J. math. fluid mech. 11 (2009) 258–273
1422-6928/09/020258-16

DOI 10.1007/s00021-008-0258-1

Journal of Mathematical Fluid Mechanics

Navier–Stokes Equations with Shear Thinning Viscosity. Regularity up to the Boundary

H. Beirão da Veiga

Abstract. In this paper we consider a class of stationary Navier–Stokes equations with shear dependent viscosity, in the shear thinning case $p < 2$, under a non-slip boundary condition. We are interested in global (i.e., *up to the boundary*) regularity results, in dimension $n = 3$, for the second order derivatives of the velocity and the first order derivatives of the pressure. As far as we know, there are no previous global regularity results for the second order derivatives of the solution to the above boundary value problem.

We consider a cubic domain and impose the non-slip boundary condition only on two opposite faces. On the other faces we assume periodicity, as a device to avoid effective boundary conditions. This choice is made so that we work in a bounded domain Ω and simultaneously with a flat boundary. The extension to non-flat boundaries is done in the forthcoming paper [7], by following ideas introduced by the author, for the case $p > 2$, in reference [5]. The results also hold in the presence of the classical convective term, provided that p is sufficiently close to the value 2.

Mathematics Subject Classification (2000). 35Q30, 35K35, 76D03, 35K55.

Keywords. Navier–Stokes equations, shear dependent viscosity, regularity up to the boundary.

1. Introduction and results

In the sequel u and π denote, respectively, the velocity and the pressure of a viscous incompressible fluid. We are mainly interested in studying and improving regularity results for solutions to the Navier–Stokes equations for flows with shear dependent viscosity, namely

$$\begin{cases} -\nabla \cdot T(u,\pi) + (u \cdot \nabla)u = f, \\ \qquad\qquad\qquad\quad \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

under suitable boundary conditions, where T denotes the Cauchy stress tensor

$$T = -\pi\, I + \nu_T(u)\mathcal{D}\, u \tag{1.2}$$

and $\mathcal{D}\, u$ denotes the symmetric gradient, i.e.,

$$\mathcal{D}\, u = \frac{1}{2}\,(\nabla\, u + \nabla\, u^T).$$

In order to fix ideas we consider the specific case

$$\nu_T(u) = (\mu + |\mathcal{D}\, u|)^{p-2}\,, \tag{1.3}$$

where $\mu > 0$ is a given constant, under the non-slip boundary condition

$$u_{|\Gamma} = 0. \tag{1.4}$$

It is worth noting that vorticity is essentially created at the boundary of the physical domain, and that this boundary is nearly always present in realistic problems. The cases $p > 2$ and $p < 2$ capture shear thickening and shear thinning phenomena, respectively. Concerning Non-Newtonian fluids, we refer the reader to [21] and [28]. In the following we consider the case $1 < p \leq 2$. For the case $p > 2$ regularity results up to the boundary are proved by us in references [3] and [4] for flat boundaries. For generic regular bounded open sets see [5] and [24]. For an improvement of the results in [4] see [6].

Below, we concentrate on the system (1.1) without the convective term; hence, on the system

$$\begin{cases} -\nabla\cdot\big((\mu + |\mathcal{D}\, u|)^{p-2}\,\mathcal{D}\, u\big) + \nabla\,\pi = f, \\ \qquad\qquad\qquad\qquad\qquad\quad \nabla\cdot u = 0. \end{cases} \tag{1.5}$$

Note that (1.5) satisfies the Stokes Principle, see [33]. For an illuminating explanation of this principle, we refer the reader to the classical work [32], p. 231, where the Stokes Principle is stated in a postulational form.

It is not difficult to show that the regularity results stated in Theorem 1.1 below still hold in the presence of the convective term $(u\cdot\nabla)u$ provided that $p > p_0$, for a value p_0 sufficiently close to the value 2. See Theorem 1.5 below and Remark 1.1 in reference [4].

In order to work with a flat boundary Γ, we are led to consider a cubic domain Ω and to impose the boundary condition (1.4) only on two of the opposite faces. On the other pair of faces we assume periodicity conditions (in this way we avoid artificial singularities due to the corner points). Alternatively, we could work in the half-space. However, in this case, the lack of the inclusion $L^q \subset L^p$, if $q > p$, leads to secondary technicalities concerning the functional framework, as shown in reference [3].

The same cubic domain and boundary condition are considered, for the case $p > 2$, in reference [4]. As in this last reference, the above simplification enables us to emphasize here the very basic ideas of our method. We remark that there is a strong parallel between the present paper (case $p < 2$) and [4] (case $p > 2$). On the other hand, in reference [5] we have extended the results proved in [4] to arbitrary regular open sets. In a similar way, the results proved below can be extended to arbitrary regular open sets $\Omega \subset \mathbb{R}^3$, as shown in the forthcoming paper [7].

