$. By appealing to the change of variables $y_1-h\to y_1$ in the second integral. And, finally, by splitting this last integral in a convenient and obvious way, we get

$$I_1=\int \big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h):\big[\widetilde{\partial_3 u}(y)(\widetilde{\theta})^2(y)\otimes\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big]\big]_{\mathrm{sym}}\,dy$$
$$-\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y):\big[\widetilde{\partial_3 u}(y)(\widetilde{\theta})^2(y)\otimes\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big]\big]_{\mathrm{sym}}\,dy$$
$$-\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y):\big\{\big[\widetilde{\partial_3 u}(y)(\widetilde{\theta})^2(y+h)\otimes\big[(\nabla\eta)(y'+h)-(\nabla\eta)(y')\big]\big]_{\mathrm{sym}}$$
$$-\big[\widetilde{\partial_3 u}(y)(\widetilde{\theta})^2(y)\otimes\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big]\big]_{\mathrm{sym}}\big\}\,dy. \tag{5.30}$$

The last integral on the right-hand side of (5.30) is bounded by

$$C\|\nabla\eta\|_\infty\|\nabla\widetilde{u}\|_p^p h^2,$$

hence it is bounded by the last term in the right-hand side of Eq. (5.28). It remains to estimate the absolute value of the difference between the first two integrals on the right-hand side of (5.30). By appealing to (5.23) one shows that this absolute value is bounded by

$$c\int\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big|\big|\widetilde{\partial_3 u}(y)\big|(\widetilde{\theta})^2(y)\big|(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big|\,dy.$$

In turn, this quantity is bounded by

$$c\int\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big|^2(\widetilde{\theta})^2(y)\,dy$$
$$+C\int\big|\widetilde{\partial_3 u}(y)\big|^2\big|(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big|^2\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}(\widetilde{\theta})^2(y)\,dy.$$

Since the last integral is bounded by

$$C\big\|D^2\eta\big\|_\infty^2\|\nabla\widetilde{u}\|_p^p h^2,$$

the estimate (5.28) follows. □

From (5.24) and (5.28) we get the following result.

Proposition 5.3. *Let $\widetilde{\phi}(y)$ be given by* (5.15) *and denote by $\mathcal{S}$ the difference between the first integral on the left-hand side of* (5.14) *and the absolute value of the second integral on the right-hand side of the same equation. Then*

$$\mathcal{S}\geqslant\bar{c}\int\big(1+\big|(\widetilde{\mathcal{D}u})(y)\big|+\big|(\widetilde{\mathcal{D}u})(y-h)\big|\big)^{p-2}\big|(\widetilde{\mathcal{D}u})(y)-(\widetilde{\mathcal{D}u})(y-h)\big|^2(\widetilde{\theta})^2(y)\,dy-C\|\nabla u\|_p^p h^2. \tag{5.31}$$

Next we consider the f-term. A classical result shows that ($s=1,2$)

$$\left|\int\widetilde{f}(y)\cdot\big(\widetilde{\phi}(y+h)-\widetilde{\phi}(y)\big)\,dy\right|\leqslant h\|\widetilde{f}\|_{p'}\|\partial_s\widetilde{\phi}\|_p. \tag{5.32}$$

Since $\widetilde{\phi}(y)$ is given by (5.16), straightforward calculations yield (recall that $0\leqslant\theta(x)\leqslant1$)

$$\left|\int\widetilde{f}(y)\cdot\big(\widetilde{\phi}(y+h)-\widetilde{\phi}(y)\big)\,dy\right|\leqslant h\|\widetilde{f}\|_{p'}\left(\int\big|\partial_s\widetilde{u}(y)-\partial_s\widetilde{u}(y-h)\big|^p(\widetilde{\theta})^2(y)\,dy\right)^{\frac{1}{p}}+h^2\|\widetilde{f}\|_{p'}\|\partial_s\widetilde{u}\|_p\big\|\nabla(\widetilde{\theta})^2\big\|_\infty. \tag{5.33}$$

At this point it looks convenient to the reader to make a full stop. In this regard we write the equation that follows from (5.14) by appealing to Proposition 5.3 and to Eq. (5.33). One has

$$\begin{aligned}
&\int\big(1+\big|(\widetilde{\mathcal{D}u})(y)\big|+\big|(\widetilde{\mathcal{D}u})(y-h)\big|\big)^{p-2}\big|(\widetilde{\mathcal{D}u})(y)-(\widetilde{\mathcal{D}u})(y-h)\big|^2(\widetilde{\theta})^2(y)\,dy\\
&\quad\leqslant\int\widetilde{\pi}(y)(\widetilde{\partial_3 u})(y-h)\cdot\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big](\widetilde{\theta})^2(y)\,dy\\
&\qquad+\int\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}(y)\big(\widetilde{u}(y)-\widetilde{u}(y-h)\big)\cdot\widetilde{\nabla}\theta(y)\,dy\\
&\qquad-\int\widetilde{\pi}(y-h)(\widetilde{\partial_3 u})(y)\cdot\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big](\widetilde{\theta})^2(y)\,dy\\
&\qquad-\int\widetilde{\pi}(y-h)\big(\widetilde{u}(y)-\widetilde{u}(y-h)\big)\widetilde{\partial_3\theta}(y)\cdot\big[(\nabla\eta)(y')-(\nabla\eta)(y'-h)\big]\widetilde{\theta}(y)\,dy\\
&\qquad+C\|\nabla u\|_p^p h^2+h\|\widetilde{f}\|_{p'}\left(\int\big|\partial_s\widetilde{u}(y)-\partial_s\widetilde{u}(y-h)\big|^p(\widetilde{\theta})^2(y)\,dy\right)^{\frac{1}{p}}+h^2\|\widetilde{f}\|_{p'}\|\nabla\widetilde{u}\|_p\big\|\nabla(\widetilde{\theta})^2\big\|_\infty.
\end{aligned} \tag{5.34}$$

By recalling, if necessary, (4.9) one easily shows that the fourth integral in the right-hand side of the above equation is bounded by

$$C\|\nabla\theta\|_\infty\|\pi\|_{p'}\|\nabla u\|_p h^2.$$

Similar estimates hold for the first and the third integrals in the right-hand side of the same equation. We set, for convenience,

$$\widetilde{A}_1^2=\int\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big|^2\widetilde{\theta}^2(y)\,dy.$$

The above arguments prove the following estimate

$$\widetilde{A}_1^2 \leqslant \int \big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}(y)\big(\widetilde{u}(y) - \widetilde{u}(y-h)\big) \cdot \widetilde{\nabla\theta}(y)\,dy + |h| \|\widetilde{f}\|_{p'} \left(\int \big|\partial_s \widetilde{u}(y) - \partial_s \widetilde{u}(y-h)\big|^p (\widetilde{\theta})^2(y)\,dy \right)^{\frac{1}{p}}$$
$$+ ch^2 \|f\|_{p'} \|\nabla u\|_p \big\|\nabla(\widetilde{\theta})^2\big\|_\infty + C\big(1 + \|\nabla\theta\|_\infty^2\big)\big(\|\nabla u\|_p^p + \|\pi\|_{p'} \|\nabla u\|_p\big)h^2. \tag{5.35}$$

By recalling the definition of constants of type C, we state the following theorem.

