Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary

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Abstract

We consider the initial boundary value problem for the 3D Navier-Stokes equations. The physical domain is a bounded open set with a smooth boundary on which we assume a condition of free-boundary type. We show that if a suitable hypothesis on the vorticity direction is assumed, then weak solutions are regular. The main tool we use in the proof is an explicit representation of the velocity in terms of the vorticity, by means of Green's matrices.

1 Introduction and results.

In this paper we consider the initial value problem for the 3D Navier-Stokes equations

$$\begin{cases} u_t + (u\cdot\nabla)\,u - \triangle u + \nabla p = 0 & \text{in } \Omega\times]0,T], \\ \nabla\cdot u = 0 & \text{in } \Omega\times]0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where the unknowns are the velocity u and the pressure p. In order to avoid inessential complications we assume that external force vanishes and that the kinematic viscosity is equal to 1. The open and bounded set $\Omega\subset\mathbb{R}^3$ -the physical domain- has a smooth boundary $\partial\Omega$, say of class $C^{3,\alpha}$, for some $\alpha>0$.

We supplement the initial value problem with the so-called "stress-free" boundary conditions

$$\begin{cases} u\cdot n = 0 & \text{on } \partial\Omega\times]0,T], \\ \omega\times n = 0 & \text{on } \partial\Omega\times]0,T], \end{cases} \tag{2}$$

where $\omega = \nabla\times u = \operatorname{curl} u$ is the vorticity field, while n denotes the exterior unit normal vector. In the case of flat boundaries, the above conditions coincide with the classical Navier boundary conditions, namely, $u\cdot n = 0$ and $n\cdot\nabla u - (n\cdot\nabla u\cdot n)\,n = 0$ (see the classical references Serrin [26] and Solonnikov and Ščadilov [**?**]; see also [3] and [4]). The boundary conditions (2) can also be used on a free-surface, see Temam [30].

The variational formulation and numerical implementation of the stationary Navier-Stokes equations with the "non-standard" boundary conditions (2) (that correspond also to a jet dye in applications to duct flows, see Conca *et al.* [14]), can be found in [1, 13, 14]. For questions of existence and regularity for the stationary problem see also Girault [20]. The boundary conditions (2) are also interesting because the treatment of the boundary layers is simpler than in the usual no-slip case, see Temam and Ziane [32] and Conca [12]. See also Clopeau, Mikelić, and Robert [11] for the 2D case. Similar conditions are also used in Lions [24] in order to study vanishing viscosity limits for the 2D problem.

The initial value problem for the Navier-Stokes equations with the above boundary conditions (2) poses the same problems as the usual one with vanishing Dirichlet boundary conditions: one can only obtain global existence of weak solutions and local existence of strong solutions. The proof can be done by adapting the usual one concerning Dirichlet boundary conditions, as in Leray [22] and Hopf [21]. See Section 2 for further details.

In the present paper we address the problem of global existence of smooth solutions, under additional hypotheses on the vorticity-direction. In particular, we extend previous results for the problem without boundaries or in the half-space. In the sequel $\theta(x, y, t)$ denotes the angle between the vorticity ω at two distinct points x and y, at time t:

$$\theta(x,y,t) \stackrel{def}{=} \angle(\widehat{\omega}(x,t), \widehat{\omega}(y,t)),$$

where, for each non-null vector v, we define $\widehat{v} \stackrel{def}{=} v/|v|$. Furthermore, c denotes an arbitrary positive constant.

The study of conditions involving the direction of vorticity, and its physical-geometric interpretation, started with Constantin and Fefferman [15], who first derived some exact formulas and employed them in order to prove regularity in the whole of $\mathbb{R}^3$. In particular, in [15] the following result is proved.

Theorem 1.1. *Let be $\Omega = \mathbb{R}^3$ and let u be a weak solution of (1) in $(0,T)$ with $u_0 \in H^1(\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0$. If*

$$\sin\theta(x,y,t) \le c\,|x-y|, \qquad a.e.\ x,y \in \mathbb{R}^3,\ a.e.\ t \in]0,T[\,, \tag{3}$$

then the solution u is strong in $[0,T]$ and, consequently, is regular.

Remark 1.2. Related results concerning vorticity direction and geometric constraints on potentially singular solutions for the 3D Euler equations (i.e. the case of vanishing viscosity) have been proved by Constantin, Fefferman, and Majda [16].

The result of Theorem 1.1 has been later improved by the authors in reference [8], by replacing the above Lipschitz condition by a 1/2-Hölder condition. More precisely, if

$$\sin\theta(x,y,t) \le c\,|x-y|^{1/2}, \qquad \text{a.e. } x,y \in \mathbb{R}^3,\ \text{a.e. } t \in]0,T[, \tag{4}$$

then the solution u is necessarily regular. Actually, in reference [8] the authors consider a family of sufficient conditions that contains (4) as the most significant case. More precisely, in [8] the following result is proved:

Theorem 1.3. *Let be $\Omega = \mathbb{R}^3$ and let u be a weak solution of (1) in $(0,T)$ with $u_0 \in H^1(\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0$. If there exists $\beta \in [1/2, 1]$ and $g \in L^a(0,T; L^b(\mathbb{R}^3))$, where*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2} \quad \textit{with} \quad a \in \left[\frac{4}{2\beta - 1}, \infty\right],$$

such that

$$\sin\theta(x,y,t) \le g(t,x)|x-y|^\beta, \qquad a.e.\ x,y \in \mathbb{R}^3,\ a.e.\ t \in]0,T[, \tag{5}$$

then the solution is necessarily regular.

More recently, one of the authors, see [5], extended the 1/2-Hölder condition in the whole of $\mathbb{R}^3$ to solutions to the boundary value problem (2) in the half-space case. In [5] the following result is proved (actually, in the above reference, the author considers the Navier's slip boundary condition. However, on flat boundaries, this condition coincides with (2)).

Theorem 1.4. *Let $\Omega = \mathbb{R}^3_+$. Suppose that $u_0 \in H^1(\Omega)$, $\nabla \cdot u_0 = 0$, and u is a weak solution to* (1)-(2) *in* $[0,T]$*. Suppose also that for some $\beta \in]0,1/2]$*

$$\sin\theta(x,y,t) \le c|x-y|^\beta, \qquad a.e.\ x,y \in \Omega,\ a.e.\ t \in]0,T[,$$

and that

$$\omega \in L^2(0,T;L^s(\Omega)), \qquad \textit{with } s = \frac{3}{\beta+1}.$$

Then, the solution u is a strong solution in $[0,T]$, *hence it is smooth.*

In reference [5] the above result is proved by appealing, separately, to the classical Dirichlet and Neumann Green's functions, in the half space. This can be done since, for flat boundaries, conditions (2) are equivalent to

$$\omega_1 = \omega_2 = 0\,; \quad \frac{\partial\,\omega_3}{\partial\,x_3} = 0\,.$$

Here, since the boundary is not flat, we have to localize the problem, a not trivial and quite technical matter. In reference [28], the author constructs global Green's matrices for a large class of boundary value problems and systems of partial differential equations. Our problem falls within this class. The first (and may be main) step in [28] consists in constructing a local version of Green's matrices, in a neighborhood of each boundary point. With the help of these local kernels, the author construct the global one. Unfortunately, it seems not possible to treat our problem by applying directly to the global Green's matrices. Hence we appeal here to the above "local" Green's matrices.
It is of interest to compare the above situation with the different one, faced in the presence of a Dirichlet boundary condition. In spite of the arbitrary (smooth) boundary, in reference [5] the fundamental estimate (51) is proved by appealing directly to the global Green's function for the Dirichlet problem, without the need of a localization argument. However, a new obstacle appears. Integration in Ω of the scalar product $-\Delta\,\omega\cdot\omega$ gives rise to the boundary integral

$$\int_{\partial\,\Omega} \frac{\partial\omega}{\partial n}\cdot\omega\,dS\,,$$

as follows from (13). Under the boundary condition (2) we are able to estimate this term in a suitable way, see the Lemma 2.2 below (if the boundary is flat, see [5], the above integral vanishes). On the contrary, under the Dirichlet boundary condition, (14) does not hold. Hence a suitable additional assumption on the above boundary integral seems necessary. See [5].

The aim of this paper is to extend the above regularity theorems to arbitrary, regular, open sets Ω. One has the following result.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set with a smooth boundary $\partial\Omega$, say of class $C^{3,\alpha}$, for some $\alpha > 0$. Suppose that $u_0 \in H^1(\Omega)$, $\nabla \cdot u_0 = 0$, and u is a weak solution to* (1)-(2) *in* $[0, T]$. *In addition, suppose either that there exists $\beta \in [1/2, 1]$ and $g \in L^a(0, T; L^b(\Omega))$, where*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2} \quad \textit{with} \quad a \in \left[\frac{4}{2\beta - 1}, \infty\right],$$

such that

$$\sin\theta(x, y, t) \le g(t, x)|x - y|^\beta, \qquad a.e.\ x, y \in \Omega,\ a.e.\ t \in]0, T[,$$

or either that there exists $\beta \in]0, 1/2]$

$$\sin\theta(x, y, t) \le c|x - y|^\beta, \qquad a.e.\ x, y \in \Omega,\ a.e.\ t \in]0, T[,$$

and that

$$\omega \in L^2(0, T; L^s(\Omega)), \qquad \textit{with } s = \frac{3}{\beta + 1}.$$

Then, the solution u is a strong solution in $[0, T]$, *hence it is smooth.*

Note that in Theorems 1.3, 1.4, and 1.5 the assumption (4) alone is a sufficient condition for regularity since weak solutions satisfy $\omega \in L^2(0, T; L^2(\Omega))$ (consider $\beta = \frac{1}{2}$, $a = b = \infty$ and $s = 2$). In addition, scaling properties show the sharpness of Theorem 1.5, since the case $\beta = 0$ and $s = 3$, corresponds to a well-known regularity class, as proved in ref. [2] and [9] for $\mathbb{R}^3$ and for a bounded domain, respectively. Moreover, by following [15], one shows that the conditions on $\sin\theta(x, y, t)$ need to be assumed only in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. For further details see Remark 3.9.

For simplicity, we present the complete proof of the above theorem only under the main assumption (4) (which corresponds to the special case $\beta = 1/2$), since this is the most significant case. In this way we avoid secondary points, that could hide the main ideas of an overall complicated and technical result. Actually, once the Proposition 3.2 is established, it is not difficult to make the necessary alterations in the subsequent results, in order to prove Theorem 1.5 in all its generality.

Hence, we shall prove with full details the following result, that is the main result of the paper (this result was announced by one of the authors in the note [7]).

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set with a smooth boundary $\partial\Omega$, say of class $C^{3,\alpha}$, for some $\alpha > 0$. Suppose that $u_0 \in H^1(\Omega)$, $\nabla \cdot u_0 = 0$, and u is a weak solution to* (1)-(2) *in* $[0, T]$. *Suppose also that*

$$\sin\theta(x, y, t) \le c\,|x - y|^{1/2}, \qquad a.e.\ x, y \in \Omega,\ a.e.\ t \in]0, T[,$$

is satisfied. Then, the solution u is a strong solution in $[0, T]$, *hence it is smooth.*

Each of the above theorems strongly appeals to ideas and techniques developed in the previous ones. In the proof of Theorem 1.6 the crucial new contribution is that one can use the representation formulas for Green's matrices derived in Solonnikov's outstanding work [27, 28] in order to treat boundaries. With the aid of these explicit formulas we introduce original local representation formulas for the velocity (in terms of the vorticity) and we are able to employ (4) in order to prove suitable estimates for the vortex stretching terms.

Plan of the paper. In Section 2 we give a proper variational formulation of the problem and we sketch the existence results for weak and strong solutions. In addition, integration by parts formulas are derived with full details. In Section 3 we use Solonnikov's theory of Green's matrices to give explicit representation of the vortex stretching term. By using Hypothesis (4) we deduce suitable estimates for the vorticity growth. Finally, in Section 4 we collect all previous results in order to prove the regularity results of Theorem 1.6. In an appendix some secondary calculations are reported for the sake of completeness.