When $p \neq 2$, there is an unusual increment in difficulty in passing from interior to boundary regularity for solutions to the system (1.5). A reflection of this fact is the lower regularity obtained for the second order derivatives of the velocity (and for the first order derivatives of the pressure) in the normal direction in comparison to the other directions. One of the main reasons is that in proving interior regularity by appealing to the classical differential quotients method, translations are admissible in all the n independent directions. This allows suitable estimates for $\nabla \mathcal{D} u$. Note that here the full gradient ∇ is obtained thanks to the possibility of appealing to translations in all the directions. Furthermore, it is easily shown that

$$c_0 |\nabla^2 u| \leq |\nabla \mathcal{D} u| \leq c_1 |\nabla^2 u|. \tag{1.6}$$

These two facts together lead to a not particularly distinct situation if we replace $\mathcal{D}u$ by ∇u in equation (1.5). However, in proving regularity up to the boundary, the two cases are completely distinct.

Concerning local regularity, or existence and local regularity for boundary value problems (including the space-periodic case) under the assumptions $p < 2$ and $n \geq 3$, we refer the reader to [1], [2], [**?**] [11], [12], [15], [17], [18], [20], [25], [29] and references therein. References [30] and [31] concern the study of electrorheological fluids, and reference [16] an Euler scheme for Newtonian fluids. It is worth noting that for $p < 2$, the presence of a $-\Delta u$ term on the left-hand side of (1.5) leads to additional regularity results. It is not difficult to show that $u \in W^{2,2}(\Omega)$. Moreover, under extremely general assumptions, see [9], we have shown that $u \in W^{1,q}(\Omega)$, for each finite q.

In the sequel we prove the following results. For notation see the next section, in particular (2.12) and (2.13). The constants c are independent of μ but may depend on p.

Theorem 1.1. *Assume that $f \in L^{p'}(\Omega)$ and let $u \in V_p$ be a solution to the problem (1.5), (1.4), where $\mu > 0$ and $1 < p < 2$. Then $D^2_* u$ and $\nabla_* \pi$ belong to $L^p(\Omega)$. Moreover,*

$$\|D^2_* u\|_p \leq c \, \|f\|_{p'}^{\frac{1}{p-1}} + c \, \| \, \mu + |\mathcal{D}u| \, \|_p \tag{1.7}$$

and

$$\|\nabla_* \pi\|_p \leq c \, \mu^{p-2} \big(\|f\|_{p'}^{\frac{1}{p-1}} + \| \, \mu + |\mathcal{D}u| \, \|_p \big) + c \, \|f\|_p \, . \tag{1.8}$$

One has the following (conditional) result.

Theorem 1.2. *Assume, in addition to the hypothesis of Theorem 1.1, that $p > \frac{3}{2}$ and that*

$$\mathcal{D} u \in L^q(\Omega), \tag{1.9}$$

for some q satisfying

$$p \leq q \leq 6.$$

Then

$$u \in W^{2,r}(\Omega), \qquad \nabla \pi \in L^r(\Omega), \tag{1.10}$$

and

$$\|u\|_{2,\,r} + \|\nabla\,\pi\|_r \le c(\mu) B\big(\|\mu + |\mathcal{D}\,u|\,\|_q^{2-p} + 1\big), \tag{1.11}$$

where

$$r = \frac{p\,q}{p(2-p)+q} \tag{1.12}$$

and

$$B = \|\mu + |\mathcal{D}u|\|_p + \|f\|_p + \|f\|_{p'}^{\frac{1}{p-1}}\,. \tag{1.13}$$

Since (1.9) holds for $q = p$, one has the following result.

Theorem 1.3. *Assume, in addition to the hypotheses in Theorem 1.1, that* $p > \frac{3}{2}$. *Then*

$$u \in W^{2,\,\frac{p}{3-p}}(\Omega), \qquad \nabla\,\pi \in L^{\frac{p}{3-p}}(\Omega), \tag{1.14}$$

and

$$\|u\|_{2,\,\frac{p}{3-p}} + \|\nabla\,\pi\|_{\frac{p}{3-p}} \le C(\mu)\big(1 + \|f\|_{p'}^{\frac{3-p}{p-1}}\big). \tag{1.15}$$

Theorem 1.2 allows a bootstrap argument, similar to that in references [3], [4], that leads to the following result.

Theorem 1.4. *Under the assumptions of Theorem 1.1, and* $p > \frac{3}{2}$, *u belongs to* $W^{1,\,s}(\Omega)$, *moreover*

$$\|u\|_{1,\,s} \le c(\mu)\big(B + B^{\frac{1}{p-1}}\big), \tag{1.16}$$

where s is given by

$$s = \frac{3\,p(p-1)}{3-p}\,. \tag{1.17}$$

Furthermore,

$$u \in W^{2,\,l}(\Omega), \qquad \nabla\,\pi \in L^l(\Omega), \tag{1.18}$$

and

$$\|u\|_{2,\,l} + \|\nabla\,\pi\|_l \le C(\mu)\big(B + B^{\frac{1}{p-1}}\big), \tag{1.19}$$

where

$$l = \frac{3p(p-1)}{p^2 - 2\,p + 3}\,. \tag{1.20}$$

Note that $l = 2$ if $p = 2$ (as expected), and that $s > p$. Moreover, $s = l^*$.
Also note that

$$\|u\|_{2,\,l} + \|\nabla\,\pi\|_l \le C(p,\,\mu,\,\|\nabla\,u\|_p)(1 + \|f\|_{p'})^{\frac{1}{p-1}}. \tag{1.21}$$

The above results may be easily applied to consider the Navier–Stokes case. See, for instance, the method followed in references [3] and [4]. This leads to the following result.