Theorem 5.5. *The following estimate holds*:

$$|h|^{-2}\widetilde{A}_1^2 \leqslant |h|^{-2} \int \big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}(y)\big(\widetilde{u}(y) - \widetilde{u}(y-h)\big) \cdot \widetilde{\nabla\theta}(y)\,dy$$
$$+ |h|^{-1} \|\widetilde{f}\|_{p'} \left(\int \big|\partial_s \widetilde{u}(y) - \partial_s \widetilde{u}(y-h)\big|^p (\widetilde{\theta})^2(y)\,dy \right)^{\frac{1}{p}} + C. \tag{5.36}$$

Note that

$$\left| \int \big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}(y)\big(\widetilde{u}(y) - \widetilde{u}(y-h)\big) \cdot \widetilde{\nabla\theta}(y)\,dy \right| \leqslant C \big\|\widetilde{u}(y) - \widetilde{u}(y-h)\big\|_2 \big\|\big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}(y)\big\|_2$$
$$\leqslant C|h| \|\partial_s \widetilde{u}\|_2 \big\|\big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}(y)\big\|_2. \tag{5.37}$$

Recall that

$$\lambda(q) = \frac{2q}{2-p+q}. \tag{5.38}$$

One has, for $1 < \lambda \leqslant \lambda(q)$,

$$\int \big|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)\big|^\lambda \widetilde{\theta}^\lambda(y)\,dy$$
$$\leqslant \int \big(1 + \big|(\widetilde{\mathcal{D}u})(y)\big| + \big|(\widetilde{\mathcal{D}u})(y-h)\big|\big)^{\frac{(2-p)\lambda}{2}} \big(1 + \big|(\widetilde{\mathcal{D}u})(y)\big| + \big|(\widetilde{\mathcal{D}u})(y-h)\big|\big)^{\frac{(p-2)\lambda}{2}} \big|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)\big|^\lambda \widetilde{\theta}^\lambda(y)\,dy. \tag{5.39}$$

By Hőlder's inequality with exponents $\frac{2}{2-\lambda}$ and $\frac{2}{\lambda}$ the following result holds.

Lemma 5.6. *Let be $p \leqslant q$ and $1 < \lambda \leqslant \lambda(q)$. Then*

$$\int \big|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)\big|^\lambda \widetilde{\theta}^\lambda(y)\,dy \leqslant \big\|1 + 2|\widetilde{\mathcal{D}u}|\big\|_{\frac{(2-p)\lambda}{2-\lambda}}^{\frac{(2-p)\lambda}{2}} (\widetilde{A}_1)^\lambda. \tag{5.40}$$

Note that $\lambda \leqslant \lambda(q)$ is equivalent to $\frac{(2-p)\lambda}{2-\lambda} \leqslant q$, moreover the corresponding equalities are equivalent.
From (5.40), with $\lambda = p$, $\lambda = \lambda(q)$, and $\lambda = r(q)$ it follows that

Corollary 5.1.

$$\int \big|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)\big|^p \widetilde{\theta}^p(y)\,dy \leqslant \big\|1 + 2|\widetilde{\mathcal{D}u}|\big\|_p^{\frac{(2-p)p}{2}} \widetilde{A}_1^p \leqslant C\widetilde{A}_1^p, \tag{5.41}$$

$$\int \big|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)\big|^{\lambda(q)} \widetilde{\theta}^{\lambda(q)}(y)\,dy \leqslant \big\|1 + 2|\widetilde{\mathcal{D}u}|\big\|_q^{\frac{(2-p)\lambda(q)}{2}} (\widetilde{A}_1)^{\lambda(q)}, \tag{5.42}$$

and

$$\int \big|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)\big|^{r(q)} \widetilde{\theta}^{r(q)}(y)\,dy \leqslant \big\|1 + 2|\widetilde{\mathcal{D}u}|\big\|_{\frac{q}{2}}^{\frac{(2-p)r(q)}{2}} (\widetilde{A}_1)^{r(q)}. \tag{5.43}$$

5.2. Estimates for the tangential derivatives of the pressure in terms of the velocity

Next we prove the following main estimate.

Lemma 5.7. *For each $\widetilde{\phi} \in C_0^2(Q_a)$ one has*

$$\begin{aligned}&\left|\int \nabla\big[\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}\big]\cdot\widetilde{\phi}\,dy\right|\\ &\quad\leqslant\left|\int\Big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y)-\big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\Big):\widetilde{\nabla(\theta\phi)}(y)\,dy\right|\\ &\qquad+\epsilon_0\big\|\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}\big\|_2\|\nabla\widetilde{\phi}\|_2+\left|\int\widetilde{f}\cdot\big(\widetilde{(\theta\phi)}(y+h)-\widetilde{(\theta\phi)}(y)\big)\,dy\right|\\ &\qquad+C|h|\big(1+\|\nabla u\|_p^{p-1}+\|\pi\|_{p'}\big)\|\nabla\widetilde{\phi}\|_p,\end{aligned}\tag{5.44}$$

where ϵ_0 and a are chosen below.

Proof. From (5.14) with $\psi=0$ and ϕ replaced by $\theta\phi$ one shows that

$$\begin{aligned}\int\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\big(\widetilde{\nabla\cdot(\theta\phi)}\big)(y)\,dy&=\int\Big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y)-\big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\Big):\widetilde{\mathcal{D}(\theta\phi)}(y)\,dy\\ &\quad+\int\widetilde{f}(y)\cdot\big(\widetilde{(\theta\phi)}(y+h)-\widetilde{(\theta\phi)}(y)\big)\,dy+c\mathcal{R},\end{aligned}\tag{5.45}$$

where $\mathcal{R}$ satisfies

$$|\mathcal{R}|\leqslant|h|\|\eta\|_{C^2}\big(1+\|\widetilde{\nabla}u\|_p\big)^{p-1}\big\|\widetilde{\partial_3(\theta\phi)}\big\|_p+|h|\|\eta\|_{C^2}\|\pi\|_{p'}\big\|\widetilde{\partial_3(\theta\phi)}\big\|_p.\tag{5.46}$$

Since $\widetilde{\partial_3(\theta\phi)}=\widetilde{\theta}\partial_3\widetilde{\phi}+\widetilde{\phi}\widetilde{\partial_3\theta}$ (recall, in particular, (4.10) for $k=3$) it follows that

$$\big\|\widetilde{\partial_3(\theta\phi)}\big\|_p\leqslant C\|\nabla\widetilde{\phi}\|_p.$$

One has

$$|\mathcal{R}|\leqslant C|h|\|\nabla\widetilde{\phi}\|_p.\tag{5.47}$$

On the other hand, by appealing to (4.12), one shows that

$$\widetilde{\nabla\cdot(\theta\phi)}=\widetilde{\theta}(\nabla\cdot\widetilde{\phi})+\widetilde{\phi}\cdot\widetilde{\nabla}\theta-\widetilde{\theta}(\nabla\eta)\cdot(\partial_3\widetilde{\phi})-(\partial_3\widetilde{\theta})(\nabla\eta)\cdot\widetilde{\phi}.$$

Hence we may decompose the left-hand side of (5.45) as

$$\begin{aligned}\int\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\big(\widetilde{\nabla\cdot(\theta\phi)}\big)\,dy&=\int\big[\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}\big](\nabla\cdot\widetilde{\phi})\,dy\\ &\quad+\int\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{(\nabla\theta)}\cdot\widetilde{\phi}\,dy-\int\big[\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}\big](\nabla\eta)\cdot(\partial_3\widetilde{\phi})\,dy\\ &\quad-\int\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)(\partial_3\widetilde{\theta})(\nabla\eta)\cdot\widetilde{\phi}\,dy.\end{aligned}\tag{5.48}$$

By means of a suitable translation one shows that

$$\begin{aligned}\int\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{(\nabla\theta)}\cdot\widetilde{\phi}\,dy&=-\int\widetilde{\pi}\big(\widetilde{\phi}(y+h)-\widetilde{\phi}(y)\big)\cdot\widetilde{(\nabla\theta)}\,dy\\ &\quad-\int\widetilde{\pi}\widetilde{\phi}(y+h)\cdot\big(\widetilde{\nabla}\theta(y+h)-\widetilde{\nabla}\theta(y)\big)\,dy.\end{aligned}\tag{5.49}$$