Added in proof. In the forthcoming paper [10], one of the authors establish new results concerning the existence of global regular solutions under suitable hypothesis on the directions of ω and $curl\,\omega$.

2 Variational formulation and energy-type estimates.

In this section we present the variational formulation of the Navier-Stokes equations with the boundary conditions (2). We start by recalling the laws of balance for some physically meaningful quantities. We assume the functions to be smooth enough to make the calculations possible. In particular, by assuming that the solutions are *strong* (see Proposition 2.9) all the formal calculations become rigorous. In the sequel, we denote by $L^p := L^p(\Omega)$, for $1 \leq p \leq \infty$ and equipped with norm $\|\,.\,\|_p$, the usual Lebesgue spaces, while $H^s := H^s(\Omega)$, for $s \geq 0$, are the classical Sobolev spaces. We shall use the same symbol for both scalar and vector function spaces. We also use the space of divergence-free tangential vector fields of $L^2(\Omega)$

$$L^2_\sigma \overset{def}{=} \left\{u \in L^2(\Omega) :\ \nabla \cdot u = 0,\ u \cdot n = 0 \text{ on } \partial\Omega\right\}.$$

We recall that the divergence is taken in the distributional sense, while the trace condition has to be understood with respect to the space $H^{-1/2}(\partial\Omega)$. We also shall use the space of more regular $H^1(\Omega)$ tangential and divergence free-vector fields:

$$H^1_\sigma \overset{def}{=} H^1(\Omega) \cap L^2_\sigma.$$

In order to give the variational formulation of (1)-(2) we make some observations to explain the integration by parts that are possible to perform within this setting. In the sequel, $\partial_i \overset{def}{=} \frac{\partial}{\partial x_i}$, while ϵ_{ijk} is the totally anti-symmetric Ricci tensor. Moreover, summation over repeated indices is assumed.

2.1 Some integral identities.

In this section we derive some integrations by parts formulas that will be used in the sequel.

We start with an identity involved in the energy budget.

Lemma 2.1. *Let u and ϕ be two vector fields, tangential to the boundary. Then*

$$-\int_\Omega \triangle u_i\, \phi_i\, dx = \int_\Omega \nabla u_i \nabla \phi_i\, dx + \int_{\partial\Omega} (\omega \times n)_i\, \phi_i\, dS + \int_{\partial\Omega} \phi_i\, u_k\, \partial_i n_k\, dS, \tag{6}$$

where $\omega = curl\,u$.

Proof. We observe that, for $i = 1, 2, 3$,

$$[\omega \times n]_i = \epsilon_{ijk}\,\omega_j\,n_k = \epsilon_{ijk}\,(\epsilon_{jlm}\,\partial_l u_m)\,n_k = (\delta_{kl}\,\delta_{im} - \delta_{km}\,\delta_{il})\,n_k\,\partial_l u_m, \qquad \text{on } \partial\Omega.$$

Hence

$$n_k\,\partial_k u_i - n_k\,\partial_i u_k = (\omega \times n)_i \qquad \text{on } \partial\Omega. \tag{7}$$

Since the vector field u is tangential to the boundary, it follows that $\frac{\partial(u \cdot n)}{\partial \tau}\big|_{|\partial\Omega} \equiv 0$, for each vector field τ tangential to the boundary. By smoothly extending the normal unit vector field n to a small neighborhood of $\partial\Omega$ (see for instance Nečas [25] for Lipschitz prolongation for $C^{0,1}$-boundaries), a straightforward argument (ϕ is tangential, as well) shows that $\phi \cdot \nabla(u \cdot n)$ vanishes on $\partial\Omega$, i.e.,

$$n_k\,\phi_i\,\partial_i u_k = -u_k\,\phi_i\,\partial_i n_k \qquad \text{on } \partial\Omega. \tag{8}$$

Finally, by appealing in particular to (7) and (8) in the classic Gauss-Green formula, we deduce formula (6). □

The second identity is concerned with the vorticity field.

Lemma 2.2. *Assume that u is divergence-free and that on $\partial\Omega$ condition* (2) *holds, i.e., $u \cdot n = 0$ and $\omega \times n = 0$. Then*

$$-\frac{\partial \omega}{\partial n} \cdot \omega = (\epsilon_{1jk}\,\epsilon_{1\beta\gamma} + \epsilon_{2jk}\,\epsilon_{2\beta\gamma} + \epsilon_{3jk}\,\epsilon_{3\beta\gamma})\,\omega_j\,\omega_\beta\,\partial_k n_\gamma. \tag{9}$$

In particular,

$$-\int_\Omega \triangle\omega \cdot \omega\,dx \le \int_\Omega |\nabla\omega|^2\,dx + c\int_{\partial\Omega} |\omega|^2\,dS. \tag{10}$$

Proof. The vorticity ω is parallel to the normal unit vector on $\partial\Omega$. Hence $\frac{\partial(\omega \times n)}{\partial \tau}\big|_{\partial\Omega} \equiv 0$ for each vector field τ tangential to the boundary. Since on the boundary ω is orthogonal to tangent vectors, it follows that $\omega \times \nabla[\,(\omega \times n)_i\,] \equiv 0$ for $i = 1, 2, 3$, on $\partial\Omega$. In more explicit coordinates we can write, for $i,\ \alpha = 1, 2, 3$,

$$\epsilon_{ijk}\,\epsilon_{\alpha\beta\gamma}\,\omega_j\,\partial_k(\omega_\beta\,n_\gamma) = 0, \qquad \text{on } \partial\Omega. \tag{11}$$

Hence, by considering Eq. (11) for (i, α) equal to $(1, 1)$, $(2, 2)$, and $(3, 3)$ we get, respectively:

$$\begin{cases} n_3\,\omega_2\,\partial_3\omega_2 + n_2\,\omega_3\,\partial_2\omega_3 - n_2\,\omega_2\,\partial_3\omega_3 - n_3\,\omega_3\,\partial_2\omega_2 + \epsilon_{1jk}\,\epsilon_{1\beta\gamma}\,\omega_j\,\omega_\beta\,\partial_k n_\gamma = 0, \\ n_1\,\omega_3\,\partial_1\omega_3 + n_3\,\omega_1\,\partial_3\omega_1 - n_3\,\omega_3\,\partial_1\omega_1 - n_1\,\omega_1\,\partial_3\omega_3 + \epsilon_{2jk}\,\epsilon_{2\beta\gamma}\,\omega_j\,\omega_\beta\,\partial_k n_\gamma = 0, \\ n_2\,\omega_1\,\partial_2\omega_1 + n_1\,\omega_2\,\partial_1\omega_2 - n_1\,\omega_1\,\partial_2\omega_2 - n_2\,\omega_2\,\partial_1\omega_1 + \epsilon_{3jk}\,\epsilon_{3\beta\gamma}\,\omega_j\,\omega_\beta\,\partial_k n_\gamma = 0. \end{cases} \tag{12}$$

Next, by adding term-by-term, equations (12) together with

$$(n_2\,\omega_2\,\partial_2\omega_2 - n_2\,\omega_2\,\partial_2\omega_2) + (n_3\,\omega_3\,\partial_3\omega_3 - n_3\,\omega_3\,\partial_3\omega_3) + (n_1\,\omega_1\,\partial_1\omega_1 - n_1\,\omega_1\,\partial_1\omega_1) = 0,$$

we show that

$$n_i\,\omega_k\,\partial_i\omega_k - (\omega_i\,n_i)(\partial_k\omega_k) + (\epsilon_{1jk}\,\epsilon_{1\beta\gamma} + \epsilon_{2jk}\,\epsilon_{2\beta\gamma} + \epsilon_{3jk}\,\epsilon_{3\beta\gamma})\,\omega_j\,\omega_\beta\,\partial_k n_\gamma = 0, \quad \text{on } \partial\,\Omega.$$

Finally, since $\nabla \cdot \omega = 0$ we get (9). Equation (10) follows by appealing to the well known Green's formula

$$-\int_\Omega \triangle\omega \cdot \omega\,dx = \int_\Omega |\nabla\omega|^2\,dx - \int_{\partial\Omega} \frac{\partial\omega}{\partial n} \cdot \omega\,dS\,, \tag{13}$$

since (9) shows that

$$\exists\, c = c(\Omega) > 0 : \qquad \left| \frac{\partial \omega(x)}{\partial n} \cdot \omega(x) \right| \leq c\, |\omega(x)|^2, \qquad \forall\, x \in \partial\Omega. \tag{14}$$

□

2.2 Weak/strong solutions and energy/enstrophy balance.

With the results of the previous section we can now give the following definition.

Definition 2.3 (Weak solution (à la Leray-Hopf))**.** *We say that $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1_\sigma)$ is a weak solution to* (1)*, with the boundary conditions* (2)*, if the two following conditions hold:*

$$\int_0^T \int_\Omega (-u\,\phi_t + \nabla u \nabla \phi + (u \cdot \nabla) u\, \phi)\; dxdt + \int_0^T \int_{\partial\Omega} \phi\, \nabla n\, u\; dSdt = \int_\Omega u_0(x)\, \phi(x,0)\, dx,$$

for each $\phi \in C^\infty([0,T] \times \overline{\Omega})$ satisfying $\nabla \cdot \phi = 0$ in $\Omega \times [0,T]$, $\phi(T) = 0$ in Ω, and $\phi \cdot n = 0$ on $\partial\Omega \times [0,T]$.

There exists $c = c(\Omega) \geq 0$ such that the energy estimate

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2\, ds \leq \|u_0\|_2^2\; e^{2ct},$$

is satisfied for all $t \in [0,T]$.

Observe that the condition $\omega \times n = 0$ on $\partial\Omega$ can be recovered by integration by parts.

Before going into existence of weak solutions, let us see one inequality that holds for smooth solutions.

Lemma 2.4. *Let u be a smooth solution of* (1)-(2) *in $[0,T]$. Then, there exists a positive constant $c = c(\Omega)$ such that*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2\, dx - c\int_{\partial\Omega} |u|^2\, dS \leq 0. \tag{15}$$

Proof. The proof follows immediately by taking the scalar product of (1) with u, by integrating over Ω, and by using results of Lemma 2.1. Note that the first order derivatives of the (extended) normal unit vector n are uniformly bounded, since the domain is smooth. □

Next we give the definition of strong solution.

Definition 2.5 (Strong solution)**.** *We say that a weak solution u is strong in $[0,T]$ if*

$$\nabla u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\,.$$

We say that a weak solution u is strong in $[0,T_1[$ if u is strong in $[0,T]$ for each $T < T_1$.

Standard trace theorems imply that for strong solutions the condition $\omega \times n = 0$ takes place in $H^{-1/2}(\partial\Omega)$. In addition, standard tools (following the same lines of the proof in [17]) show uniqueness of strong solutions in the much wider class of weak solutions.

In order to show existence of strong solutions, one can consider the balance equation for the vorticity: By applying the curl operator to (1) we get

$$\begin{cases} \omega_t + (u \cdot \nabla)\,\omega - \triangle\omega = (\omega \cdot \nabla)\,u & \text{in } \Omega\times]0,T], \\ \nabla \cdot \omega = 0 & \text{in } \Omega\times]0,T], \end{cases} \tag{16}$$

and the system is supplemented with the boundary condition $(\omega \times n)_{|\partial\Omega} = 0$.

In order to deduce enstrophy balance, we take the scalar product of $(16)_1$ with ω, and we integrate over Ω. By appealing to (10)we show the following result.

Lemma 2.6. *Let u be a strong solution of (1)-(2) in $[0,T]$. Then, there exists a positive constant $c = c(\Omega)$ such that*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2 dx + \int_\Omega |\nabla\omega|^2\, dx - c\int_{\partial\Omega} |\omega|^2\, dS \leq \left|\int_\Omega (\omega\cdot\nabla)u\cdot\omega\, dx\right|. \tag{17}$$

Inequality (17) allows us to bound (at least for small times/small data) the vorticity in natural function spaces. As is well known, the presence in the right-hand-side of the vortex stretching term (that, at least at first glance, behaves like the integral of $|\omega|^3$) is the main obstacle to proving global existence results for strong solutions, even for the Cauchy problem in $\mathbb{R}^3$.