Theorem 1.5. *Modify equation* (1.5) *by adding to the left-hand side the convective term* $(u \cdot \nabla)u$ *and let* u *be a solution to this modified equation under the boundary condition* (1.4). *There is a value* $p_0 < 2$ *such that the above regularity results hold (with modified estimates for the corresponding norms), provided that* $p > p_0$. *Actually*

$$p_0 = \frac{15}{8}\,.$$

For a discussion on this point see the last section.

Added in proof. In the meantime, by following the main lines established here, the results were improved in the two subsequent papers [10] and [8]. Actually, two new ideas allow interesting improvements. In reference [10] the author improves the results by appealing to anisotropic embedding theorems of Sobolev type, a fruitful idea in the general context of regularity up to the boundary for p-fluid flows. Further, in reference [8], we use this last idea together with a new device that overcomes the need of results like that stated in the Lemma 3.2 below, typical in treating the shear thinning case, see [17]. This leads to an improvement of the results established in references [10] and [8]. Furthermore, in reference [9] (as above, see equation (1.21)) we obtain very accurate estimates in terms of f.

2. Notation, weak solutions and some auxiliary results

In the sequel Ω denotes the 3-dimensional cube $\Omega = (\,]0,1[\,)^3$.

Further, we set

$$\Gamma_- = \{x : |x_1|,\, |x_2| < 1\,, x_3 = 0\,\}\,, \quad \Gamma_+ = \{x : |x_1|,\, |x_2| < 1\,, x_3 = 1\,\}\,.$$

The Dirichlet boundary condition will be imposed only on

$$\Gamma = \Gamma_- \cup \Gamma_+.$$

The problem will be assumed periodic, with period equal to 1, in both the x_1 and the x_2 directions. In the following the significant boundary is Γ. Actually $\Gamma = \partial\,\Omega$ provided that Ω and Γ are indefinitely reflected in the x_1 and x_2 directions. Sometimes we use the term "boundary" to denote Γ. For convenience we set

$$x' = (x_1, x_2).$$

By x'-periodic we mean periodic of period 1 in both x_1 and x_2. Further, we set $\partial_i\, f = \frac{\partial\, f}{\partial\, x_i}$, $\partial^2_{i\,j}\, f = \frac{\partial^2\, f}{\partial\, x_i\, \partial\, x_j}$.

We use the same notation for functional spaces and norms for both scalar and vector fields. The symbol $\|\cdot\|_p$ denotes the canonical norm in $L^p(\Omega)$ and $\|\cdot\|$, that in $L^2(\Omega)$. $W^{1,p}(\Omega)$ denotes the usual Sobolev space.

We set

$$V_p = \big\{ v \in W^{1,p}(\Omega) : (\nabla \cdot v)_{|\Omega} = 0\,;\; v_{|\Gamma} = 0\,;\, v \text{ is } \; x' - \text{periodic} \big\}\,. \tag{2.1}$$

Note that, by appealing to inequalities of Korn's type, one gets the following result.

Lemma 2.1. *There is a positive constant c such that the estimate*

$$\|\nabla v\|_p + \|v\|_p \leq c\,\|\mathcal{D}v\|_p \tag{2.2}$$

holds for each $v \in V_p$. Hence the two quantities above are equivalent norms in V_p.

For the proof see, for instance, [27], Proposition 1.1.

Lemma 2.1 is one of the cornerstones of our proof, in the absence of (1.6).

Definition 2.1. Assume that

$$f \in (V_p)'. \tag{2.3}$$

We say that u is a *weak solution* to problem (1.5), (1.4) if $u \in V_p$ satisfies

$$\frac{1}{2}\int_\Omega \nu_T(u)\mathcal{D}u \cdot \mathcal{D}v\,dx = \int_\Omega f \cdot v\,dx \tag{2.4}$$

for all $v \in V_p$.

The typical proofs of the existence of weak solutions appeal to techniques coming from the minimization of convex functionals, see [20] and [2], or from the related theory of monotone operators. From this last point of view, basic ideas are described in [23]. See Theorems 2.1 and 2.2, Chap. 2, Sect. 2, in this last reference. For a more general situation we also refer to the clear treatment in references [18] and [29].