Hence the second integral on the right-hand side of (5.48) satisfies

$$\left|\int\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{(\nabla\theta)}\cdot\widetilde{\phi}\,dy\right|\leqslant|h|\|\widetilde{\nabla}\theta\|_{C^1}\|\widetilde{\pi}\|_{p'}\|\nabla\widetilde{\phi}\|_p.\tag{5.50}$$

A similar device applied to the last integral on the right-hand side of (5.48) shows that

$$\left|\int\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)(\partial_3\widetilde{\theta})(\nabla\eta)\cdot\widetilde{\phi}\,dy\right|\leqslant c|h|\|\nabla\eta\|_{C_1}\|\partial_3\widetilde{\theta}\|_{C_1}\|\widetilde{\pi}\|_{p'}\|\nabla\widetilde{\phi}\|_p.\tag{5.51}$$

On the other hand

$$\left|\int\big[\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}\big](\nabla\eta)\cdot(\partial_3\widetilde{\phi})\,dy\right|\leqslant\big\|\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}\big\|_2\|\nabla\eta\|_{C_0}\|\widetilde{\nabla}\phi\|_2.\tag{5.52}$$

From (5.48), (5.50), (5.51) and (5.52) it follows that

$$\int \big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\big(\widetilde{\nabla\cdot(\theta\phi)}\big)\,dy=-\int \nabla\big[\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}\big]\cdot\widetilde{\phi}\,dy+\mathcal{R}_2, \tag{5.53}$$

for each $\widetilde{\phi}\in C_0^2(Q_a)$, where $\mathcal{R}_2$ satisfies the estimate

$$|\mathcal{R}_2|\leqslant C|h|\|\widetilde{\pi}\|_{p'}\|\nabla\widetilde{\phi}\|_p+\epsilon_0\big\|\big(\widetilde{\pi}(y)-\widetilde{\pi}(y-h)\big)\widetilde{\theta}\big\|_2\|\nabla\widetilde{\phi}\|_2, \tag{5.54}$$

for an arbitrarily small positive ϵ_0, provided that $a\leqslant a(\epsilon_0)$. We appeal to the fact that $\nabla\eta(0)=0$. From (5.53), (5.54) and (5.45), (5.47) the estimate (5.44) follows. □

Next we prove the following result.

Lemma 5.8. *The following estimate holds*:

$$\begin{aligned}&\bigg|\int \big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y)-\big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big):\widetilde{\nabla(\theta\phi)}\,dy\bigg|\\&\quad\leqslant c\widetilde{A}_1\|\nabla\widetilde{\phi}\|_2+C|h|\big(1+\|\nabla u\|_p\big)^{p-1}\|\nabla\widetilde{\phi}\|_p.\end{aligned} \tag{5.55}$$

Proof. From (4.11) it follows that

$$\widetilde{\nabla(\theta\phi)}=\widetilde{\theta}\nabla\widetilde{\phi}-\widetilde{\theta}\big[(\nabla\eta)\otimes\partial_3\widetilde{\phi}\big]+\widetilde{\phi}\otimes\widetilde{\nabla}\theta,$$

for each $y\in Q_a$. Moreover

$$\big|\widetilde{\theta}\nabla\widetilde{\phi}-\widetilde{\theta}\big[(\nabla\eta)\otimes\partial_3\widetilde{\phi}\big]\big|\leqslant C|\widetilde{\theta}||\nabla\widetilde{\phi}|. \tag{5.56}$$

Hence, by appealing to (5.23), it follows that

$$\begin{aligned}&\bigg|\int \big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y)-\big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big):\widetilde{\nabla(\theta\phi)}\,dy\bigg|\\&\quad\leqslant C\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big||\widetilde{\theta}||\nabla\widetilde{\phi}|\,dy\\&\qquad+\bigg|\int \big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y)-\big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big):(\widetilde{\phi}\otimes\widetilde{\nabla}\theta)\,dy\bigg|.\end{aligned} \tag{5.57}$$

Next, since $p\leqslant 2$, it readily follows that

$$\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\big|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)\big||\widetilde{\theta}||\nabla\widetilde{\phi}|\,dy\leqslant c\widetilde{A}_1\|\nabla\widetilde{\phi}\|_2, \tag{5.58}$$

which is the desired estimate for the first integral on the right-hand side of (5.57).

We could appeal to similar devices to obtain as well a useful estimate for the second integral in the right-hand side of (5.57). However, the lack of $\widetilde{\theta}(y)$ in this integral would imply some tricky arguments. We rather prefer to introduce a more elegant device to obtain the desired estimate. Denote by I the referred integral. An obvious translation shows that

$$I=\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y):\big(\widetilde{\phi}(y)\otimes\widetilde{\nabla}\widetilde{\theta}(y)\big)\,dy-\int \big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y):\big(\widetilde{\phi}(y+h)\otimes\widetilde{\nabla}\widetilde{\theta}(y+h)\big)\,dy.$$

By appealing to an obvious decomposition of $(\widetilde{\phi}(y+h)\otimes\widetilde{\nabla}\widetilde{\theta}(y+h))-(\widetilde{\phi}(y)\otimes\widetilde{\nabla}\widetilde{\theta}(y))$, it readily follows that

$$|I|\leqslant c|h|\|\widetilde{\nabla}\widetilde{\theta}\|_{C^1}\big(1+\big\|\widetilde{(\nabla u)}\big\|_p\big)^{p-1}\big(\|\widetilde{\phi}\|_p+\|\nabla\widetilde{\phi}\|_p\big).$$

Hence

$$\bigg|\int \big(\big(1+\big|\widetilde{\mathcal{D}u}(y)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y)-\big(1+\big|\widetilde{\mathcal{D}u}(y-h)\big|\big)^{p-2}\widetilde{\mathcal{D}u}(y-h)\big):(\widetilde{\phi}\otimes\widetilde{\nabla}\theta)\,dy\bigg|\leqslant C|h|\big(1+\|\nabla u\|_p\big)^{p-1}\|\nabla\widetilde{\phi}\|_p. \tag{5.59}$$

By appealing to Eqs. (5.57), (5.58) and (5.59) one proves (5.55). □

Next, by appealing to an obvious decomposition of the $\theta\phi$ terms, one shows that

$$\bigg|\int \widetilde{f}\cdot\big(\widetilde{(\theta\phi)}(y+h)-\widetilde{(\theta\phi)}(y)\big)\,dy\bigg|\leqslant C|h|\|f\|_{p'}\|\nabla\widetilde{\phi}\|_p. \tag{5.60}$$

The following result follows from (5.44), (5.55) and (5.60).

Lemma 5.9. *Given $\epsilon_0 > 0$ there is $a(\epsilon_0) > 0$ (independent of the point x_0) such that for $a \leqslant a(\epsilon_0)$, one has*

$$\left| \int \nabla\big[\big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}\big] \cdot \widetilde{\phi}\, dy \right| \leqslant c\widetilde{A}_1 \|\nabla\widetilde{\phi}\|_2 + \epsilon_0 \big\|\big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}\big\|_2 \|\nabla\widetilde{\phi}\|_2 + C|h|\big(1 + \|\nabla u\|_p^{p-1} + \|\pi\|_{p'} + \|f\|_{p'}\big)\|\nabla\widetilde{\phi}\|_p, \tag{5.61}$$

for each $\widetilde{\phi} \in C_0^2(Q_a)$.

The following theorem follows from the above estimates.