To employ inequality (17) we must observe that it concerns the L^2-norm of the vorticity and its first order derivatives, while the definition of strong solutions involves the full first and second order derivatives of u. In order to deduce suitable estimates we shall show that it is possible to bound the gradient of velocity, by the curl (at least in the L^2-setting). More precisely, we have the following result.

Lemma 2.7. *Let $u \in H^1_\sigma$ be a function satisfying (2). Then, there exists a positive constant $c = c(\Omega)$ such that*

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \leq c(\Omega)\int_\Omega |u|^2\, dx + \int_\Omega |\omega|^2 dx. \tag{18}$$

In addition, if $\omega \in H^1(\Omega)$, then $u \in H^2$ and its H^2-norm can be bounded by $\|\omega\|_{H^1}$.

Proof. Since $\nabla \cdot u = 0$ in Ω, one has

$$-\triangle u = \operatorname{curl}\operatorname{curl} u = \operatorname{curl}\omega\,.$$

In particular,

$$\begin{cases} -\triangle u = \operatorname{curl}\omega & \text{in } \Omega, \\ u\cdot n = 0 & \text{on } \partial\Omega, \\ \omega \times n = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

Next, we multiply both sides of the first equation (19) by u, and integrate over Ω. By appealing to Lemma 2.1 it follows that

$$\int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u_i\, u_k\, \partial_i n_k\, dS = \int_\Omega \operatorname{curl}\omega\cdot u\, dx\,.$$

This last equation can be written in the equivalent form

$$\int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial\Omega} u_i \, u_k \, \partial_i n_k \, dS = \int_{\partial\Omega} (\omega \times n) \cdot u \, dS + \int_{\Omega} |\omega|^2 \, dx \,. \qquad (20)$$

The boundary integral on the right hand side of (20) vanishes. On the other hand, smoothness of $\partial\Omega$ implies that the second integral on the left hand side of (20) is bounded by a multiple of $\int_{\partial\Omega} |u|^2 \, dS$. Hence, the standard trace inequality implies (18).

The L^2-regularity of second order derivatives follows by standard arguments. □

Remark 2.8. In order to use inequality (18), we need a bound for the L^2-norm of u to ensure the H^1-*a-priori* estimate for the solution. Since we are considering the time-evolution problem, the above bound follows from the energy estimate $(23)_1$, in the next section. However, if Ω is convex, then this last device is superfluous since the integrand that appears in the surface integral in the left-hand-side of (20) is (almost) everywhere non-negative. With this assumption it is also possible to prove existence (and regularity) of solutions to the stationary Stokes and Navier-Stokes equations with non-standard boundary conditions (2), see [20] for an approach with vector-valued potentials. In addition, with a different variational formulation, existence and uniqueness of weak solutions to the stationary Navier-Stokes equations with the "non-standard" boundary conditions (2) can be given in simply connected domains. For related question of non-uniqueness, see Foiaş and Temam [19] with the characterization of curl/div-free vector fields in non-simply connected domains.

In the following, a main point is that the system (19), more precisely, the system

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \partial\Omega, \\ \omega \times n = 0 & \text{on } \partial\Omega. \end{cases} \qquad (21)$$

is of Petrovksĭ type (see [28]). In Petrovksĭ's systems –roughly speaking– different equations and unknowns have the same "differentiability order," see p. 126 in [27]. This fact allows us to use in the sequel the "simplified" representation formula (25), in which just a single Green's matrix is present. We also recall that Petrovksĭ's systems are an important subclass of Agmon-Douglis-Nirenberg (ADN) elliptic systems, having the same good properties of self-adjoint ADN systems. In addition, for these systems the H^2-regularity can be used to prove the full regularity of solutions, provided that the data are smooth. In particular, this implies (by employing a boot-strap argument) that if $\partial\Omega$ is smooth, then strong solutions of (1) are smooth, say C^∞.

For the reader's convenience we give here some remarks on the above subject. In reference [28], see p. 126, in connection with the particular system of equations and boundary value problem under study, the author considers a set of integer "weights" t_i, s_i, σ_j. The system is called of Petrovksĭ type if $s_i = 0$ and $\sigma_j < 0$, for all i and j. Let us consider the system (21), in the case of a flat boundary and assume that the x_3 direction is normal to the boundary. In this case the above weights are given by $t_1 = t_2 = t_3 = 2$, $s_1 = s_2 = s_3 = 0$, and $\sigma_1 = \sigma_2 = -1$, $\sigma_3 = -2$. Hence the system (21) is of Petrovksĭ type. On the contrary, if we consider the Stokes problem

$$\begin{cases} -\triangle u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega, \\ \omega \times n = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{22}$$

then one has two additional weights, $s_4 = -1$ and $t_4 = 1$. Hence the system is not of Petrovksĭ type.

For an introduction to the above subject we recommend the reader to look up in the proof of proposition 2.2 in [31], where the Stokes system is considered under the Dirichlet boundary condition. Under this boundary condition the t_i and the s_i, $i = 1, \ldots, 4$ are as above, moreover $\sigma_1 = \sigma_2 = \sigma_3 = -1$. Hence the system is still not of Petrovksĭ type (the weights σ_k are denoted in [31] by r_k).

2.3 Existence of solutions.

We conclude this section by giving a sketch of the proof of the existence results for weak and strong solutions.

By using standard techniques, the two differential inequalities (15) and (17) can be used to prove the existence of weak and strong solutions. In fact, by taking into account the trace inequality, we prove the following differential inequalities:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega |u|^2 dx + \dfrac{1}{2}\int_\Omega |\nabla u|^2\, dx \leq c(\Omega)\int_\Omega |u|^2 dx, \\ \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega |\omega|^2 dx + \dfrac{1}{2}\int_\Omega |\nabla \omega|^2\, dx \leq c(\Omega)\int_\Omega |\omega|^2 dx + \left|\int_\Omega (\omega\cdot\nabla)u\cdot\omega\, dx\right|. \end{cases} \tag{23}$$

By using a Faedo-Galerkin approximation method, with the techniques introduced by Hopf [21], (see e.g. Temam [31] or Constantin and Foiaş [17]) one can easily show the following result.

Proposition 2.9. *Let $u_0 \in L^2_\sigma$ be given. Then, for each $T > 0$ there exists at least one weak solution of the 3D Navier-Stokes equations* (1) *with the boundary conditions* (2)*. In addition, if $u_0 \in H^1_\sigma$ then there is a $T^* = T^*(\|\nabla u_0\|_2) > 0$ such that a unique strong solution in $[0, T^*[$ exists.*

From Lemma 2.7 it follows that if we are able to bound the L^2-norm of the curl of a weak solution u, we are also able to bound the full gradient of this solution. This is the reason why we can use the vorticity equation. A standard continuation argument, that will be used in the proof of Theorem 1.6, reduces our task to showing that if a weak solution satisfies Hypothesis (4) in $(0, T)$, then $\omega(x, t)$ belongs to $L^\infty(0, T; L^2(\Omega))$.

3 Solonnikov's theory on the Green's Matrices: A sharp estimate of the vortex stretching.

The results in this section are the core of our proof. In fact, by appealing to the integral bounds shown below, the proof of Theorem 1.6 will follow by a standard continuation argument (see

the next section).
In order to give upper bounds for the vortex stretching we prove a suitable estimate for the integral that appears in the right-hand-side of the vorticity balance equation (17). The estimate on the vortex stretching term will be derived by using an explicit representation of the solution to the boundary value problem (19), which generalizes that introduced in reference [5] in the half-space case, and by using identities similar to those introduced in [15].

3.1 Preliminaries on Green's functions.

Since $\partial\Omega$ is smooth and compact, we may fix a positive, real δ such that for each point of $x \in \Omega_\delta$, where

$$\Omega_\delta \stackrel{def}{=} \{x \in \Omega \text{ such that } d(x, \partial\Omega) \leq \delta\},$$

there exists a unique point (the orthogonal projection) $\mathbb{P}x \in \partial\Omega$ such that

$$d(x, \mathbb{P}x) = \min_{y \in \partial\Omega} d(x, y),$$

where $d(\,.\,,\,.\,)$ is the Euclidean distance in $\mathbb{R}^3$.

Given $x_0 \in \Omega$ and we distinguish between two cases: 1) $x_0 \in \Omega_\kappa$; 2) $x_0 \in \Omega \backslash \Omega_\kappa$, for some positive $0 < \kappa \leq \delta$ that we shall fix later.

We aim at proving (see Proposition 3.2) a bound for

$$\big|(\omega(x_0) \cdot \nabla)\, u(x_0) \cdot \omega(x_0)\big|, \tag{24}$$

independent of x_0, recall (17). To this end we express the velocity u in terms of the vorticity ω, by appealing to the boundary value problem (19). Since this system is of Petrovksĭ type, there exists a single Green's matrix $\mathcal{G}(x, y)$ (see [28]) such that:

$$u(x) = \int_\Omega \mathcal{G}(x, y)\, \mathrm{curl}\, \omega(y)\, dy. \tag{25}$$

The matrix $\mathcal{G}(x, y)$ can be written as

$$\mathcal{G}(x, y) = \mathsf{G}(x, y) + g(x, y),$$

where the first term on the right hand side, which contains the "leading-order terms," satisfies the estimates

$$\exists\, c > 0: \qquad |D_x^\alpha D_y^\beta \mathsf{G}(x, y)| \leq \frac{c}{|x - y|^{|\alpha| + |\beta| + 1}}, \qquad \forall\, x, y \in \Omega,\ x \neq y, \tag{26}$$

while the second term $g(x, y)$ consists of lower order terms, as $|x - y| \to 0$; see again Solonnikov [27, 28]. In order to prove our results, we need more explicit representation formulas for $\mathsf{G}(x, y)$. So, let us be precise about some details, proved in reference [28], to which we constantly refer.

First, we localize our problem by appealing to the construction of the Green's matrices made in [27]. More precisely (for a proof and further details see p. 150 in [27]), it is possible to find a finite covering $\{\omega_a\}_{a=1,\dots,N}$, $N \in \mathbb{N}$, of $\overline{\Omega}$ such that:

a) $\omega_a \subseteq \overline{\Omega}$;

b) The regions ω_a -which do not intersect the boundary $\partial\Omega$- are cubes defined (for $i = 1, 2, 3$) by $|x_i - x_i^a| \leq d_1$, with $x^a \in \Omega$, and $d(\omega^a, \partial\Omega) \geq d_1$. The set of indices of these *interior* regions is denoted by $\mathcal{I}$.

The remaining ω^a are given, in local coordinates $\{z^a\}$ with centers at points $x^a \in \partial\Omega$, by inequalities

$$|z_i^a| \leq d_2 \quad i = 1, 2; \qquad 0 \leq z_3 - F_a(z_1^a, z_2^a) \leq 2d_2\,,$$

where $F^a \in C^{3,\alpha}$ define $\partial\Omega$ as a Cartesian surface (graph) near the points x_a by equations $z_3^a = F^a(z_1^a, z_2^a)$ defined in square domains $|z_i^a| \leq d_2$, with $i = 1, 2$. The set of indices of these *boundary* regions is denoted by $\mathcal{B}$.

c) There is a partition of the unity consisting of smooth functions $\{\chi_a(y)\}_a$ associated to the covering $\{\omega^a\}_a$, with $\sum_a \chi_a(y) \equiv 1$, $\forall\, y \in \overline{\Omega}$ such that $\mathrm{supp}[\chi_a(y)] \subset \omega^a$ and $\bigcup_a \omega^a \supset \overline{\Omega}$.

The coordinates $\{z^a\}$ are connected to x by an orthogonal transformation $z^a = U^a(x - x^a)$ in order that the z_3^a-axis is directed along the normal interior direction at the point $x^a \in \partial\Omega$. The transformation $\xi^a = \mathcal{F}^a(z^a)$ is defined by

$$\begin{cases} \xi_1^a = z_1^a, \\ \xi_2^a = z_2^a, \\ \xi_3^a = z_3^a - F^a(z_1^a, z_2^a), \end{cases} \tag{27}$$

and maps ω^a into the cube $|\xi_i^a| \leq d_2$, for $i = 1, 2$ and $0 \leq \xi_3^2 \leq 2d_2$. It also maps $\omega^a \cap \partial\Omega$ onto $\xi_3^a \equiv 0$. Finally, the transformation

$$T^a = \mathcal{F}^a \circ U^a,$$

which connects x and ξ^a, has Jacobian identically equal to 1.