By replacing v by u in equation (2.4) one gets

$$\frac{1}{2}\int_\Omega (\mu + |\mathcal{D}u|)^{p-2}\,|\mathcal{D}u|^2\,dx = \langle f, u\rangle\,, \tag{2.5}$$

where the symbols $\langle\cdot,\cdot\rangle$ denote a duality pairing. It readily follows that

$$2^{p-3}\int_{|\mathcal{D}\,u|\geq\mu} |\mathcal{D}u|^p\,dx \leq \langle f,\,u\rangle \leq \|f\|_{-1,p'}\,\|u\|_{1,\,p}\,, \tag{2.6}$$

where, in general, we denote by q' the dual exponent of q, namely

$$q' = \frac{q}{q-1}\,. \tag{2.7}$$

Consequently,

$$\|\mathcal{D}u\|_p^p \leq 2^{p-3}\,\|f\|_{-1,p'}\,\|u\|_{1,\,p} + |\Omega|\,\mu^p.$$

Finally, by appealing to (2.2), it follows that

$$\|\nabla u\|_p^{p-1} \leq c(\|f\|_{-1,p'} + \mu^{p-1}). \tag{2.8}$$

By restriction of (2.4) to divergence-free test-functions v with compact support in Ω, and by De Rham's theorem, there follows the existence of a distribution π (determined up to a constant) such that

$$\nabla\,\pi = -\nabla\cdot\big[(\mu + |\mathcal{D}u|)^{p-2}\,\mathcal{D}u\,\big] + f. \tag{2.9}$$

Equation (2.9) shows that the first equation (1.5) holds in the distributions sense.

The following result is well known.

Lemma 2.2. *If a distribution g is such that $\nabla g \in W^{-1,\alpha}(\Omega)$ then $g \in L^\alpha(\Omega)$ and*

$$\|g\|_{L^\alpha_\#} \leq c\, \|\nabla g\|_{W^{-1,\alpha}}\,, \tag{2.10}$$

where $L^\alpha_\# = L^\alpha/\mathbb{R}$.

From (2.9) and (2.8), together with the above lemma, it readily follows that

$$\|\pi\|_{L^{p'}_\#} \leq c\big(\|f\|_{-1,p'} + \mu^{p-1}\big). \tag{2.11}$$

We end this section by introducing some more notation.

We denote by $D^2 u$ the set of all the second derivatives of u. The meaning of expressions like $\|D^2 u\|$ is clear. The symbol $D^2_* u$ denotes any of the second order derivatives $\partial^2_{i\,k}\, u_j$ except for the derivatives $\partial^2_{3\,3}\, u_j$, if $j = 1$ or $j = 2$. Moreover,

$$|D^2_*\, u|^2 := |\partial^2_{3\,3}\, u_3|^2 + \sum_{\substack{i,j,k=1\\ (i,k)\neq(3,3)}}^{3} |\partial^2_{i\,k}\, u_j|^2\,. \tag{2.12}$$

Similarly, ∇_* may denote any first order partial derivative, except for $\partial/\partial x_3$. In particular,

$$|\nabla_*\, \pi|^2 := \sum_{j=1}^{2} |\partial_j\, \pi|^2\,. \tag{2.13}$$

Some integrability exponents play a crucial role in our proofs and are, for the reader's convenience, introduced here.

In the sequel p denotes an exponent that lies in the interval

$$1 < p \leq 2\,. \tag{2.14}$$

In general, for $1 < r < 3$ we define the Sobolev embedding exponent r^* by the equation

$$\frac{1}{r^*} = \frac{1}{r} - \frac{1}{3}\,. \tag{2.15}$$

We denote by c, c_0, c_1, etc. generic positive constants that may change from equation to equation. Constants of this type are independent of the positive parameter μ (assumed bounded from above).

3. Regularity of the tangential derivatives

In this section we prove the estimates (1.7) and (1.8).

In the sequel, in order to avoid arguments already developed in similar contexts (see, for instance, [3] or [4]) we replace the use of the translation method in the tangential directions by differentiation in these same directions.

In the sequel $s = 1, 2$. Hence x_s denotes the two tangential directions to Γ. Consequently translations in these two directions are admissible in the usual sense.

In order to fix ideas we state some well known results in the context of the particular case considered here. We define the tensor S as

$$S = (\mu + |D|)^{p-2}\, D\,, \tag{3.1}$$

where D is an arbitrary tensor. One has

$$\frac{\partial S_{i\,j}}{\partial D_{k\,l}}\, C_{i\,j}\, C_{k\,l} \geq (p-1)(\mu + |D|)^{p-2}\, |C|^2\,, \tag{3.2}$$

for all tensors C. Summation on repeated indexes is assumed except for the index s below.

Define, for $s = 1,\, 2$,

$$J_s(u) =: \int_\Omega \nabla \cdot \left[(\mu + |\mathcal{D}|)^{p-2}\, \mathcal{D}\right] \cdot \partial^2_{s\,s}\, u\, dx. \tag{3.3}$$

For convenience, here and in the sequel, we set

$$\mathcal{D} = \mathcal{D}\, u\,.$$

Remark. Under periodic boundary conditions there are no distinction between the coordinates x_s, $s = 1, 2, 3$. Hence, in the right-hand side of (3.3), the single tangential derivatives $\partial^2_{s\,s} u$, $s = 1, 2$, may be simply replaced by $\Delta\, u$. See [17].