Theorem 5.10. *For sufficiently small positive values of a (which are independent of the particular point x_0) one has*

$$\big\|\big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}\big\|_2 \leqslant c\widetilde{A}_1 + C|h|\big(1 + \|\nabla u\|_p^{p-1} + \|\pi\|_{p'} + \|f\|_{p'}\big). \tag{5.62}$$

Proof. Eq. (5.61) shows that $\nabla[(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\widetilde{\theta}] \in W^{-1,2}(Q_a)$ and that the corresponding norm is bounded by the right-hand side of Eq. (5.63) below. A main point here is that $\widetilde{\theta}$ has compact support in J_a. To fix ideas the reader may assume, once and for all, that

$$\operatorname{supp} \widetilde{\theta} \subset \overline{Q}_{\frac{a}{2}},$$

and that translation amplitudes satisfies $|h| < \frac{a}{2}$. Next, by appealing to Lemma 3.1 (see also Appendix B), one shows that

$$\big\|\big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}\big\|_2 \leqslant \epsilon_0 \big\|\big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}\big\|_2 + c\widetilde{A}_1 + C|h|\big(1 + \|\nabla u\|_p^{p-1} + \|\pi\|_{p'} + \|f\|_{p'}\big). \tag{5.63}$$

This proves (5.62). □

From Eqs. (5.36), (5.37) and (5.62) it follows that

$$|h|^{-2}\widetilde{A}_1 \leqslant C|h|^{-1}\big\|\widetilde{u}(y) - \widetilde{u}(y-h)\big\|_{2,Q_a} + |h|^{-1}\|f\|_{p'} \left(\int \big|\partial_s\widetilde{u}(y) - \partial_s\widetilde{u}(y-h)\big|^p (\widetilde{\theta})^2(y)\, dy \right)^{\frac{1}{p}} + C.$$

From the above estimate we get

Theorem 5.11. *The following estimate holds*:

$$|h|^{-2} \int \big(1 + \big|\widetilde{\mathcal{D}}u(y)\big| + \big|\widetilde{\mathcal{D}}u(y-h)\big|\big)^{p-2} \big|\widetilde{\mathcal{D}}u(y) - \widetilde{\mathcal{D}}u(y-h)\big|^2 \widetilde{\theta}^2(y)\, dy \leqslant C\big\|\partial_s\widetilde{u}(y)\big\|_{2,Q_a} + C|h|^{-1}\|f\|_{p'} \left(\int \big|\partial_s\widetilde{u}(y) - \partial_s\widetilde{u}(y-h)\big|^p (\widetilde{\theta})^2(y)\, dy \right)^{\frac{1}{p}} + C. \tag{5.64}$$

Remark 5.2. It is worth noting that the particular features of the problem under hands require a special care, and some new device, in order to apply the translation's method here. On the other hand, going on with the explicit expressions of the differential quotients would be detrimental to the reader, since the main ideas would stay in hiding among intricate expressions. On the other hand, the work already done by appealing to the differential quotients technique is largely sufficient to allow the interested reader to carry on the proofs by this technique. We also refer to [7], where this technique is continuously applied. The above situation leads us to come to a compromise: From now on, we replace the differential quotients by the corresponding derivatives.

For convenience, we define the non-negative quantity $\widetilde{A}$ by

$$\widetilde{A}^2 = \int \big(1 + 2\big|(\widetilde{\mathcal{D}}u)(y)\big|\big)^{p-2} \big|(\partial_s\widetilde{\mathcal{D}}u)(y)\big|^2 (\widetilde{\theta})^2(y)\, dy. \tag{5.65}$$

By taking into account (5.64) and the above remark, we may write

$$\widetilde{A}^2 \leqslant C\big\|\partial_s\widetilde{u}(y)\big\|_{2,Q_a} + C\|f\|_{p'} \left(\int \big|\partial_{ss}^2\widetilde{u}(y)\big|^p (\widetilde{\theta})^2(y)\, dy \right)^{\frac{1}{p}} + C. \tag{5.66}$$

Eqs. (5.41), (5.42), (5.43), and (5.62) show the following result.

Theorem 5.12. *The following estimates hold*:

$$\big\|(\nabla_* \widetilde{\mathcal{D}u})\widetilde{\theta}\big\|_p^p \leqslant C\widetilde{A}^p, \tag{5.67}$$

$$\big\|(\nabla_* \widetilde{\mathcal{D}u})\widetilde{\theta}\big\|_{\lambda(q)}^{\lambda(q)} \leqslant c\big(1 + \|\widetilde{\mathcal{D}u}\|_q^{\frac{(2-p)\lambda(q)}{2}}\big)\widetilde{A}^{\lambda(q)}, \tag{5.68}$$

$$\big\|(\nabla_* \widetilde{\mathcal{D}u})\widetilde{\theta}\big\|_{r(q)}^{r(q)} \leqslant c\big(1 + \|\widetilde{\mathcal{D}u}\|_{\frac{q}{2}}^{\frac{(2-p)r(q)}{2}}\big)\widetilde{A}^{r(q)}, \tag{5.69}$$

and

$$\big\|(\nabla_* \widetilde{\pi})\widetilde{\theta}\big\|_2^2 \leqslant C\big(1 + \|f\|_{p'}^2 + \widetilde{A}^2\big). \tag{5.70}$$

Finally (roughly speaking), we prove that the tangential derivatives of the full gradient ∇u can be estimated in terms of the tangential derivatives of the symmetric gradient $\mathcal{D}u$. See Remark 3.1 in reference [7].

We start by the following auxiliary result, where the bounded open set D has a "Lipschitz" boundary consisting on the union of two disjoint pieces, S_1 and S_2, both with not vanishing 2-dimensional measure. The exponent p may be any real $p > 1$.

Lemma 5.13. *There is a linear continuous map from $\widetilde{f}_0 \in L^p(D)$ into $\widetilde{w} \in W^{1,p}(D)$ such that $\nabla \cdot \widetilde{w} = \widetilde{f}_0$ in D and $\widetilde{w} = 0$ on S_1. In particular,*

$$\|\widetilde{w}\|_{1,p} \leqslant c\|\widetilde{f}_0\|_p. \tag{5.71}$$

Proof. Extend the domain D to a fixed domain $\widetilde{D}$ "throughout" S_2. Then extend $\widetilde{f}_0$ to $\widetilde{D}$ in such a way that the extension $\widetilde{F}_0$ has vanishing mean value in $\widetilde{D}$ (for instance, $\widetilde{F}_0$ constant outside D). Then, it is well known that there is $\widetilde{W_0} \in W^{1,p}(\widetilde{D})$ such that $\nabla \cdot \widetilde{W_0} = \widetilde{F}_0$ in $\widetilde{D}$ and $\widetilde{W_0} = 0$ on $\partial\widetilde{D}$. For proofs see [29, Chapter III, Section 3], and for quite complete references see [29, Chapter III, Section 7]. We define $\widetilde{w}$ as the restriction of $\widetilde{W_0}$ to D. □

Theorem 5.14. *For each $\beta > 1$, one has*

$$\big\|\widetilde{\theta}\nabla_*(\nabla\widetilde{u})\big\|_{\beta, Q_a} \leqslant c\big\|\widetilde{\theta}\nabla_*(\mathcal{D}\widetilde{u})\big\|_{\beta, Q_a} + C, \tag{5.72}$$

in Q_a, for each admissible value of a.

Note that $\widetilde{\theta}$ can be replaced by any positive power of $\widetilde{\theta}$.