The fact that the domain Ω is smooth and bounded implies that we may choose the two strictly positive numbers d_1 and d_2 (small enough) such that

$$\frac{1}{2}|x - y| \leq |T^a x - T^a y| \leq 2|x - y|, \qquad \forall\, x,\, y \in \omega^a \quad \forall\, a \in \mathcal{B}. \tag{28}$$

By means of this change of coordinates the Green's matrix $\mathcal{G}(x, y)$ can be expressed in terms of the explicit Green's matrices $Z^a(\,.\,,\,.\,)$ and $G^a(\,.\,,\,.\,)$ that are known respectively for the whole space or for the half-space, leading to the following representation formula:

$$\begin{aligned} u(x) = &\int_\Omega \sum_{a\in\mathcal{I}} \chi_a(y)\, Z^a(x, y)[\mathrm{curl}\,\omega(y)]\, \zeta\left(\frac{|x - y|}{d_3}\right) dy + \\ &+ \int_\Omega \sum_{a\in\mathcal{B}} \chi_a(y)\, G^a(T^a x, T^a y)[\mathrm{curl}\,\omega(y)]\, \zeta\left(\frac{|T^a x - T^a y|}{d_3}\right) dy \\ &+ \int_\Omega g(x, y)\, [\mathrm{curl}\,\omega(y)]\, dy, \end{aligned} \tag{29}$$

with

$$d_3 = (1/4) \min\{d_1, d_2\}.$$

Here $Z^a(. , .)$ is the Green's matrix related to the Poisson problem in the whole-space:

$$Z^a_{ij}(\xi,\eta) = \frac{\delta_{ij}}{4\pi}\frac{1}{|\xi-\eta|}, \qquad i,j = 1,\dots,3$$

and δ_{ij} denotes the Kronecker's delta such that $\delta_{ij} = 1$, if $i = j$, and 0 otherwise. The function $G^a(. , .)$ is the Green's matrix associated to the Poisson problem in the half-space with suitable (Navier) boundary conditions:

$$G^a_{ij}(\xi,\eta) = \frac{\delta_{ij}}{4\pi}\left(\frac{1}{|\xi-\eta|} - \epsilon_j\frac{1}{|\xi-\overline{\eta}|}\right),$$

with $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$. The "bar" denotes the "reflected point"

$$[\,\overline{\eta}\,]_j \overset{def}{=} \epsilon_j\,\eta_j \qquad j = 1,2,3.$$

We recall that the introduction of "reflected point" derives from the use of *virtual charges* to treat problems with boundaries, classical in the potential theory for electrostatic problems; see, e.g., Courant-Hilbert [18].

The function $\zeta \in C^\infty(\mathbb{R})$ is a monotonic non-increasing cut-off function such that

$$0 \le \zeta(r) \le 1 \qquad \text{and} \qquad \zeta(r) = \begin{cases} 1 & \text{if } r \le \dfrac{1}{4}, \\ 0 & \text{if } r \ge \dfrac{3}{4}. \end{cases}$$

Finally, as recalled above, the matrix $g(\xi,\eta)$ consists of lower-order-terms (i.e. terms that are not of the leading order as those in $Z(\xi,\eta)$ and $G(\xi,\eta)$), say

$$\exists\, c,\gamma > 0: \qquad |D^\alpha_\xi D^\beta_\eta g(\xi,\eta)| \le \frac{c}{|\xi-\eta|^{|\alpha|+|\beta|+1-\gamma}}, \qquad \forall\, x,y\in\Omega,\; x\neq y, \tag{30}$$

where $\gamma > 0$ depends on the Hölder regularity of the solutions to (19), see [28]. Consequently γ depends just on the regularity of the boundary $\partial\Omega$, since the differential operator and the boundary operators in (19) have constant coefficients. Recall that, as already remarked at the end of Section 2.3, we may assume that the right-hand side $\operatorname{curl}\omega$ is regular in $(0,T)$.

Remark 3.1. In order to understand the explicit formulas for $Z(x,y)$ and $G(x,y)$, we recall that we are dealing with boundary conditions involving the vorticity which, on flat boundaries, become the usual Navier-slip boundary condition. For system (19) this boundary condition -in local coordinates- become a Neumann boundary condition for the first two components of u (or, equivalently, for the normal derivative of the velocity in the tangential directions) and a Dirichlet boundary condition for the velocity in the normal direction (the third component in our reference frame.) Hence, in the flat-boundary case (see [5]), problem (19) reduces to

$$\begin{cases} -\triangle u = \operatorname{curl}\omega & \text{in } \mathbb{R}^3_+, \\ u_3 = 0 & \text{on } \xi_3 = 0, \\ \dfrac{\partial u_j}{\partial \xi_3} = 0, \quad j = 1,2 & \text{on } \xi_3 = 0. \end{cases}$$

This basic problem leads to the construction of the principal part $G(\xi,\eta)$ of the Green's matrix. For a classical treatment of the Green's function in these particular cases see also Lévy [23] and Courant-Hilbert [18].

3.2 Some explicit formulas for the vortex stretching.

In this section we appeal to the explicit representation formula (29) to estimate the vortex stretching term. We start from the integrals involving the leading order terms and -for the sake of completeness- we shall treat in an appendix all the lower order terms.

A crucial point of this paper is the following proposition.

Proposition 3.2. *There exists a non-negative constant C, uniformly bounded for $x \in \Omega$, such that*

$$\begin{aligned}\Big|(\omega(x)\cdot\nabla)\,u(x)\cdot\omega(x)\Big| \leq C\,|\omega(x)|^2 \Bigg[&\|\omega\|_2 + \\ &+\int_\Omega \Big[\big|\mathrm{Det}\big(\widehat{\omega}(x),\widehat{\omega}(y),\widehat{Tx-Ty}\big)\big| + \big|\mathrm{Det}\big(\widehat{\omega}(x),\widehat{\omega}(y),\sigma(y')\big)\big|\Big]\,\frac{|\omega(y)|\,dy}{|Tx-Ty|^3} \\ &+\int_\Omega \Big[\big|\mathrm{Det}\big(\widehat{\overline{\omega}}(x),\widehat{\omega}(y),\widehat{\overline{Tx_0-\overline{Ty}}}\big)\big| + \big|\mathrm{Det}\big(\widehat{\overline{\omega}}(x),\widehat{\omega}(y),\sigma(y')\big)\big|\Big]\,\frac{|\omega(y)|\,dy}{|Tx-\overline{Ty}|^3}\Bigg].\end{aligned} \tag{31}$$

This proposition will be proved separately for points "near the boundary" and for points "far from the boundary;" see (32) and (47). New ideas concern the treatment of points near the boundary. Estimates for points far from the boundary can be derived easily from the results in the whole of the space, or by a substantial simplification of the argument used to treat points near to the boundary. Nevertheless, just for completeness, we shall also give the guidelines for proving (31) for points x far from the boundary.

In order to properly define "near" and far" set

$$d \stackrel{def}{=} \min\{\delta,\, d_1,\, d_2\},$$

were δ, d_1, d_2, and d_3 are defined in the previous section.

3.2.1 Proof of Proposition 3.2 for points "near the boundary."

We now suppose that x_0 is an arbitrary (but fixed) point, near the boundary. More precisely, we assume that

$$x_0 \in \Omega_{2d/3}. \tag{32}$$

As previously claimed, by means of a rigid rotation (recall that the Navier-Stokes equations are invariant by means of rigid transformations) we can use a reference frame with origin at $\mathbb{P}(x_0)$ and such that $e_3 = x_0 - \mathbb{P}x_0$. The e_1 and e_2-directions -tangential to $\partial\Omega$- are chosen in order to have a right-handed triple of unit vectors. In this system of coordinates the boundary point $\mathbb{P}x_0$ becomes the origin $(0,0,0)$. The change of coordinates is made by flattening the domain near x_0 in the direction of the normal unit vector passing through x_0 and having this line as vertical axis for the corresponding square in e_3-variables. With this choice of coordinates the transformation is simply given by

$$Tx = \begin{cases} x_1 \\ x_2 \\ x_3 - F(x_1,x_2), \end{cases} \tag{33}$$

where, for notation convenience, from now on we denote z by x (observe that (33) is a special case of (27)). Note that here there are no rotation U, see formula (27) (more precisely, U is the identity).

It is worthwhile observing that the transformation T depends on the point $x_0 \in \Omega$, even if we do not write it explicitly. In particular, contrary to the tools used to prove existence of Green's matrices, in the sequel we appeal to a different transformation T for each point $x_0 \in \Omega$. Note that (see [27, 28]), for a given regular domain Ω, the parameters that characterize the transformation T can be chosen independently of the particular point x_0. In fact, these parameters depend only on the diameter of the local subset ω^a and on the local regularity of the boundary $\partial\Omega$, which however is characterized by global parameters (for instance the curvature is globally bounded).

In addition, we shall make use of just one chart in connection to each single point x_0, in order to bound (24) uniformly with respect to the point x_0. This is justified by taking into account the above independence of the main parameters, with respect to the particular point x_0. More precisely, we make use of two sets ω^1 and ω^2 such that:

1. $x_0 \in \omega^1$, where ω^1 is defined by

$$\omega^1 = \big\{|x_i| \leq 2d/3 \quad i = 1, 2; \quad 0 \leq x_3 - F(x_1, x_2) \leq 4d/3\big\},$$

   where $x_3 = F(x_1, x_2)$ denotes the analytical expression of the boundary $\partial\Omega$ near $\mathbb{P}x_0$ (recall the third equation (33)).

2. $x_0 \notin \omega^2$, where ω^2 is defined by

$$\omega^2 = \big\{x \in \Omega : d/3 \leq |x_i| \quad i = 1, 2; \text{ or if } |x_i| < d/3, \text{ then } x_3 > F(x_1, x_2) + d\big\}.$$

Remark 3.3. The definition of these two sets implies that $d(x_0, \omega^2) \geq d/3 > 0$.

Moreover, the local change of coordinates is given by "flattening" the boundary by means of the smooth function F. In particular, $F \in C^{3,\alpha}$ satisfies $F(0,0) = F_{x_1}(0,0) = F_{x_2}(0,0) = 0$ and the transformation T is bounded from below and from above in a Lipschitz way by (28). Hence couples of points that are "near," are (uniformly) mapped into couples of points that are "near," and reciprocally. We finally observe that under the transformation T one has

$$Tx_0 = x_0.$$

Actually, T acts as the identity on the vertical line passing through x_0 and $\mathbb{P}x_0$.

By using these tools for x near to x_0 -say for $d(x, x_0) < d/16-$ formula (29) becomes

$$\begin{aligned} u(x) = & \int_{\omega^1} \chi_1(y)\, G(Tx, Ty)\, [\operatorname{curl}\omega(y)]\, \zeta\left(\frac{|Tx - Ty|}{d_4}\right) dy \\ & + \int_{\omega^2} \chi_2(y)\, \mathcal{G}_2(x, y)\, [\operatorname{curl}\omega(y)]\, dy + \int_\Omega g(x, y) \operatorname{curl}\omega(y)\, dy, \\ = & \; J_1(x) + J_2(x) + J_3(x), \end{aligned} \tag{34}$$

where d_4 is defined by

$$d_4 = (1/4) \min\{d, d_2\}. \tag{35}$$

Recall that $\text{supp}[\chi_2(y)] \subset \omega_2$ and note that $\mathcal{G}_2(x,y)$ collects terms of leading order (multiplied possibly by a cut-off function) which satisfy -at worse- the estimate in (26). The matrix $g(x,y)$ (which contains lower-order terms), $J_2(x)$, and $J_3(x)$ will be treated in the appendix. Note that, since we are working "near to the boundary," the $Z(x,y)-$terms are not present.