By two integrations by parts, and by taking into account that $\mathcal{D}u$ is symmetric one shows, after some manipulations, that

$$J_s(u) = \int_\Omega \partial_s\, \left[(\mu + |\mathcal{D}|)^{p-2}\, \mathcal{D}\right] : \partial_s\, \mathcal{D}\, dx. \tag{3.4}$$

Consequently,

$$J_s(u) = \int_\Omega \frac{\partial}{\partial D_{k\,l}} \left[(\mu + |D|)^{p-2}\, D_{i\,j}\right] (\partial_s\, \mathcal{D}_{k\,l})(\partial_s\, \mathcal{D}_{i\,j}) dx, \tag{3.5}$$

where derivatives with respect to $D_{k\,l}$ are evaluated at the point $D = \mathcal{D}$. Hence, by (3.2), the following result follows.

Lemma 3.1. *Let be $s = 1, 2$. Then*

$$J_s(u) \geq (p-1)\, I_s(u), \tag{3.6}$$

where

$$I_s(u) = \int_\Omega (\mu + |\mathcal{D}|)^{p-2}\, |\partial_s\, \mathcal{D}\, u|^2\, dx. \tag{3.7}$$

Next multiply both sides of the first equation (1.5) by $\partial^2_{s\,s}\, u$ and integrate over Ω. By appealing to (3.3) and(3.6), it readily follows that

$$I_s(u) \leq \frac{1}{p-1}\, \|f\|_{p'}\, \|\partial^2_{s\,s}\, u\,\|_p\,, \tag{3.8}$$

for $s = 1,\, 2$. It is worth noting that the single derivatives that appear in the expression $\partial_s\, \mathcal{D}\, u$ can not be point wisely estimated by that in $\partial_s\, \mathcal{D}\, u$. This was the obstacle which requires (see references [3] and [4]) the addition of a $-\Delta\, u$ term to the left-hand side of the main equation (1.5), in order to estimate all the $D_*\, u$ derivatives; recall the impossibility of appealing to (1.6), when considering boundary value problems. A new, simple but crucial idea, is realizing that (2.2) applies with $v = \partial_s\, u$, if $s = 1, 2$. This device yields the following estimate:

$$\|\partial_s\, \nabla\, u\|_p + \|\partial_s u\|_p \leq c\, \|\partial_s\, \mathcal{D}\, u\|_p \leq c\, \|\nabla_*\, \mathcal{D}\, u\|_p\,. \tag{3.9}$$

Hence,

$$I_s(u) \leq \frac{1}{p-1}\, \|f\|_{p'}\, \|\partial_s\, \mathcal{D}\, u\|_p\,, \tag{3.10}$$

for $s = 1,\, 2$.

In particular, by appealing to the the constraint $\nabla \cdot u = 0$, the estimate (3.10) allows us to extend the estimates proved from $\nabla_*\, \mathcal{D}\, u$ to $\nabla_*\, \nabla\, u$. More precisely, one has the following result.

Lemma 3.2. *Let be* $s = 1,\, 2$. *Then*

$$\|D^2_*\, u\|_p \leq c(\|\nabla_*\, \nabla\, u\|_p + \|\nabla_* u\|_p) \leq c\, \|\nabla_*\, \mathcal{D}\, u\|_p\,. \tag{3.11}$$

Proof of equation (1.7). Let $a \geq 0$ and $b > 0$ be two reals and let $0\, \leq q \leq r$. Then

$$a^q \leq b^{q-r}\, a^r + b^q.$$

In fact, if $a \leq b$ then $a^q \leq b^q$. On the other hand, if $b \leq a$, then $a^q = a^r\, a^{q-r} \leq a^r\, b^{q-r}$.

The clever idea of appealing to the above very simple inequality in order to estimate the L^p norm below is borrowed from Lemma 6 of Diening and Růžička (see [17]).

We get, from the above inequality,

$$\|\partial_s\, \mathcal{D}\, u\|^p_p \leq I_s(u) + \|\, \mu + |\mathcal{D}u|\, \|^p_p\,. \tag{3.12}$$

From (3.10) and (3.12) it readily follows that

$$I_s(u) \leq c\, \|f\|^{p'}_{p'} + c\, \|\, \mu + |\mathcal{D}u|\, \|^p_p\,. \tag{3.13}$$

Hence, by appealing to (3.11), we prove (1.7). □

Proof of equation (1.8). Next, by differentiation of the first equation (1.5) with respect to x_s, one gets

$$\nabla\,\partial_s\,\pi = -\nabla\cdot\partial_s\big((\mu+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\big)+\partial_s\,f. \tag{3.14}$$