Proof. Set

$$\widetilde{v} = \widetilde{\theta}\partial_s\widetilde{u},$$

$s = 1, 2$. From $\nabla \cdot u = 0$ and from (4.10) it follows that

$$\nabla \cdot \widetilde{u} = (\partial_1\eta)(\partial_3\widetilde{u}_1) + (\partial_2\eta)(\partial_3\widetilde{u}_2).$$

Straightforward calculations show that

$$\begin{cases} (\nabla \cdot \widetilde{v})_{|Q_a} = \widetilde{f}_0, \\ \widetilde{v}_{|\Lambda_a} = 0, \end{cases} \tag{5.73}$$

where

$$\widetilde{f}_0(y) = \widetilde{\theta}\big[(\partial_1\eta)(\partial_{s3}\widetilde{u}_1) + (\partial_2\eta)(\partial_{s3}\widetilde{u}_2)\big] + R$$

and

$$\big|R(y)\big| \leqslant c\big(\big|\nabla\widetilde{\theta}(y)\big| + \big|D^2\eta(y)\big|\big)\big|\nabla\widetilde{u}(y)\big|.$$

Hence,

$$\|\widetilde{f}_0\|_\beta \leqslant \|\nabla\eta\|_\infty\big(\|\widetilde{\theta}\partial_{13}\widetilde{u}\|_\beta + \|\widetilde{\theta}\partial_{23}\widetilde{u}\|_\beta\big) + C. \tag{5.74}$$

Next we define $\widetilde{w}$ as in Lemma 5.13, where $D = Q_a$, $\widetilde{D} = J_a$ and $S_1 = \Lambda_a$. Set

$$\widetilde{g} = \widetilde{v} - \widetilde{w}.$$

One has

$$\begin{cases} (\nabla \cdot \widetilde{g})_{|Q_a} = 0, \\ \widetilde{g}_{|\Lambda_a} = 0. \end{cases} \tag{5.75}$$

From (5.75) it follows that there is a constant c (independent of the particular $\widetilde{g}$) such that

$$\|\widetilde{g}\|_{1,\beta} \leqslant c\|\mathcal{D}\widetilde{g}\|_{\beta}. \tag{5.76}$$

The proof follows essentially by appealing to a classical result of Nečas. See, for instance, [47, Lemma 1.1 and Proposition 1.1].

By taking into account the definition of $\widetilde{g}$ and (5.71), it follows that

$$\|\nabla\widetilde{v}\|_{\beta} \leqslant c\big(\|\mathcal{D}\widetilde{v}\|_{\beta} + \|\widetilde{f}_0\|_{\beta}\big).$$

Finally, by taking into account that $\widetilde{v} = \widetilde{\theta}\partial_s\widetilde{u}$, that

$$\big|\big\|\nabla(\widetilde{\theta}\partial_s\widetilde{u})\big\| - \big\|\widetilde{\theta}\nabla(\partial_s\widetilde{u})\big\|\big| \leqslant C,$$

that

$$\big|\big\|\mathcal{D}(\widetilde{\theta}\partial_s\widetilde{u})\big\| - \big\|\widetilde{\theta}\mathcal{D}(\partial_s\widetilde{u})\big\|\big| \leqslant C,$$

and the estimate (5.74), one proves (5.72). Recall that $\|\nabla\eta\|_\infty \leqslant \epsilon_0$, for arbitrarily small ϵ_0. □

Theorem 5.15. *One has*

$$\widetilde{A} \leqslant C\big(1 + \|f\|_{p'} + \|\nabla_*\widetilde{u}\|_{2,Q_a}^{\frac{1}{2}}\big). \tag{5.77}$$

Proof. From (4.16) it follows that

$$\big\|\widetilde{\theta}\nabla_*(\mathcal{D}\widetilde{u})\big\|_{p,Q_a} \leqslant \big\|\widetilde{\theta}\nabla_*(\widetilde{\mathcal{D}u})\big\|_{p,Q_a} + \epsilon_0\big\|\widetilde{\theta}\nabla_*(\nabla\widetilde{u})\big\|_{p,Q_a} + C\|\nabla\widetilde{u}\|_{p,Q_a}.$$

By applying (5.72) to the second term on the right-hand side of the above inequality, and by choosing ϵ_0 sufficiently small, we prove that

$$\big\|\widetilde{\theta}\nabla_*(\mathcal{D}\widetilde{u})\big\|_{p,Q_a} \leqslant c\big\|\widetilde{\theta}\nabla_*(\widetilde{\mathcal{D}u})\big\|_{p,Q_a} + C. \tag{5.78}$$

Next, from (5.66), (5.72) and (5.78), we get

$$\widetilde{A}^2 \leqslant C\big(1 + \|\nabla_*\widetilde{u}\|_{2,Q_a} + \|f\|_{p'}\big\|\widetilde{\theta}\nabla_*(\widetilde{\mathcal{D}u})\big\|_{p,Q_a}\big).$$

Finally, by appealing to (5.67), we show that

$$\widetilde{A}^2 \leqslant C\big(1 + \|\nabla_*\widetilde{u}\|_{2,Q_a} + \|f\|_{p'}\widetilde{A}\big).$$

This proves (5.77). □

5.3. Estimates for the "tangential derivatives" in terms of the data

For the reader's convenience we recall once more that ∇_* denotes the gradient with respect to the variables y_j, $j = 1, 2$. Hence

$$\big|\nabla_*(\widetilde{\nabla u})(y)\big|^2 = \sum_{j=1,2}\sum_{i,k=1}^{3}\big(\partial_j(\widetilde{\nabla u})_{ik}\big)^2,$$

$$\big|\nabla_*(\widetilde{\mathcal{D}u})(y)\big|^2 = \sum_{j=1,2}\sum_{i,k=1}^{3}\big(\partial_j(\widetilde{\mathcal{D}u})_{ik}\big)^2,$$

and

$$\big|\nabla_*\widetilde{\pi}(y)\big|^2 = \sum_{j=1,2}(\partial_j\widetilde{\pi})^2.$$

Theorem 5.16. *The following estimates hold*:

$$\big\|\widetilde{\theta}\nabla_*(\widetilde{\nabla u})\big\|_{p,Q_a} \leqslant C\big(1+\|f\|_{p'}+\|\nabla_*\widetilde{u}\|_{2,Q_a}^{\frac{1}{2}}\big), \tag{5.79}$$

$$\big\|\widetilde{\theta}\nabla_*(\widetilde{\nabla u})\big\|_{\lambda(q),Q_a} \leqslant C\big(1+\|\mathcal{D}u\|_q^{\frac{2-p}{2}}\big)\big(1+\|f\|_{p'}+\|\nabla_*\widetilde{u}\|_{2,Q_a}^{\frac{1}{2}}\big), \tag{5.80}$$

$$\big\|\widetilde{\theta}\nabla_*(\widetilde{\nabla u})\big\|_{r(q),Q_a} \leqslant C\big(1+\|\mathcal{D}u\|_{\frac{q}{2}}^{\frac{2-p}{2}}\big)\big(1+\|f\|_{p'}+\|\nabla_*\widetilde{u}\|_{2,Q_a}^{\frac{1}{2}}\big), \tag{5.81}$$

and

$$\|\widetilde{\theta}\nabla_*\widetilde{\pi}\|_{2,Q_a} \leqslant C\big(1+\|f\|_{p'}+\|\nabla_*\widetilde{u}\|_{2,Q_a}^{\frac{1}{2}}\big). \tag{5.82}$$

Proof. From Eqs. (5.72) and (5.67) (together with (4.16) and related devices already explained), and (5.77), we show that (5.79) holds. We also appeal here to (4.14). Similarly, by appealing to (5.68) and (5.69), one proves (5.80) and (5.81). Finally, from Eqs. (5.70) and (5.77) one shows (5.82). □

Remarks.