Let us focus on the first integral in the right-hand-side (rhs in the sequel) of equation (34). We first integrate by parts, obtaining

$$\begin{aligned}
J_1(x) &\stackrel{def}{=} \int_{\omega^1} \chi_1(y)\, G(Tx,Ty)\, [\text{curl}\,\omega(y)]\, \zeta\left(\frac{|Tx-Ty|}{d_4}\right) dy \\
&= \int_{\omega^1} \omega(y)\, \text{curl}\left[\chi_1(y)\, G(Tx,Ty)\, \zeta\left(\frac{|Tx-Ty|}{d_4}\right)\right] dy + \\
&+ \int_{\partial\omega^1} [\omega(y) \times n]\ \chi_1(y)\, G(Tx,Ty)\, \omega(y)\, \zeta\left(\frac{|Tx-Ty|}{d_4}\right) dS.
\end{aligned}$$

Observe that the presence of the cut-off function $\chi_1(y)$ implies that the boundary integral needs not to be evaluated on the whole $\partial\omega^1$ but just on $\partial\Omega \cap \partial\omega^1$. Due to the boundary conditions $\omega \times n_{|\partial\Omega} = 0$ this surface integral vanishes identically. Hence, we are left with the following identity[1]

$$\begin{aligned}
J_1(x) &= \int_{\omega^1} \omega(y)\, \text{curl}\, G(Tx,Ty)\ \left[\chi_1(y)\, \zeta\left(\frac{|Tx-Ty|}{d_4}\right)\right] dy \\
&\quad + \int_{\omega^1} \omega(y)\, G(Tx,Ty)\ \times \nabla\left[\chi_1(y)\, \zeta\left(\frac{|Tx-Ty|}{d_4}\right)\right] dy, \\
&\stackrel{def}{=} J_1^1(x) + J_1^2(x).
\end{aligned} \tag{36}$$

For the moment let us consider the leading term $J_1^1(x)$. The term $J_1^2(x)$ will be treated in the appendix A.1. To deal with $J_1^1(x)$, we use the index notation, with the Einstein's convention of summation over repeated indices. Recall that $[v \times w]_j = \epsilon_{jkl} v_k\, w_l$ for vectors $v, w \in \mathbb{R}^3$ and $[\text{curl}\, u]_j = [\nabla \times u(x)]_j = \epsilon_{jkl} \frac{\partial u_l(x)}{\partial x_k}$. A detailed expression is then

$$\begin{aligned}
&\left[J_1^1(x)\right]_j = \\
&= -\frac{1}{4\pi} \int_{\omega^1} \omega_l(y)\, \epsilon_{jkl} \left[\frac{T_m x - T_m y}{|T_m x - T_m y|^3} \frac{\partial T_m y}{\partial y_k} - \epsilon_j \frac{T_m x - \overline{T_m y}}{|T_m x - \overline{T_m y}|^3} \frac{\partial \overline{T_m y}}{\partial y_k}\right] \chi_1(y)\, \zeta\left(\frac{|Tx-Ty|}{d_4}\right) dy, \\
&= -\frac{1}{4\pi} \int_{\omega^1} \omega_l(y)\, \epsilon_{jkl} \left[\frac{T_m x - T_m y}{|Tx - Ty|^3} - \epsilon_j \epsilon_m \frac{T_m x - \overline{T_m y}}{|T_m x - \overline{T_m y}|^3}\right] \frac{\partial T_m y}{\partial y_k}\ \chi_1(y)\, \zeta\left(\frac{|Tx-Ty|}{d_4}\right) dy.
\end{aligned}$$

Since $J_1^1(x)$ is one of the terms that enters in the representation formula (34), we need to differentiate it with respect to the x_i-variables, and to multiply by $\omega_i(x)\,\omega_j(x)$, in order to be able to estimate its contribution to the term $(\omega(x_0) \cdot \nabla)\, u(x_0)\, \omega(x_0)$. To this end, and to simplify the manipulations, we separate the terms *with* and *without* "reflected quantities." Consequently

$$\frac{\partial \left[J_1^1(x)\right]_j}{\partial x_i}\ \omega_i(x)\, \omega_j(x) = \frac{\partial a_j(x)}{\partial x_i}\ \omega_i(x)\, \omega_j(x) + \frac{\partial b_j(x)}{\partial x_i}\ \omega_i(x)\, \omega_j(x), \tag{37}$$

[1] The differential operators "nabla" and "curl" act on the y variables.

where (see [5], equation (43))

$$\begin{cases} a_j(x) \stackrel{def}{=} -\dfrac{1}{4\pi}\epsilon_{jkl}\displaystyle\int_{\omega^1} \omega_l(y)\,\dfrac{T_m x - T_m y}{|T_m x - T_m y|^3}\dfrac{\partial T_m y}{\partial y_k}\;\chi_1(y)\,\zeta\left(\dfrac{|Tx-Ty|}{d_4}\right)\,dy, \\ b_j(x) \stackrel{def}{=} \dfrac{1}{4\pi}\epsilon_{jkl}\epsilon_j\epsilon_m\displaystyle\int_{\omega^1} \omega_l(y)\,\dfrac{T_m x - \overline{T_m y}}{|T_m x - \overline{T_m y}|^3}\dfrac{\partial T_m y}{\partial y_k}\;\chi_1(y)\,\zeta\left(\dfrac{|Tx-Ty|}{d_4}\right)\,dy. \end{cases} \tag{38}$$

We start by dealing with the term involving $a_j(x)$ and we have the following result.

Lemma 3.4. *Assume that x_0 satisfies (32), and define the functions $a_j(x)$, for $j = 1, 2, 3$, as above. Then, there exists a positive constant c, independent of x_0, such that*

$$\begin{aligned} \left|\frac{\partial a_j(x_0)}{\partial x_i}\,\omega_j(x_0)\,\omega_j(x_0)\right| &\le c\,|\omega(x_0)|^2\Bigg(\|\omega\|_2 + \\ &+ \int_{\omega_1}\Big[\big|\mathrm{Det}\big(\widehat{\omega}(x_0),\widehat{\omega}(y),\widehat{Tx_0 - Ty}\big)\big| + \big|\mathrm{Det}\big(\widehat{\omega}(x_0),\widehat{\omega}(y),\sigma(y')\big)\big|\Big]\,\frac{|\omega(y)|\,dy}{|Tx-Ty|^3}\Bigg). \end{aligned} \tag{39}$$

Proof. Taking the derivative of $a_j(x)$ with respect to x_i we get

$$\begin{aligned} \frac{\partial a_j(x)}{\partial x_i} =& -\frac{1}{4\pi}\epsilon_{jkl}\int_{\omega^1}\omega_l(y)\,\frac{\delta_{pm}}{|Tx-Ty|^3}\frac{\partial T_m y}{\partial y_k}\frac{\partial T_p x}{\partial x_i}\;\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right)\,dy \\ & -\frac{1}{4\pi}\epsilon_{jkl}\int_{\omega^1}\omega_l(y)\,\frac{(T_p x - T_p y)(T_m x - T_m y)}{|Tx-Ty|^5}\,\frac{\partial T_m y}{\partial y_k}\frac{\partial T_p x}{\partial x_i}\;\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right)\,dy \\ & -\frac{1}{4\pi}\epsilon_{jkl}\int_{\omega^1}\omega_l(y)\,\frac{T_m x - T_m y}{|Tx-Ty|^3}\,\frac{\partial T_m y}{\partial y_k}\;\chi_1(y)\,\zeta'\left(\frac{|Tx-Ty|}{d_4}\right)\,\frac{T_p x}{d_4|Tx-Ty|}\,\frac{\partial T_p x}{\partial x_i}\,dy \\ & \stackrel{def}{=} A^1_{ij}(x) + A^2_{ij}(x) + A^3_{ij}(x). \end{aligned} \tag{40}$$

These three terms should be multiplied by $\omega_i(x)\,\omega_j(x)$. We start by considering the first one, i.e.,

$$A^1_{ij}(x)\,\omega_i(x)\,\omega_j(x).$$

Observe that, due to the formula that defines the function T (recall (33))

$$\frac{\partial T_r y}{\partial y_s} = \delta_{rs} + \sigma_s(y)\delta_{3r}\,, \qquad r, s = 1, 2, 3,$$

where $\sigma_s(y) = \sigma_s(y_1, y_2) = -\frac{\partial F(y_1,y_2)}{\partial y_s}$ is independent of y_3. Hence, we write $\sigma(y')$, where $y' = (y_1, y_2)$. Note that $\sigma_3(y_1, y_2) = 0$ and also that $\sigma(y') = o(|y'|)$. Hence, by choosing a possibly smaller $d > 0$, we may suppose that $|\sigma(y)| \le 1$. In addition, note that

$$\sigma_i(x_0) = \sigma_i(x_0{}') = \sigma_i(0, 0) = 0, \qquad i = 1, 2, 3. \tag{41}$$

In order to make the calculations clearer, we distinguish between terms coming from the diagonal of the matrix $\frac{\partial T_r y}{\partial y_s}$ (we call them "non-σ-terms"), which are independent of σ, from those deriving from the off-diagonal part (we call them "σ-terms"), which depend on $\sigma(y)$. Neglecting the σ-terms and due to the properties of the Ricci tensor ϵ_{jkl} (for convenience, in this case, we write $\simeq$ instead of $=$) we have

$$A^1_{ij}(x)\,\omega_i(x)\,\omega_j(x) \simeq -\frac{1}{4\pi}\epsilon_{jkl}\,\delta_{ik}\,\omega_i(x)\,\omega_j(x)\int_{\omega^1}\frac{\omega_l(y)}{|Tx-Ty|^3}\;\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right)\,dy \equiv 0.$$

The σ-terms will be treated later on, after having considered the $A^2_{ij}(x)\,\omega_i(x)\,\omega_j(x)$ terms.

As above, we start again with the non-σ-terms (the treatment is now similar to the corresponding terms in the whole space case and this is the reason for the order in which we consider the various terms.) The non-σ-term of $A^2_{ij}(x)$-type is given by

$$A^2_{ij}(x)\,\omega_i(x)\,\omega_j(x) \simeq$$
$$\simeq -\frac{3}{4\pi}\int_{\omega^1} \epsilon_{jkl}\,\omega_i(x)\,\omega_j(x)\,\omega_k(y)\,\frac{(T_ix-T_iy)(T_kx-T_ky)}{|Tx-Ty|^5}\;\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right)\,dy.$$

By appealing to the properties of the Ricci tensor, we can rewrite the latter term as follows

$$A^2_{ij}(x)\,\omega_i(x)\,\omega_j(x) \simeq$$
$$\simeq -\frac{3}{4\pi}\int_{\omega^1} (\widehat{Tx-Ty}\cdot\omega(x))\;\mathrm{Det}\big(\widehat{Tx-Ty},\omega(y),\omega(x)\big)\;\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right)\frac{dy}{|Tx-Ty|^3}. \tag{42}$$

This shows, when $x=x_0$, that the absolute value of rhs in (42) is bounded by the first integral in the rhs of (39). To this end, recall (28), observe that $|\widehat{Tx-Ty}|=1$, and also that the non-negative quantities $\chi_1(.)$ and $\zeta(.)$ are bounded by 1.

We now come back to the σ-terms of $A^1_{ij}(x)\,\omega_i(x)\,\omega_j(x)$. These σ-terms involve, in principle, three type of terms which come from the product $\frac{\partial T_ry}{\partial y_s}\frac{\partial T_px}{\partial x_i}$. However, since all quantities must be evaluated at $x=x_0$, and also by recalling (41), we are left simply with the single term

$$\sigma_s(y_1,y_2)\delta_{3r}\,\delta_{pi}.$$

In coordinate notation, the σ-term of $A^1_{ij}(x_0)\,\omega_i(x_0)\,\omega_j(x_0)$ is given by

$$-\frac{1}{4\pi}\omega_i(x_0)\,\omega_j(x_0)\int_{\omega^1}\epsilon_{jkl}\omega_l(y)\frac{\delta_{pm}\delta_{3m}\delta_{pi}}{|Tx_0-Ty|^3}\sigma_k(y')\;\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\,dy.$$

Hence $p=m=i=3$. It follows that the above σ-term is given by:

$$\frac{1}{4\pi}\omega_3(x_0)\int_{\omega^1}\epsilon_{jkl}\omega_j(x_0)\,\sigma_k(y')\,\omega_l(y)\;\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\frac{dy}{|Tx_0-Ty|^3},$$
$$=\frac{1}{4\pi}\omega_3(x_0)\int_{\omega^1}\mathrm{Det}\big(\omega(x_0),\omega(y),\sigma(y')\big)\;\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\frac{dy}{|Tx_0-Ty|^3}.$$

Consequently, we can bound the last term by the second integral in the rhs of (39).