On the other hand, one easily shows that

$$\begin{aligned}&\partial_s\,\left((\mu+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\right)\\ &\quad=(\mu+|\mathcal{D}\,u|)^{p-2}\,\partial_s\,\mathcal{D}\,u+(p-2)(\mu+|\mathcal{D}\,u|)^{p-3}\,|\mathcal{D}\,u|^{-1}(\mathcal{D}\,u\cdot\partial_s\,\mathcal{D}\,u)\mathcal{D}\,u.\end{aligned} \tag{3.15}$$

Hence,

$$|\partial_s\,\left((\mu+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\right)|\leq(3-p)(\mu+|\mathcal{D}\,u|)^{p-2}\,|\partial_s\,\mathcal{D}\,u|\,, \tag{3.16}$$

almost everywhere in Ω. Hence, by (3.14), and by appealing to Lemma 2.2, we prove that

$$\|\partial_s\,\pi\|_p\leq c\,\|(\mu+|\mathcal{D}\,u|)^{p-2}\,\partial_s\mathcal{D}\,u\,\|_p+c\,\|f\|_p\,, \tag{3.17}$$

for $s\neq 3$. By appealing to (1.7), one proves (1.8). □

4. Normal derivatives of the velocity. The basic linear system

We follow here the pioneering paper [3]. Let us consider, for almost all $x\in\Omega$, the 2×2 matrix $A=A(x)$ with elements $a_{j\,l}$ given by

$$a_{j\,l}=(\mu+|\mathcal{D}\,u|)^{p-2}\,\delta_{j\,l}+2(p-2)(\mu+|\mathcal{D}\,u|)^{p-3}\,|\mathcal{D}\,u|^{-1}\,\mathcal{D}_{l3}\,\mathcal{D}_{j3}\,, \tag{4.1}$$

for $j,l\leq 3$. Note that $a_{j\,l}=a_{l\,j}$. One has the following result (the proof is immediate).

Lemma 4.1. *For almost all $x\in\Omega$ one has*

$$\sum_{j,l=1}^{2}a_{j\,l}\xi_j\xi_l=(\mu+|\mathcal{D}\,u|)^{p-2}\,|\xi|^2-2(2-p)(\mu+|\mathcal{D}\,u|)^{p-3}\,|\mathcal{D}\,u|^{-1}\,\left[(\mathcal{D}\,u)\cdot\xi\right]_3^2\,. \tag{4.2}$$

In particular,

$$\sum_{j,l=1}^{2}a_{j\,l}\xi_j\xi_l\geq 2\left(p-\frac{3}{2}\right)(\mu+|\mathcal{D}\,u|)^{p-2}\,|\xi|^2. \tag{4.3}$$

Hence the following result holds.

Lemma 4.2. *If $p\geq\frac{3}{2}$ the matrix $A(x)$ is positive definite for almost all $x\in\Omega$. More precisely*

$$\det\,A\geq\left[2\left(p-\frac{3}{2}\right)(\mu+|\mathcal{D}\,u|)^{p-2}\right]^2. \tag{4.4}$$

By appealing to (3.15), the j^{th} equation (1.5) may be written in the form

$$\begin{aligned} &- (\mu + |\mathcal{D}\, u|)^{p-2} \sum_{k=1}^{3} \partial^2_{k\,k}\, u_j \\ &- 2(p-2)(\mu + |\mathcal{D}\, u|)^{p-3} |\mathcal{D}\, u|^{-1} \sum_{l,m,k=1}^{3} \mathcal{D}_{lm} \mathcal{D}_{jk}\, \partial^2_{m\,k}\, u_l + 2\partial_j \pi = 2 f_j\,, \end{aligned} \tag{4.5}$$

where $\mathcal{D}_{ij} = (\mathcal{D}\, u)_{ij} = \partial_j\, u_i + \partial_i\, u_j$ and $1 \le j \le 3$. Let us write the first two equations (4.5), $j = 1, 2$, as follows:

$$\begin{aligned} &(\mu + |\mathcal{D}\, u|)^{p-2}\, \partial^2_{3\,3}\, u_j + 2(p-2)(\mu + |\mathcal{D}\, u|)^{p-3}\, |\mathcal{D}\, u|^{-1}\, \mathcal{D}_{j3} \sum_{l=1}^{2} \mathcal{D}_{l3}\, \partial^2_{3\,3}\, u_l \\ &= F_j(x) + 2\, \partial_j \pi - f_j\,, \end{aligned} \tag{4.6}$$

where the $F_j(x)$, $j \neq 3$, are given by

$$\begin{aligned} F_j(x) :&= -(\mu + |\mathcal{D}\, u|)^{p-2} \sum_{k=1}^{2} \partial^2_{k\,k}\, u_j \\ &- 2(p-2)(\mu+|\mathcal{D} u|)^{p-3} |\mathcal{D} u|^{-1} \left\{ \mathcal{D}_{33}\, \mathcal{D}_{j3} \partial^2_{3\,3} u_3 + \sum_{\substack{l,m,k=1 \\ (m,k) \neq (3,3)}}^{3} \mathcal{D}_{lm} \mathcal{D}_{jk} \partial^2_{m\,k} u_l \right\}. \end{aligned} \tag{4.7}$$

In the sequel, equations (4.6), $j = 1, 2$, will be treated as a 2×2 linear system in the unknowns $\partial^2_{3\,3}\, u_j$, $j \neq 3$. Note that, with an obviously simplified notation, the measurable functions F_j satisfy

$$|F_j(x)| \le c(\mu + |\mathcal{D}\, u|)^{p-2}\, |D^2_*\, u(x)|, \tag{4.8}$$

a.e. in Ω.