- Note that we may replace the norms $\|\widetilde{\nabla u}\|_p$, $\|\widetilde{\pi}\|_{p'}$ and $\|\widetilde{f}\|_{p'}$, in Q_a, by the norms $\|\nabla u\|_p$, $\|\pi\|_{p'}$ and $\|f\|_{p'}$ in Ω_a, hence by these last norms in the whole of Ω.
- The constants C depend on the C^2-norms of η and θ in Q_a. However the C^2-norm of η is bounded from above on Γ, hence is independent of the particular point x_0. On the other hand the particular truncation function θ may be fixed once and for all in our proofs as a regular function equal to 1 for $|x'| \leqslant \frac{a}{2}$ and with compact support inside I_a. This shows that the dependence of the constants C on θ is just a dependence on a.
- Whenever we appeal to a "sufficiently small" ϵ_0, recall (4.8), a smaller, positive, upper bound on the values of the parameter a must be assumed. However this situation happens a finite number of times. Hence a strictly positive lower bound for a exists. Further, as already shown, this value does not depend on the point x_0.

6. The linear system for the normal derivatives of the tangential components of the velocity

We set

$$\xi(x) = \partial^2_{33}u(x),$$
$$\xi'(x) = \big(\xi_1(x), \xi_2(x)\big),$$

and

$$M(x) = \big|\mathcal{D}u(x)\big|.$$

Note that derivatives are with respect to the x-variables. Due to (4.8), we may replace (on "right-hand sides" of estimates) derivatives $\partial_k\eta$, for $k=1,2$ simply by ϵ_0. Recall that $\partial_3\eta = 0$. In the same line, $c\epsilon_0$ and ϵ_0^2 can be replaced by ϵ_0.

We will use without a particular warning that

$$\partial_3\widetilde{g} = \widetilde{\partial_3 g}. \tag{6.1}$$

Lemma 6.1. *One has a.e. in Q_a,*

$$|\widetilde{\xi}_3| \leqslant \big|\nabla_*(\widetilde{\nabla u})\big| + \epsilon_0|\widetilde{\xi}'|. \tag{6.2}$$

Proof. From $\nabla\cdot u = 0$ it follows that

$$\widetilde{\xi}_3 = -\partial_3(\widetilde{\partial_1 u_1} + \widetilde{\partial_2 u_2}). \tag{6.3}$$

On the other hand, from (4.10),

$$\partial_3(\widetilde{\partial_m u_l}) = \partial_m\widetilde{\partial_3 u_l} - (\partial_m\eta)\partial_3(\widetilde{\partial_3 u_l}). \tag{6.4}$$

Hence, for $m, l \neq 3$,

$$|\partial_3\widetilde{\partial_m u_l}| \leqslant \big|\nabla_*(\widetilde{\nabla u})\big| + \epsilon_0|\widetilde{\xi}'|.$$

By taking into account (6.3), the thesis follows. □

Lemma 6.2. *One has a.e. in Q_a,*

$$\left|\widetilde{D_*^2 u}(y)\right| \leqslant \left|\nabla_*\widetilde{(\nabla u)}\right| + \epsilon_0|\widetilde{\xi}'|. \tag{6.5}$$

Proof. From (4.10)

$$\mathcal{T}(\partial_j\partial_k u_l) = \partial_k(\widetilde{\partial_j u_l}) - (\partial_k\eta)\partial_3(\widetilde{\partial_j u_l}).$$

By appealing to the above estimates the thesis follows easily. Note that if $j = k = l = 3$ the result follows from (6.2). □

Straightforward calculations show that

$$\partial_k\big(\big(1+|\mathcal{D}u|\big)^{p-2}\mathcal{D}u\big) = \big(1+|\mathcal{D}u|\big)^{p-2}\partial_k\mathcal{D}u + (p-2)\big(1+|\mathcal{D}u|\big)^{p-3}|\mathcal{D}u|^{-1}(\mathcal{D}u\cdot\partial_k\mathcal{D}u)\mathcal{D}u. \tag{6.6}$$

By appealing to (6.6), the jth equation (2.1) may be written in the form

$$-\big(1+|\mathcal{D}u|\big)^{p-2}\sum_{k=1}^{3}\partial_{kk}^2 u_j - 2(p-2)\big(1+|\mathcal{D}u|\big)^{p-3}|\mathcal{D}u|^{-1}\sum_{l,m,k=1}^{3}\mathcal{D}_{lm}\mathcal{D}_{jk}\partial_{mk}^2 u_l + 2\partial_j\pi = 2f_j, \tag{6.7}$$

where $\mathcal{D}_{ij} = (\mathcal{D}u)_{ij}$. Let us write the first two equations (6.7), $j = 1, 2$, as follows

$$\big(1+|\mathcal{D}u|\big)^{p-2}\partial_{33}^2 u_j + 2(p-2)\big(1+|\mathcal{D}u|\big)^{p-3}|\mathcal{D}u|^{-1}\mathcal{D}_{j3}\sum_{l=1}^{2}\mathcal{D}_{l3}\partial_{33}^2 u_l = F_j(x) + 2\partial_j\pi - 2f_j, \tag{6.8}$$

where the $F_j(x)$, $j \neq 3$, are given by

$$F_j(x) := -\big(1+|\mathcal{D}u|\big)^{p-2}\sum_{k=1}^{2}\partial_{kk}^2 u_j - 2(p-2)\big(1+|\mathcal{D}u|\big)^{p-3}|\mathcal{D}u|^{-1}\left\{\mathcal{D}_{33}\mathcal{D}_{j3}\partial_{33}^2 u_3 + \sum_{\substack{l,m,k=1\\(m,k)\neq(3,3)}}^{3}\mathcal{D}_{lm}\mathcal{D}_{jk}\partial_{mk}^2 u_l\right\}. \tag{6.9}$$

In the sequel, Eqs. (6.8), $j = 1, 2$, will be treated as a 2×2 linear system in the unknowns $\partial_{33}^2 u_j$, $j \neq 3$. Note that, with an obviously simplified notation, the measurable functions F_j satisfy

$$\left|F_j(x)\right| \leqslant c\big(1+|\mathcal{D}u|\big)^{p-2}\left|D_*^2 u(x)\right|, \tag{6.10}$$

a.e. in Ω_a.

Hence, from (6.5) it follows that

$$|\widetilde{F}_j| \leqslant C(1+\widetilde{M})^{p-2}\big(\left|\nabla_*\widetilde{(\nabla u)}\right| + \epsilon_0|\widetilde{\xi}'|\big) \leqslant C(1+\widetilde{M})^{\frac{p-2}{2}}\left|\nabla_*\widetilde{(\nabla u)}\right| + \epsilon_0 C(1+\widetilde{M})^{p-2}|\widetilde{\xi}'|. \tag{6.11}$$

Next we consider the 2×2 linear system (6.8) in terms of the y variables, i.e., the system

$$(1+\widetilde{M})^{p-2}\widetilde{\xi}_j - 2(2-p)(1+\widetilde{M})^{p-3}\widetilde{M}^{-1}\widetilde{\mathcal{D}}_{j3}\sum_{l=1}^{2}\widetilde{\mathcal{D}}_{l3}\widetilde{\xi}_l = \widetilde{F}_j + \widetilde{\partial_j\pi} - \widetilde{f}_j, \tag{6.12}$$

and we show that this system can be point-wisely solved for the unknowns $\widetilde{\xi}_j$, $j = 1, 2$, for almost all $y \in Q_{\frac{a}{2}}$. The elements $\widetilde{a}_{jl}$ of the matrix system $\widetilde{A}$ are given by

$$\widetilde{a}_{jl} = (1+\widetilde{M})^{p-2}\delta_{jl} + 2(p-2)(1+\widetilde{M})^{p-3}\widetilde{M}^{-1}\widetilde{\mathcal{D}}_{l3}\widetilde{\mathcal{D}}_{j3},$$

for $j, l \neq 3$. Note that $\widetilde{a}_{jl} = \widetilde{a}_{lj}$. One easily shows that

$$\sum_{j,l=1}^{n-1}\widetilde{a}_{jl}\lambda_j\lambda_l = (1+\widetilde{M})^{p-2}|\lambda|^2 - 2(2-p)(1+\widetilde{M})^{p-3}\widetilde{M}^{-1}\big[(\widetilde{\mathcal{D}}u)\cdot\lambda\big]_3^2.$$

In particular

$$\sum_{j,l=1}^{2}\widetilde{a}_{jl}\lambda_j\lambda_l \geqslant 2\left(p-\frac{3}{2}\right)\big(1+|\widetilde{\mathcal{D}}u|\big)^{p-2}|\lambda|^2. \tag{6.13}$$

Hence the following result holds.