Next we consider the σ-terms that appear in the expression of $A^2_{ij}(x_0)\,\omega_i(x_0)\,\omega_j(x_0)$. We note that, due to the fact that $\sigma(x_0)=0$, we are left only with the following term

$$-\frac{1}{4\pi}\omega_i(x_0)\,\omega_j(x_0)\epsilon_{jkl}\int_{\omega^1}\omega_l(y)\,\frac{(T_px_0-T_py)(T_mx_0-T_my)}{|Tx_0-Ty|^5}\,\delta_{3m}\delta_{pi}\sigma_k(y')\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\,dy.$$

Since $p=i$ and $m=3$, the above expression becomes

$$=-\frac{1}{4\pi}\int_{\omega^1}((\widehat{Tx_0-Ty})\cdot\omega(x_0))\,\epsilon_{jkl}\,\omega_j(x_0)\,\sigma_k(y')\,\omega_l(y)\,\frac{T_3x_0-T_3y}{|Tx_0-Ty|^4}\;\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\,dy,$$

which, in turn, is equal to

$$= -\frac{1}{4\pi}\int_{\omega^1} ((\widehat{Tx_0 - Ty})\cdot\omega(x_0))\,\mathrm{Det}\big(\omega(x_0),\omega(y),\sigma(y')\big)\;\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\frac{(T_3x_0-T_3y)\,dy}{|Tx_0-Ty|^4}. \tag{43}$$

Hence this term is still bounded by the second term in Eq. (39).

Finally we consider the $A^3_{ij}(x)$ term. The contribution of this term is easier to handle, since the function $\zeta'(s)$ is identically zero if its argument s is, in absolute value, smaller than $1/4$. This implies that in the integral that defines $A^3_{ij}(x)$ the potentially singular contribution coming from points y such that $|Tx_0 - Ty|$ vanishes, is cut off. This shows that $|A^3_{ij}(x)|$ can be bounded in terms of d_4 for all x. (Recall also that the derivatives $\frac{\partial T_p}{\partial x_i}$ are uniformly bounded, see (28).) Then, we have the following estimate:

$$\begin{aligned} |A^3_{ij}(x_0)\,\omega_i(x_0)\,\omega_j(x_0)| &\le C(d_4)\,|\omega_i(x_0)|\,|\omega_j(x_0)|\int_{\omega^1}|\omega(y)|\,dy \\ &\le C(d_4)\,|\omega(x_0)|^2\int_{\Omega}|\omega(y)|\,dy \le C\|\omega\|_2|\omega(x_0)|^2, \end{aligned} \tag{44}$$

where $C(d_4)$ is a bounded function, depending only on the bounded domain Ω. Note also that $d_4 > 0$ is a fixed number, see (35).

The proof of Lemma 3.4 is now accomplished. □

We treat now the $b_j(x)$ terms, that involve the "reflected" quantities.

Lemma 3.5. *Assume that x_0 satisfies* (32), *and recall the definition* (38) *for the functions $b_j(x)$, for $j = 1, 2, 3$. Then, there exists a positive constant c, independent of x_0, such that*

$$\begin{aligned} \left|\frac{\partial b_j(x_0)}{\partial x_i}\,\omega_j(x_0)\,\omega_j(x_0)\right| \le c\,|\omega(x_0)|^2\Bigg(&\|\omega\|_2 + \\ +\int_{\omega_1}\Big[\big|\mathrm{Det}\big(\widehat{\overline{\omega}}(x_0),\widehat{\omega}(y),\widehat{Tx_0-\overline{Ty}}\big)\big| &+ \big|\mathrm{Det}\big(\widehat{\overline{\omega}}(x_0),\widehat{\omega}(y),\sigma(y')\big)\big|\Big]\frac{|\omega(y)|\,dy}{|Tx_0-\overline{Ty}|^3}\Bigg). \end{aligned} \tag{45}$$

Proof. By following the notation of the previous lemma we write

$$\frac{\partial b_j(x)}{\partial x_i} \overset{def}{=} B^1_{ij}(x) + B^2_{ij}(x) + B^3_{ij}(x),$$

where each term is obtained from the corresponding term of $A^k_{ij}(x)$, see (40), by changing its sign and by replacing $Tx - Ty$ everywhere by $Tx - \overline{Ty}$ (except in the argument of the cut-off function ζ and in ζ'.) Hence, the non-σ-terms of $B_{ij}(x)$-type satisfy

$$B^1_{ij}(x)\,\omega_i(x)\,\omega_j(x) \simeq \epsilon_{jkl}\epsilon_j\epsilon_k\,\omega_j(x)\,\omega_k(x)\int_{\omega^1}\omega_l(y)\,\frac{1}{|Tx-\overline{Ty}|^3}\;\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right)dy \equiv 0,$$

as follows from the properties of the Ricci tensor, together with that of ϵ_j.

On the other hand, at $x = x_0$, the σ-term $B^1_{ij}(x)$ is given by

$$\begin{aligned}
&\frac{1}{4\pi}\omega_i(x_0)\,\omega_j(x_0)\int_{\omega^1}\epsilon_{jkl}\,\epsilon_j\,\epsilon_m\,\omega_l(y)\frac{\delta_{pm}\delta_{3m}\delta_{pi}}{|Tx_0-\overline{Ty}|^3}\sigma_k(y')\,\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)dy\\
&=-\frac{1}{4\pi}\omega_3(x_0)\int_{\omega^1}\mathrm{Det}\big(\overline{\omega}(x_0),\omega(y),\sigma(y')\big)\,\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\frac{dy}{|Tx_0-\overline{Ty}|^3}.
\end{aligned}$$

This term will be estimated below.

The $B^2_{ij}(x)$-term is treated by adapting the previous calculations made to estimate the $A^2_{ij}(x)$-term. In particular, we must take into account the action of the ϵ_m term. One has

$$\begin{aligned}
&B^2_{ij}(x_0)\,\omega_i(x_0)\,\omega_j(x_0)=\\
&=\frac{3}{4\pi}\int_{\omega^1}((\widehat{Tx_0-\overline{Ty}})\cdot\omega(x_0))\,\mathrm{Det}\big(\widehat{\overline{Tx_0-\overline{Ty}}},\omega(y),\overline{\omega}(x_0)\big)\;\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\frac{dy}{|Tx_0-\overline{Ty}|^3}\\
&-\frac{1}{4\pi}\int_{\omega^1}((\widehat{Tx_0-\overline{Ty}})\cdot\omega(x_0))\,\mathrm{Det}\big(\overline{\omega}(x_0),\omega(y),\sigma_k(y')\big)\,\chi_1(y)\,\zeta\left(\frac{|Tx_0-Ty|}{d_4}\right)\frac{(T_3x_0-\overline{T_3y})\,dy}{|Tx_0-\overline{Ty}|^4},
\end{aligned}$$

where the second term on the rhs corresponds to the σ-terms: see the equations (42) and (43). By appealing to the inequality

$$|Tx_0-\overline{Ty}|\geq|Tx_0-Ty|, \tag{46}$$

we prove that both $|B^1_{ij}(x_0)\,\omega_i(x_0)\,\omega_j(x_0)|$ and $|B^2_{ij}(x_0)\,\omega_i(x_0)\,\omega_j(x_0)|$ are bounded by the rhs of (45). Note that $|\overline{\omega}(x_0)|=|\omega(x_0)|$.

Finally, by using again (46), and by recalling the remarks already made for the A^3_{ij} terms, one easily shows that

$$|B^3_{ij}(x_0)\,\omega_i(x_0)\,\omega_j(x_0)|\leq C\,\|\omega\|_2|\omega(x_0)|^2.$$

The proof of Proposition 3.2, for points near the boundary, is accomplished by appealing to the estimates proved in this section. □

Remark 3.6. For the reader's convenience, we summarize the main steps done until now. By appealing to (34) we have shown that

$$(\omega(x)\cdot\nabla)\,u(x)\,\omega(x)=\omega_i(x)\left(\frac{\partial J_1(x)}{\partial x_i}+\frac{\partial J_2(x)}{\partial x_i}+\frac{\partial J_3(x)}{\partial x_i}\right)\omega_j(x).$$

The $J_1(x)$ term (see (36) and (37)), which is the main term, gives rise to the following equality

$$(\omega(x)\cdot\nabla)\,u(x)\,\omega(x)=\left(\frac{\partial a_j(x)}{\partial x_i}+\frac{\partial b_j(x)}{\partial x_i}+\text{“lower order terms”}\right)\omega_i(x)\,\omega_j(x).$$

By using Lemmas 3.4-3.5 we ended the proof of Proposition 3.2. As shown above (with the aid of (29)) the terms $J_2(x)$ and $J_3(x)$ give rise to “lower order terms” and, for convenience, they are treated in the first part of the appendix.

3.2.2 Proof of Proposition 3.2 for points "far from the boundary."

In the case of points x_0 that are not "near the boundary,"

$$x_0 \notin \Omega_{2d/3}, \tag{47}$$

we do not need to appeal to the change of coordinates T. We define sets

$$\omega^1 \stackrel{def}{=} \left\{ x \in \Omega : \ d(x, x_0) < \frac{4d}{9} \right\},$$

and

$$\omega^2 \stackrel{def}{=} \left\{ x \in \Omega : \ d(x, x_0) \geq \frac{2d}{9} \right\}.$$

Note that $d(x_0, \omega^2) \geq 2d/9 > 0$. Further, we define functions $a_j(x)$, $j = 1, 2, 3$, as done in Eq. (38), where now $\frac{\partial T_m y}{\partial y_k}$ is replaced by δ_{mk}. Note that for points far from the boundary there are no b_j terms.

Lemma 3.7. *Assume that x_0 satisfies* (47). *Then, there exists a positive constant c, independent of x_0, such that*

$$\left| \frac{\partial a_j(x_0)}{\partial x_i} \, \omega_j(x_0) \, \omega_j(x_0) \right| \leq c \, |\omega(x_0)|^2 \left(\|\omega\|_2 + \int_\Omega \left| \mathrm{Det}\big(\widehat{\omega}(x_0), \widehat{\omega}(y), \widehat{x_0 - y}\big) \right| \frac{|\omega(y)| \, dy}{|x-y|^3} \right).$$

Proof. The proof is a simplification of that of Lemma 3.4 and we just present a sketch of it.

Now the leading order term of the Green's matrix is that occurring in the whole space case and the calculations are very similar to those in [15]. The representation formula for the solution of system (19) is now (for x near x_0)

$$\begin{aligned} u(x) = & \int_{\omega^1} \chi_1(y) \, Z(x,y) \, [\mathrm{curl}\, \omega(y)] \, \zeta\left(\frac{|x-y|}{d_4} \right) dy \\ & + \int_{\omega^2} \chi_2(y) \, \mathcal{G}_2(x,y) \, [\mathrm{curl}\, \omega(y)] \, dy + \int_\Omega g(x,y) \, [\mathrm{curl}\, \omega(y)] \, dy. \end{aligned} \tag{48}$$

Remark 3.8. The functions $\mathcal{G}_2(x,y)$ and $g(x,y)$ are not those in Eq. (34). However, we use the same symbols since they have the same main properties of the corresponding functions in (34).