We denote by $\widetilde{F}_j$ the right-hand sides

$$\widetilde{F}_j(x) := F_j(x) + 2\, \partial_j\, \pi - 2\, f_j\,, \tag{4.9}$$

that appear in the above 2×2 system (4.6).

Let us study the 2×2 system (4.6) in terms of the unknowns $\partial^2_{3\,3}\, u_j$, $j = 1, 2$, for almost all fixed $x \in \Omega$. The elements $a_{j\,l}$ of the matrix system $A = A(x)$ are given by (4.1). In particular Lemma 4.2 applies. By setting $\xi_l = \partial^2_{3\,3}\, u_l$, we get from (4.6), i.e. from

$$\sum_{l=1}^{2} a_{j\,l}\, \xi_l = \widetilde{F}_j\,, \tag{4.10}$$

that

$$\sum_{l,j=1}^{2} a_{j\,l}\,\xi_l\xi_j = \sum_{j=1}^{2} \widetilde{F}_j\,\xi_j\,. \tag{4.11}$$

Consequently

$$2\left(p-\frac{3}{2}\right)(\mu+|\mathcal{D}\,u|)^{p-2}\sum_{l=1}^{2}|\partial^2_{3\,3}\,u_l| \le \left(\sum_{j=1}^{2}|\widetilde{F}_j|^2\right)^{1/2}, \tag{4.12}$$

almost everywhere in Ω. By appealing to (4.8) and (4.9) one shows that

$$\left(p-\frac{3}{2}\right)\sum_{l=1}^{2}|\partial^2_{3\,3}\,u_l| \le c\,\left(|D^2_*\,u(x)| + c(\mu+|\mathcal{D}\,u|)^{2-p}(|\nabla^*\,\pi|+|f|)\right), \tag{4.13}$$

almost everywhere in Ω. Since $(\mu+|\mathcal{D}\,u|)^{2-p} \in L^{\frac{q}{2-p}}$ and $\nabla^*\pi \in L^p$, Hölder's inequality shows that the right-hand side of (4.13) is integrable with power r. More precisely, by appealing to (1.7) and (1.8), and to the inequality $r \le p$, it readily follows that

$$\left(p-\frac{3}{2}\right)\sum_{l=1}^{2}\|\partial^2_{3\,3}\,u_l\|_r \le c\Big(\|f\|_{p'}^{\frac{1}{p-1}} + A + \|\mu+|\mathcal{D}\,u|\,\|_q^{2-p}\big(\|f\|_{p'}^{\frac{1}{p-1}} + A + \|f\|_p\big)\Big), \tag{4.14}$$

where

$$A = \|\mu+|\mathcal{D}u|\|_p\,.$$

Hence,

$$\sum_{l=1}^{2}\|\partial^2_{3\,3}\,u_l\|_r \le c\,B\big(\|\mu+|\mathcal{D}\,u|\,\|_q^{2-p}+1\big), \tag{4.15}$$

where B is given by (1.13). This shows that $\|u\|_{2,\,r}$ satisfies (1.11).

The regularity of $\frac{\partial\pi}{\partial x_3}$, hence the global regularity of $\nabla\pi$, is easily obtained from (4.5). In fact, this equation, written for $j=3$, furnishes an explicit expression for $\frac{\partial\pi}{\partial x_3}$ in terms of functions already estimated. Actually,

$$|\partial_3\,\pi| \le c(\mu+|\mathcal{D}\,u(x)|)^{p-2}\,|D^2\,u(x)| + |f(x)|, \tag{4.16}$$

almost everywhere in Ω. Hence

$$\partial_3\,\pi\ \in L^r(\Omega).$$

Moreover (1.11) holds.

Theorem 1.3 follows immediately by setting $p=q$ in Theorem 1.2. The expression on the right-hand side of (1.15) follows by appealing in particular to (2.8). This same device can be used in the other estimates. Straightforward manipulations show that the the right-hand side of (4.15) is bounded by the left-hand side of (1.15). This yields (1.15).

Finally we prove Theorem 1.4. By Theorem 1.2, and by a well know Sobolev embedding theorem, one has

$$u \in W^{1,q}(\Omega) \Rightarrow u \in W^{1,r^*}(\Omega), \tag{4.17}$$

moreover

$$\|u\|_{1,r^*} \leq c(\mu) B (1 + \|\mathcal{D}\, u\|_q)^{2-p}\,, \tag{4.18}$$

where

$$\frac{1}{r^*} = \frac{1}{r} - \frac{1}{3} = \frac{2-p}{q} + \frac{1}{p} - \frac{1}{3}\,.$$

Set $q_1 = p$ and define q_{n+1}, for $n \geq 1$, by

$$\frac{1}{q_{n+1}} = \frac{2-p}{q_n} + \frac{1}{p} - \frac{1}{3}.$$

The increasing sequence q_n converges to the value s obtained by setting $q_n = q_{n+1} = s$ in the above definition. Actually, s is given by (1.17).