Lemma 6.3. *If $p > \frac{3}{2}$ the matrix $\widetilde{A}(y)$ is positive definite for almost all $y \in \Omega_a$. More precisely*

$$\det \widetilde{A} \geqslant \left[2\left(p - \frac{3}{2}\right)\left(1 + |\widetilde{\mathcal{D}u}|\right)^{p-2} \right]^2. \tag{6.14}$$

This lemma allows the following estimate.

Lemma 6.4. *One has a.e. in Q_a,*

$$|\widetilde{\xi}| \leqslant C\left|\nabla_*\widetilde{(\nabla u)}\right| + C(1 + \widetilde{M})^{2-p}\left(|\widetilde{\nabla_*\pi}| + |\widetilde{f}|\right). \tag{6.15}$$

Proof. From (6.12), i.e. from

$$\sum_{l=1}^{2} \widetilde{a}_{jl}\widetilde{\xi}_l = \widetilde{F}_j + \widetilde{\partial_j\pi} - \widetilde{f}_j, \tag{6.16}$$

together with (6.14), it follows that

$$\sum_{l,j=1}^{2} \widetilde{a}_{jl}\widetilde{\xi}_l\widetilde{\xi}_j = \sum_{j=1}^{2} (\widetilde{F}_j + 2\widetilde{\partial_j\pi} - 2\widetilde{f}_j)\widetilde{\xi}_j \tag{6.17}$$

holds. Consequently

$$(1 + \widetilde{M})^{p-2}|\widetilde{\xi}'| \leqslant |\widetilde{F}_j + \widetilde{\partial_j\pi} - \widetilde{f}_j|, \tag{6.18}$$

a.e. in Q_a. By appealing to (6.11) we show that

$$(1 + \widetilde{M})^{p-2}|\widetilde{\xi}'| \leqslant C(1 + \widetilde{M})^{p-2}\left|\nabla_*\widetilde{(\nabla u)}\right| + \epsilon_0 C(1 + \widetilde{M})^{p-2}|\widetilde{\xi}'| + c|\widetilde{\nabla_*\pi}| + c|\widetilde{f}|. \tag{6.19}$$

Hence (6.15) holds. We also appeal here to (6.2). □

Corollary 6.1. *For any admissible positive a one has in Q_a,*

$$\|\widetilde{\xi}\|_{r(q)} \leqslant C\left\|\nabla_*\widetilde{(\nabla u)}\right\|_{r(q)} + C\left(1 + \|\mathcal{D}u\|_q^{2-p}\right)\left(\|\widetilde{\nabla_*\pi}\|_2 + \|\widetilde{f}\|_2\right). \tag{6.20}$$

In particular, for $j = 1, 2$,

$$\|\partial_3\widetilde{\partial_3 u_j}\|_{r(q), Q_{\frac{a}{2}}} \leqslant C\left(1 + \|\mathcal{D}u\|_{q,Q_a}^{2-p}\right)\left(1 + \|\nabla_*\widetilde{u}\|_{2,Q_a}^{\frac{1}{2}} + \|f\|_{p'}\right). \tag{6.21}$$

Proof. Since

$$\left\|(1 + \widetilde{M})^{2-p}\widetilde{\nabla_*\pi}\right\|_{r(q)} \leqslant \|1 + \widetilde{M}\|_q^{2-p}\|\widetilde{\nabla_*\pi}\|_2,$$

the estimate (6.20) follows easily from (6.15).

Next, write (6.20) in $Q_{\frac{a}{2}}$ and estimate the quantities $\|\nabla_*\widetilde{(\nabla u)}\|_{r(q),Q_{\frac{a}{2}}}$ and $\|\widetilde{\nabla_*\pi}\|_{2,Q_{\frac{a}{2}}}$ by appealing to (5.81) and (5.82). Take into account that $\widetilde{\theta}$ is non-negative and equal to 1 on $Q_{\frac{a}{2}}$. It readily follows (6.21). We have used that $0 \leqslant \frac{2-p}{2} \leqslant 2 - p$. □

Further, from (5.80), a similar argument shows that

$$\|\nabla_*\widetilde{\nabla u}\|_{\lambda(q),Q_{\frac{a}{2}}} \leqslant C\left(1 + \|\mathcal{D}u\|_{q,Q_a}^{\frac{2-p}{2}}\right)\left(1 + \|f\|_{p'} + \|\nabla_*\widetilde{u}\|_{2,Q_a}^{\frac{1}{2}}\right). \tag{6.22}$$

7. Proof of Theorem 2.1

We start by stating the following particular case of more general results proved by Troisi in reference [58], to which we refer for details.

Proposition 7.1. *Let Q_0 be an open, bounded, "sufficiently regular" set, and let $v \in W^{1,1}(Q_0)$. Assume that*

$$\partial_k v \in L^{p_k}(Q_0), \quad \text{for } k = 1, 2, 3, \tag{7.1}$$

where

$$\frac{1}{\overline{p}} := \frac{1}{3}\sum_{k=1}^{3}\frac{1}{p_k} - \frac{1}{3}. \tag{7.2}$$

Then $v \in L^{\overline{p}}(Q_0)$ and

$$\|v\|_{\overline{p}} \leqslant c\prod_{k=1}^{3}\|\partial_k v\|_{p_k}^{\frac{1}{3}} + c\|v\|_p. \tag{7.3}$$

Obviously, we may replace $\|v\|_p$ by any other L^s norm, $s \geqslant 1$.

An essential point in order to get the limit exponent l in Theorem 2.2 is that the constant c on the right-hand side of (7.3) does not depend on the values of the exponents p_k used in the sequel. This property holds provided that $\overline{p}$ lies bounded away from 3. This follows essentially from Eq. (1.15) in the above reference (note, however, that all the values p_k used in the sequel lie bounded away from 3).

Further, note that the exponent $\mathcal{Q}(q)$, see (2.3), satisfies

$$\frac{1}{\mathcal{Q}(q)} = \frac{1}{3}\left(\frac{2}{\lambda(q)} + \frac{1}{r(q)} - 1\right).$$

Proof of Theorem 2.1. We apply Troisi's Theorem, in $Q_{\frac{a}{2}}$, to the single components of $\widetilde{\nabla u}$. By appealing to (6.21) and to (6.22) we show that

$$\|\widetilde{\nabla u}\|_{\mathcal{Q}(q),Q_{\frac{a}{2}}} \leqslant C\big(1 + \|\nabla_*\widetilde{u}\|_{2,Q_a}^{\frac{1}{2}} + \|f\|_{p'}\big)\big(1 + \|\mathcal{D}u\|_{q,Q_a}^{\frac{2(2-p)}{3}}\big). \tag{7.4}$$

From (7.4), by passing from the y to the x variables, it follows that,

$$\|\nabla u\|_{\mathcal{Q}(q),\Omega_{\frac{a}{2}}} \leqslant C\big(1 + \|\nabla u\|_{2,\Omega_a}^{\frac{1}{2}} + \|f\|_{p'}\big)\big(1 + \|\nabla u\|_{q,\Omega_a}^{\frac{2(2-p)}{3}}\big). \tag{7.5}$$

Clearly, (7.4) also holds if Q_a is contained in Ω. Actually much stronger interior estimates hold (obtained in a much easier way).