We come back to Eq. (36) and we make the same calculations starting with (48). As usual taking the derivative with respect to x_j and multiplying by $\omega_i(x) \, \omega_j(x)$ we define terms that correspond to the $a_j(x)$ in (38). Essentially we have only the terms "without reflections," the main difference is that in this (simpler) case we have not σ-terms, since no rectifications in required at interior points. Hence the estimates are proved in the same way. □

Remark 3.9. In all the expression appearing in the statement (and in the derivation) of Proposition 3.2 we need to have the vorticity-direction $\widehat{\omega}$ well-defined. In all computations of Section 3 we are implicitly assuming that ω is always non-vanishing. To be rigorous one has to fix a positive constant K, and to decompose the vorticity as $\omega = \omega_1 + \omega_2$, where

$$\omega_1(x) = \begin{cases} \omega(x), & \text{if} \quad |\omega(x)| \leq K, \\ 0, & \text{if} \quad |\omega(x)| > K, \end{cases}$$

while $\omega_2(x) = \omega(x) - \omega_1(x)$. Then, the vortex-stretching term can be split into the sum of eight terms

$$\big([\omega_1 + \omega_2] \cdot \nabla\big)[u_1 + u_2] \cdot [\omega_1 + \omega_2],$$

with obvious notation. Most of the resulting terms are not difficult to handle since they involve the bounded part ω_1 of the vorticity. Only the $(2, 2, 2)$ term needs the use of Hypothesis (4) in order to be estimated as in Proposition 3.2. For this term the quantity $\widehat{\omega}$ is well-defined, and all calculations are completely justified. Full details how to implement this essential technical part can be found in [15] and in Section 4 of [8]. It is straightforward to apply the same ideas to the present context.

3.3 Using the hypothesis on the vorticity direction.

We now use the Hypothesis (4), in order to control the various terms that derive from our representation of the vortex-stretching term. We prove the following result.

Proposition 3.10. *There exists a non-negative function $\mathcal{S} : \Omega \to \mathbb{R}$ belonging to $L^3(\Omega)$ such that, for each $x_0 \in \Omega$,*

$$\begin{aligned}\left|\frac{\partial a_j(x_0)}{\partial x_i}\,\omega_i(x_0)\,\omega_j(x_0)\right| &\leq C|\omega(x_0)|^2\Big(\|\omega\|_2 + \mathcal{S}(x_0)\Big),\\ \left|\frac{\partial b_j(x_0)}{\partial x_i}\,\omega_i(x_0)\,\omega_j(x_0)\right| &\leq C|\omega(x_0)|^2\Big(\|\omega\|_2 + \mathcal{S}(x_0)\Big).\end{aligned} \tag{49}$$

Moreover,

$$\|\mathcal{S}\,\|_3 \leq C\|\omega\,\|_2.$$

The above constants $C = C(\Omega)$ are independent of x_0.

Proof. Let us consider the rhs of Eq. (39). By using (4) we obtain

$$\big|\mathrm{Det}\big(\widehat{\omega}(x), \widehat{\omega}(y), \widehat{Tx - Ty}\big)\big| \leq \sin\theta(x, y, t) \leq C\,|x - y|^{1/2}\,, \qquad \forall\, x, y \in \Omega,$$

almost everywhere for $t \in [0, T]$. Hence, by recalling (28), we show that

$$|\omega(x_0)|^2 \int_{\omega_1} \big|\mathrm{Det}\big(\widehat{\omega}(x_0), \widehat{\omega}(y), \widehat{Tx_0 - Ty}\big)\big|\,dy \leq c\,|\omega(x_0)|^2 \int_{\omega^1} \frac{|\omega(y)|}{|x_0 - y|^{5/2}}\,dy.$$

Set

$$\mathcal{S}(x) \stackrel{def}{=} \int_\Omega \frac{|\omega(y)|}{|x - y|^{5/2}}\,dy.$$

Note that by Hardy-Littlewood-Sobolev inequality, see e.g. [29], it follows that $\mathcal{S} \in L^3(\Omega)$ since $\omega \in L^2(\Omega)$ for almost every $t \in (0, T)$.

The last term in the rhs of Eq. (39) is treated in a similar way, by recalling that σ is bounded. The first equation (49) is proved.

We pass now to the reflected $b_i(x)$ terms. We want to estimate the rhs of Eq. (45). The relevant point is to prove that

$$\int_{\omega_1} \big|\mathrm{Det}\big(\widehat{\overline{\omega}}(x_0), \widehat{\omega}(y), \widehat{\overline{Tx_0 - \overline{Ty}}}\big)\big|\frac{|\omega(y)|}{|Tx_0 - \overline{Ty}|^3}\,dy \leq C\int_{\omega^1} \frac{|\omega(y)|}{|x_0 - y|^{5/2}}\,dy, \tag{50}$$

with C independent of x_0. The second term in the rhs of Eq. (45) can be treated as the above one.

In order to prove this inequality, the obstacle arises from the fact that we have to compare the direction of the vorticity at the point y with that at the point x_0, *after reflection.*

We use now the fact that the exterior unit normal vector at $\mathbb{P}x_0$ satisfies $n(\mathbb{P}x_0) = -e_3$ and also that

$$\widehat{\omega}(\mathbb{P}x_0) = e_3 \quad \text{or} \quad \widehat{\omega}(\mathbb{P}x_0) = -e_3\,,$$

since $\mathbb{P}x_0$ belongs to the vertical line passing through x_0. First we observe that $\sin\angle\big(\widehat{\omega}(x_0), \pm e_3\big) = \sin\angle\big(\widehat{\overline{\omega}}(x_0), \pm e_3\big)$, due to the fact that reflection on the boundary (as in the half-space case) changes the sign of the third component (that in e_3-direction). Consequently, the sinus of the angle between the reflected vector and the direction e_3 is that of the angle identified by the vorticity without reflection.

Next, by using the Hypothesis (4) it follows that

$$\sin\angle\big(\widehat{\overline{\omega}}(x_0), \pm e_3\big) \leq c|x_0 - (\mathbb{P}x_0)|^{1/2} = c([x_0]_3)^{1/2},$$

where $[x_0]_3$ denotes the third component of x_0.

Now we identify the angle between unit vectors with the length of a geodesic connecting them on a spherical unit surface. In this way we see that

$$\angle\big(\widehat{\overline{\omega}}(x_0), \widehat{\omega}(y)\big) \leq \angle\big(\widehat{\overline{\omega}}(x_0), e_3\big) + \angle\big(e_3, n(\Pi y)\big) + \angle\big(n(\Pi y), \widehat{\omega}(y)\big),$$

where $\Pi y \in \partial\Omega$ is the point of the boundary obtained by projecting y on $\partial\Omega$, along the direction of $x_0 - \mathbb{P}x_0 = e_3$, see Figure **??**.

First, note that (roughly speaking) the angle between the direction of the normal unit vectors $n(\mathbb{P}x_0)$ and $n(\Pi y)$ is small, if $|x_0 - y|$ is small. In fact, the magnitude of the angle is determined by the curvature of the boundary, which is uniformly bounded. Hence, the angle $\angle(n(\mathbb{P}x_0), n(\Pi y))$ is (at least) bounded by $c\,|\mathbb{P}x_0 - \Pi y|$. Then, since $\mathbb{P}x_0$ is the origin, and since the first two components of x_0 vanish, it follows that

$$\begin{aligned}
|\mathbb{P}x_0 - \Pi y|^2 &= \sum_{i=1}^{3}([\mathbb{P}x_0]_i - [\Pi y]_i)^2 = \sum_{i=1}^{3}([\Pi y]_i)^2 \\
&\leq y_1^2 + y_2^2 + c(d)(y_1^2 + y_2^2) \leq y_1^2 + y_2^2 + c(d)(y_1^2 + y_2^2) + ([x_0]_3 - y_3)^2 \\
&\leq c_1(d)\sum_{i=1}^{3}([x_0]_i - [y]_i)^2 = c_1(d)|x_0 - y|^2.
\end{aligned}$$

Recall (again by the regularity of the boundary) that $|[\Pi y]_3| \leq c(d)\sqrt{y_1^2 + y_2^2}$. Finally

$$\sin\angle\big(n(\mathbb{P}x_0), n(\Pi y)\big) \leq c\,|x_0 - y|.$$

Then, Hypothesis (4), together with the above remarks on the distance between y and Πy, imply that

$$\sin\angle\big(\widehat{\overline{\omega}}(x_0), \widehat{\omega}(y)\big) \leq c\Big([x_0]_3^{1/2} + |x_0 - y| + \Big|2y_3^2 + c(d)(y_1^2 + y_2^2)\Big|^{1/4}\Big).$$

By using the calculus inequalities $(1+a^2)^{1/4} \leq 1+\sqrt{|a|}$ and $|a+|b|\,| \leq |a+b|+|a-b|$, we can increase the rhs of the last expression as follows:

$$\Big([x_0]_3^{1/2} + |x_0-y| + \Big|2y_3^2 + c(d)(y_1^2+y_2^2)\Big|^{1/4}\Big) \leq c\Big([x_0]_3^{1/2} + |y_3|^{1/2} + |x_0-y| + \Big|(y_1^2+y_2^2)\Big|^{1/4}\Big)$$
$$\leq c\Big(|x_0-y|^{1/2} + |x_0-\overline{y}|^{1/2} + |x_0-y|\Big).$$

Now we observe that

$$|x_0-\overline{y}| \leq 2|Tx_0 - T\overline{y}| \leq 2|Tx_0 - \overline{Ty}| + 2|\overline{Ty} - T\overline{y}|.$$

In addition, by using the explicit expression (33) for the transformation T, we get

$$|\overline{Ty} - T\overline{y}| \leq 2|F(y')| = 2|F(y') - F(x_0)| \leq c(d)\,|x_0-y|.$$

Then, since x_0 and y belong to ω^1, their distance is bounded by the "small" number $d>0$. Hence, the term $|x_0-y|$ can be absorbed into $|x_0-y|^{1/2}$, by increasing the constants c.

Finally, by collecting all the previous inequalities, we get, for $x_0,\, y \in \omega^1$:

$$\sin\angle\big(\widehat{\overline{\omega}}(x_0), \widehat{\omega}(y)\big) \leq c\,(|x_0-\overline{y}|^{1/2} + |x_0-y|^{1/2}) \leq c\big(|Tx_0 - T\overline{y}|^{1/2} + |x_0-y|^{1/2}\big).$$

By using (46) and (28) it readily follows that

$$\frac{|Tx_0-\overline{Ty}|^{1/2} + |x_0-y|^{1/2}}{|Tx_0-\overline{Ty}|^3} \leq \frac{|Tx_0-\overline{Ty}|^{1/2}}{|Tx_0-\overline{Ty}|^3} + \frac{|x_0-y|^{1/2}}{|Tx_0-Ty|^3} \leq \frac{c}{|x_0-y|^{3-1/2}}.$$

We have finally proved that:

$$\big|B_{ij}^2(x_0)\,\omega_i(x_0)\,\omega_j(x_0)\big| \leq c\,|\omega(x_0)|^2 \int_{\omega^1} \frac{|\omega(y)|}{|x_0-y|^{3-1/2}}\,dy,$$

and this ends the proof of the Proposition 3.10. □

4 Proof of the main result.

We have now at disposal all the results needed to give the proof Theorem 1.6. By using the results of the previous section, we deduce the following result.

Proposition 4.1. *Let us assume that Hypothesis* (4) *holds and that u is a strong solution in $[0,T[$. Then, to each $\varepsilon>0$ there corresponds a positive $C_\varepsilon>0$ such that the following inequality holds:*

$$\left|\int_\Omega (\omega(x)\cdot\nabla u(x))\cdot\omega(x)\,dx\right| \leq \varepsilon\|\nabla\omega\|_2^2 + C_\varepsilon(\|\omega\|_2^4 + \|\omega\|_2^3), \quad a.e.\ t\in[0,T[. \tag{51}$$

Proof. The above inequality follows easily from the uniform bounds previously proved, together with Hölder's inequality. Actually, one shows that

$$\left|\int_\Omega (\omega(x_0)\cdot\nabla u(x_0))\cdot\omega(x_0)\,dx_0\right| \leq C\int_\Omega |\omega(x_0)|^2\big[1+\mathcal{S}(x_0)+\|\omega\|_2\big]\,dx_0$$
$$\leq C\,\|\,|\omega|^2\,\|_{3/2}\,\big\|1+\|\omega\|_2+\mathcal{S}\,\big\|_3$$
$$\leq C\|\omega\|_3^2\,\|\omega\|_2$$
$$\leq C\|\omega\|_2^2(\|\omega\|_2+\|\nabla\omega\|_2)\,,$$

where the last inequality is obtained by using convex interpolation and the Sobolev inequality $\|f\|_6 \leq C(\|f\|_2 + \|\nabla f\|_2)$. Finally, an application of Young's inequality ends the proof. $\square$

The proof of the main result is now a simple consequence of Proposition 4.1.