By setting $q = p$ we get

$$\|\nabla u\|_{\frac{6p}{8-3p},\Omega_{\frac{a}{2}}} \leqslant C\big(1 + \|\nabla u\|_{2,\Omega_a}^{\frac{1}{2}} + \|f\|_{p'}\big). \tag{7.6}$$

Since

$$\frac{6p}{8-3p} \geqslant 2,$$

for $p \geqslant \frac{4}{3}$, it readily follows (by a standard argument) that $\|\nabla u\|_{2,\Omega} \leqslant C$. Hence, we may drop the $\|\nabla u\|_{2,\Omega_a}^{\frac{1}{2}}$ term from the right-hand side of (7.5). This leads to

$$\|\nabla u\|_{\mathcal{Q}(q),\Omega_{\frac{a}{2}}} \leqslant C\big(1 + \|f\|_{p'}\big)\big(1 + \|\nabla u\|_{q,\Omega_a}^{\frac{2(2-p)}{3}}\big). \tag{7.7}$$

It readily follows that (2.7) holds, where C depends on the (fixed) number N of sets of type $\Omega_{\frac{a}{2}}$ (plus the number of spheres contained in the interior of Ω) sufficient to cover Ω.

In a similar way, from (5.81) and (6.21), we show (2.8). □

Further, from (5.82), it follows that

$$\|\nabla_*\widetilde{\pi}\|_{2,Q_{\frac{a}{2}}} \leqslant C\big(1 + \|f\|_{p'}\big). \tag{7.8}$$

On the other hand, by writing Eq. (6.7) for $j = 3$ we obtain an explicit expression for $\partial_3\pi$. In particular, it follows that

$$|\partial_3\pi| \leqslant c\big(1 + \big|M(x)\big|\big)^{p-2}\big|D^2u(x)\big| + \big|f(x)\big|. \tag{7.9}$$

Since $p<2$, $|\partial_3\pi|\leqslant C|D^2u(x)|+|f(x)|$. By transforming the inequality (7.9) from the x to the y variables one gets (for instance, for $y\in Q_{\frac{a}{2}}$)

$$\left|\partial_3\widetilde{\pi}(y)\right|\leqslant C\left|\widetilde{\left(D^2u\right)}(y)\right|+\left|\widetilde{f}(y)\right|. \tag{7.10}$$

This equation together with (2.8), and (7.8), show that $\|\nabla\pi\|_{r(q),\Omega}$ is bounded by the right-hand side of (2.8).

8. The boot-strap argument. Proof of Theorem 2.2

The proof follows that one in reference [4]. Since $\mathcal{D}u\in L^p(\Omega)$, it follows from (2.7) that $\mathcal{D}u\in L^{\mathcal{Q}(p)}(\Omega)$, where $\mathcal{Q}(p)=\frac{6p}{8-3p}$. Since this last exponent is greater than p, we may start an induction argument. Recall that our constants C are independent of the integrability exponents used here.

Define the strictly increasing sequence

$$\begin{cases} q_1=p,\\ q_{n+1}=\mathcal{Q}(q_n). \end{cases} \tag{8.1}$$

Note that the exponent $\bar{q}$ given by (2.4) is the limit $\bar{q}=\lim_{n\to\infty}q_n$. In particular, $\bar{q}$ is the fixed point of the map $q\to\mathcal{Q}(q)$.

From (2.7) it follows that

$$\|u\|_{1,q_{n+1}}\leqslant C\left(1+\|f\|_{p'}\right)\left(1+\|u\|_{1,q_n}^{\frac{2(2-p)}{3}}\right). \tag{8.2}$$

With an obvious notation, we write this equation in the form

$$a_{n+1}\leqslant b\left(1+a_n^{\alpha}\right).$$

Note that $0<\alpha<1$. By arguing as in [5] we prove that

$$\|u\|_{1,q_n}:=a_n<2\left(b+b^{\frac{1}{1-\alpha}}\right),$$

at least for sufficiently large values of n. Consequently, $\|u\|_{1,\bar{q}}$ is bounded by the right-hand side of the above equation. This shows that

$$\|u\|_{1,\bar{q}}\leqslant C\left(1+\|f\|_{p'}^{\frac{3}{2p-1}}\right).$$

The estimate (2.10) follows. It is worth noting that the boot-strap argument can be avoid, by following [9, Section 6].

Further, the estimate (2.11) follows easily by applying once more the estimate (2.8), now with $q=\bar{q}$, and by taking into account (2.10). Note that $r(\bar{q})=l$. Very similar devices show that $\nabla\pi\in L^l$.

Appendix A

The proof of Theorem 2.3 is done by following the short proof of Theorem 1.1 in reference [9, Section 6]. As in some of our previous papers, the proof in the presence of the convective term $(u\cdot\nabla)u$ follows in a straightforward way from the corresponding result obtained without such a term. However, with respect to the proof in reference [9], it is worth noting that in this last reference the constant C, that appears in (2.11), depends only on $\|\nabla u\|_p$. Here (see (5.10)) C also depends on $\|\pi\|_{p'}$. A fundamental point in the proof given in [9] is that the introduction of the convective term does not change the energy estimate obtained for $\|\nabla u\|_p$. In fact, this estimate is obtained by multiplication by u followed by integration on Ω, and the contribution of the convective term here vanishes. Hence, in order to be sure that the proof given in reference [9] applies here, we have to take into account the dependence of C on $\|\pi\|_{p'}$. We overcome this obstacle by showing that for $p\geqslant\frac{9}{5}$ one has

$$\|\pi\|_{p'}\leqslant c\|\nabla u\|_p^2.$$

This result is sufficient to our purpose, since $\frac{9}{5}<p_0$.

In the case of the full Navier–Stokes equations (2.12) one has an additional term $(u\cdot\nabla)u$ on the right-hand side of Eq. (3.11). This leads to an additional term $\|u^2\|_{p'}$ on the right-hand side of (3.12). If $p\geqslant\frac{9}{5}$, this last term is bounded by $c\|\nabla u\|_p^2$. Hence, if (u,π) is a weak solution to the full Navier–Stokes equations (2.12), then

$$\|\pi\|_{p'}\leqslant c\left(\|\nabla u\|_p^2+\|f\|_{-1,p'}+1\right).$$

The main point here is that, on the right-hand side of the above estimate, one has f and not $F=f-(u\cdot\nabla)u$ (see (3.9)).

Appendix B

Often, in Lemma 3.1, the additional assumption $g \in L^\alpha$ is required. We claim that it is sufficient to assume that g is a distribution in Ω (see also [20] for a similar claim). Nevertheless, for completeness, we show here a different proof of (5.62), based on the following result.

Proposition B.1. *Let p be a scalar field in L^2. Then, there is a constant c such that*

$$\|p - \overline{p}\| \leqslant c\|\nabla p\|_{-1}, \tag{B.1}$$

where $\overline{p}$ is defined by

$$\overline{p} = |\Omega|^{-1} \int_\Omega p\,dx. \tag{B.2}$$

For an exhaustive proof of the above proposition see the end of Appendix II, p. 1111, in [3] (warning: in this last reference the symbol $\overline{p}$ denotes $p - \overline{p}$).

Set, for convenience,

$$\psi(y) = \big(\widetilde{\pi}(y) - \widetilde{\pi}(y-h)\big)\widetilde{\theta}(y).$$

Clearly $\widetilde{\pi} \in L^2(\Omega) \subset L^{p'}(\Omega)$. Hence, by the above proposition,

$$\|\psi - \overline{\psi}\|_2 \leqslant c\|\nabla\psi\|_{-1,2}.$$

By appealing to

$$\overline{\psi} = \frac{1}{|Q_a|}\int \widetilde{\pi}(y)\big(\widetilde{\theta}(y) - \widetilde{\theta}(y+h)\big)\,dy,$$

the desired estimate follows.

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