Proof of Theorem 1.6. Let us suppose -*per absurdum*- that the weak solution u is strong in $[0, T_1[$, for some $T_1 < T$ and that u cannot be continued as a smooth solution beyond T_1. By scalar multiplication of both sides of (16) followed by integration in Ω (recall also $(23)_2$), and by appealing to Proposition 4.1, the following differential inequality holds:

$$\frac{d}{dt}\|\omega(t)\|_2^2 + \|\nabla\omega\|_2^2 \leq C\Big(1 + \|\omega(t)\|_2 + \|\omega(t)\|_2^2\Big)\,\|\omega(t)\|_2^2, \qquad \text{a.e. } t \in [0, T_1[.$$

Consequently, Gronwall's lemma implies that

$$\limsup_{t\to T_1^-} \|\omega(t)\|_2 < +\infty.$$

Hence, by Lemma 2.7, $\|\nabla u(t)\|_2$ is uniformly bounded in $[0, T_1]$, i.e., u is a strong solution in $[0, T_1]$. By standard arguments one proves that the solution u is regular in $[0, T_1 + \epsilon[$, for some positive ϵ, contradicting the maximality of T_1.

$\square$

Appendices

As announced in Section 3 we report here the (simple) calculations that lead to the estimates of some of the "secondary terms."

A.1

We start by considering the term $J_1^2(x)$, defined in Eq. (36), whose explicit expression is

$$-\frac{1}{4\pi}\epsilon_{jkl}\int_{\omega^1} \omega_l(y)\,G(Tx,Ty)\,\left[\partial_k\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right) + \chi_1(y)\,\zeta'\left(\frac{|Tx-Ty|}{d_4}\right)\frac{T_py}{d_4|Tx-Ty|}\frac{\partial T_py}{\partial y_k}\right]\,dy.$$

Next we differentiate with respect to x_j and multiply by $\omega_i(x)\,\omega_j(x)$. For convenience we split $\partial_j J_1^2(x)$ as follows:

$$\int_{\omega^1}\epsilon_{jkl}\,\omega_l(y)\,\partial_{x_j}G(Tx,Ty)\,\left[\partial_k\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right) + \chi_1(y)\,\zeta'\left(\frac{|Tx-Ty|}{d_4}\right)\frac{T_py}{d_4|Tx-Ty|}\frac{\partial T_py}{\partial y_k}\right]\,dy +$$
$$+\int_{\omega^1}\epsilon_{jkl}\,\omega_l(y)\,G(Tx,Ty)\,\left[\partial_k\chi_1(y)\,\partial_{x_j}\zeta\left(\frac{|Tx-Ty|}{d_4}\right) + \chi_1(y)\partial_{x_j}\left[\zeta'\left(\frac{|Tx-Ty|}{d_4}\right)\right]\frac{T_py}{d_4|Tx-Ty|}\frac{\partial T_py}{\partial y_k}\right]\,dy.$$

Note that the first and the second derivatives of the cut-off function ζ vanish if the argument is small enough (recall "for instance" the estimate (44) of $A_{ij}^3(x_0)$). Hence, we have to consider just the term

$$\frac{1}{4\pi}\int_{\omega^1}\epsilon_{jkl}\,\omega_l(y)\,\partial_{x_j}G(Tx,Ty)\,\partial_k\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right)\,dy =$$
$$= \frac{1}{4\pi}\int_{\omega^1}\epsilon_{jkl}\,\omega_l(y)\left[\frac{T_mx-T_my}{|Tx-Ty|^3} - \epsilon_j\epsilon_m\frac{T_mx-\overline{T_my}}{|Tx-\overline{Ty}|^3}\right]\frac{\partial T_mx}{\partial x_j}\partial_k\chi_1(y)\,\zeta\left(\frac{|Tx-Ty|}{d_4}\right)\,dy,$$

which is bounded by

$$C \int_{\omega^1} \frac{|\omega(y)|}{|Tx - Ty|^2}\, dy.$$

A.2

Now we consider the terms present in formulas (34)-(48), which are not treated in the previous sections. The function $g(x,y)$ includes only terms that are not of leading order, and it satisfies (30). In particular, the contribution of the lower order term $\nabla_x J_3(x)$ (where $J_3(x)$ is defined in Eq. (34)) can be bounded as follows:

$$\begin{aligned}\left|\nabla_x \int_\Omega g(x,y)\,\mathrm{curl}\,\omega(y)\,dy\right| &\le c\int_\Omega \frac{1}{|x-y|^{2-\gamma}}|\nabla\times\omega(y)|\,dy\\ \le c\left(\int_\Omega \frac{dy}{|x-y|^{4-2\gamma}}\right)^{1/2} &\|\nabla\times\omega\|_{L^2(\Omega)}.\end{aligned}$$

The last integral defines a function of x that is uniformly bounded (due to the fact that Ω is a bounded domain with compact closure), provided that $\gamma > 1/2$. Consequently,

$$\begin{aligned}\left|\int_\Omega (\omega(x)\cdot\nabla_x)\left[\int_\Omega g(x,y)\,\mathrm{curl}\,\omega(y)\,dy\right]\omega(x)\,dx\right| &\le c\|\nabla\omega\|_{L^2(\Omega)}\|\omega\|^2_{L^2(\Omega)}\\ &\le \varepsilon\|\nabla\omega\|^2_{L^2(\Omega)} + C_\varepsilon\|\omega\|^4_{L^2(\Omega)},\end{aligned}$$

with $\varepsilon > 0$ arbitrarily small.

To end up, we now consider the term

$$\int_{\omega^2} \chi_2(y)\,\mathcal{G}_2(x_0,y)\,[\mathrm{curl}\,\omega(y)]\,dy.$$

We observe that this term appears in both (34)-(48). In both these equations the function $\mathcal{G}_2(x,y)$ satisfies

$$\exists\, C = C(d) > 0: \quad |\mathcal{G}_2(x,y)_{|x=x_0}|,\ |\nabla_x\mathcal{G}_2(x,y)_{|x=x_0}| \le C, \qquad \forall\, y\in\omega^2,$$

since x_0 and ω^2 are far "enough" from each other (recall Fig. **??**). Finally, just in the way used to prove the above results, we show that there exists $C > 0$, independent of x_0, such that

$$\left|\big(\omega(x_0)\cdot\nabla\int_{\omega^2}\chi_2(y)\,\mathcal{G}_2(x,y)\,\mathrm{curl}\,\omega(y)\,dy\big)\cdot\omega(x_0)\right| \le C|\omega(x_0)|^2\int_\Omega |\mathrm{curl}\,\omega(y)|\,dy, \qquad \forall\, x_0\in\Omega.$$

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References

[1] C. Bègue, C. Conca, F. Murat, and O. Pironneau, *Les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression. (French),* Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IX (Paris, 1985–1986), 179–264, Pitman Res. Notes Math. Ser., 181, Longman Sci. Tech., Harlow, 1988.

[2] H. Beirão da Veiga, *A new regularity class for the Navier-Stokes equations in* $\mathbf{R}^n$, Chinese Ann. Math. Ser. B **16** (1995), no. 4, 407–412.

[3] H. Beirão da Veiga, *Regularity for Stokes and generalized Stokes systems under non homogeneous slip type boundary conditions*, Adv. Diff. Eq., **9**, (2004), no. 9-10, 1079-1114.

[4] H. Beirão da Veiga, *Regularity of solutions to a nonhomogeneous boundary value problem for general Stokes systems in* $\mathbb{R}^n_+$, Math. Annalen, **331**, (2005), 203-217.

[5] H. Beirão da Veiga, *Vorticity and regularity for flows under the Navier boundary condition*, Comm. Pure Appl. Anal. **5** (2006), 907–918.

[6] H. Beirão da Veiga, *Vorticity and regularity for viscous incompressible flows under the Dirichlet boundary condition. Results and related open problems*, J. Math. Fluid Mech. **9** (2007),506-516.

[7] H. Beirão da Veiga, *Vorticity and smoothness in incompressible viscous flows. Boundary value problems*, Proceedings of the Conference "Kyoto Conference on the Navier-Stokes Equations and their Applications," RIMS Kokyuroku Bessatsu, Vol. B1, pag. 71–77. Y. Giga, H. Kozono, H. Okamoto, and Y. Shibata, editors. Kyoto, 2007.

[8] H. Beirão da Veiga and L.C. Berselli, *On the regularizing effect of the vorticity direction in incompressible viscous flows*, Differential Integral Equations **15** (2002), 345–356.

[9] L. C. Berselli, *On a regularity criterion for the solutions to the 3D Navier-Stokes equations,* Differential Integral Equations **15** (2002), 1129–1137.

[10] L. C. Berselli, *Some geometric constraints and the problem of global regularity for the Navier-Stokes equations,* to appear.

[11] T. Clopeau, A. Mikelić, and R. Robert, *On the vanishing viscosity limit for the* 2D *incompressible Navier-Stokes equations with the friction type boundary conditions*, Nonlinearity **11** (1998), 1625–1636.

[12] C. Conca, *On the application of the homogenization theory to a class of problems arising in fluid mechanics.* J. Math. Pures Appl. (9) **64** (1985), 31–75.

[13] C. Conca, F. Murat, and O. Pironneau, *The Stokes and Navier-Stokes equations with boundary conditions involving the pressure,* Japan. J. Math. (N.S.) **20** (1994), 279–318.

[14] C. Conca, C. Parés, O. Pironneau, and M. Thiriet, *Navier-Stokes equations with imposed pressure and velocity fluxes,* Internat. J. Numer. Methods Fluids **20** (1995), 267–287.

[15] P. Constantin and C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math. J. **42** (1993), 775–789.

[16] P. Constantin, C. Fefferman, and A.J. Majda, *Geometric constraints on potentially singular solutions for the* 3*-D Euler equations.* Comm. Partial Differential Equations **21** (1996), 559–571.

[17] P. Constantin and C. Foiaş, *Navier-Stokes Equations*, University of Chicago Press, Chicago, IL, 1988.

[18] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 2, Interscience Publishers, New York, 1962.

[19] C. Foiaş and R. Temam, *Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation. (French),* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 28–63.

[20] V. Girault, *Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in* $\mathbf{R}^3$, Math. Comp. **51** (1988), 55–74.

[21] E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213–231.

[22] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.

[23] P. Lévy, *Sur l'allure des fonctiones de Green et de Neumann dans le voisinage du contour*, Acta Math. **642** (1920), 207–267.

[24] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, 1969.

[25] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson et Cie, Éditeurs, Paris, 1967.

[26] J. Serrin, *Mathematical principles of classical fluid mechanics, Handbuch der Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I (Mitherausgeber C. Truesdell)*, pp. 125–263, Springer-Verlag, Berlin, 1959.

[27] V.A. Solonnikov, *On Green's Matrices for Elliptic Boundary Problem I*, Trudy Mat. Inst. Steklov **110** (1970), 123-170.

[28] V.A. Solonnikov, *On Green's Matrices for Elliptic Boundary Problem II*, Trudy Mat. Inst. Steklov **116** (1971), 187-226. V.A. Solonnikov and V.E. Ščadilov, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Proc. Steklov Inst. Math.,**125** (1973), 186-199.

[29] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N.J., 1970, Princeton Mathematical Series, No. 30.

[30] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Second Edition.* Springer-Verlag, New York, 1997, Applied Mathematical Sciences, 68.

[31] R. Temam, *Navier-Stokes equations. Theory and numerical analysis. Reprint of the 1984 edition.* AMS Chelsea Publishing, Providence, RI, 2001.

[32] R. Temam, and M. Ziane, *Navier-Stokes equations in three-dimensional thin domains with various boundary conditions,* Adv. Differential Equations **1** (1996), 499–546.