Moreover, from the estimate

$$\|u\|_{1,q_{n+1}} \leq c(\mu) B (1 + \|u\|_{1,q_n})^{2-p}, \tag{4.19}$$

and by taking into account that $0 < 2-p < 1$, it readily follows that $\|u\|_{1,\,q_n}$ is uniformly bounded by the right-hand side of (1.16). Hence (1.16) holds.

Finally, by appealing to Theorem 1.2 with $q = s$, we prove (1.19).

5. The Navier–Stokes equation

Since

$$\int_\Omega (u \cdot \nabla) u \cdot u\, dx = 0,$$

it readily follows that all the estimates stated in section 2 for weak solutions hold for solutions u to the complete Navier–Stokes equations

$$\begin{cases} -\nabla \cdot \big((\mu + |\mathcal{D}\, u|)^{p-2} \mathcal{D}\, u \big) + \nabla\, \pi = F, \\ \hfill \nabla \cdot u = 0, \end{cases} \tag{5.1}$$

where

$$F = f - (u \cdot \nabla) u.$$

In particular, by (2.8),

$$\|u\|_{W^{1,p}} \leq c\big(\|f\|_{p'}^{\frac{1}{p-1}} + \mu \big). \tag{5.2}$$

Moreover, by (1.21), it follows that

$$\|u\|_{W^{2,l}} \leq C (1 + \|\big(u \cdot \nabla\, u\|_{p'}^{\frac{1}{p-1}} \big), \tag{5.3}$$

where the constant C depends on p, μ, $\|\nabla\, u\|_p$ and $\|f\|_{p'}$. It is clear that the first term C on the right-hand side of (5.3) is irrelevant here. Hence, for the reader's convenience, we drop this term. In the same line of simplifications, we replace C in the second term on the right-hand side by 1. Hence we simply write

$$\|u\|_{2,l} \preceq \|(u\cdot\nabla)u\|_{p'}^{\frac{1}{p-1}}\,, \tag{5.4}$$

and left details to the reader. Let us write equation (5.4) in the equivalent form

$$\|u\|_{2,\,l} \preceq \left(\int |u|^{p'}\,|\nabla u|^{(1-\alpha)p'}\,|\nabla u|^{\alpha\,p'}\,dx\right)^{\frac{1}{p}}. \tag{5.5}$$

Next we estimate the above integral by appealing to Hőlder's inequality with exponent q' applied to $|u|^{p'}\,|\nabla u|^{(1-\alpha)p'}$ and exponent q applied to $|\nabla u|^{\alpha\,p'}$. We determine α and q by the equations

$$\alpha\,p' = \frac{s}{q}\,, \qquad \frac{1}{p\,q} = \frac{1}{s}\,.$$

It follows $\alpha = p-1$ and $q = \frac{3(p-1)}{3-p}$. In this way we prove that

$$\|u\|_{2,\,l} \preceq K(u)^{\lambda}\,\|\nabla\, u\|_{s}\,, \tag{5.6}$$

where

$$K(u) = \int |u|^{p'\,q'}\,|\nabla u|^{(1-\alpha)p'\,q'}\,dx \tag{5.7}$$

and $\lambda = \frac{1}{p\,q'}$ (however, the exact value of λ is irrelevant here).

Actually,

$$K(u) = \int |u|^{\frac{3\,p}{2(2p-3)}}\,|\nabla u|^{\frac{3\,p(2-p)}{2(2p-3)}}\,dx. \tag{5.8}$$

By assuming $p > \frac{9}{5}$ and by Hőlder's inequality with exponents $\frac{2(2p-3)}{3-p}$ and $\frac{2(2p-3)}{5p-9}$ one gets

$$K(u) \leq \|u\|_{p^*}^{\frac{3\,p}{2(2p-3)}}\left(\int |\nabla u|^{\beta}\,dx\right)^{\frac{5p-9}{2(2\,p-3)}}, \tag{5.9}$$

where

$$\beta = \frac{2(2-p)}{3-p}\,\frac{p(p-1)}{5p-9}\,.$$

One has $\beta \leq p$ if and only if $p \geq \frac{15}{8}$. Now, the reader easily verifies that if $p > \frac{15}{8}$ we may obtain (5.6) with a bounded $K(u)^{\lambda}$ (similar, but different from the previous one. Nevertheless we use the same symbol) and s replaced by a smaller exponent t. In this way we get an estimate of the form

$$\|u\|_{2,l} \preceq K(u)^{s}\,\|\nabla\, u\|_{t}\,, \tag{5.10}$$

where $t < s$. Consequently there is a γ, $0 < \gamma < 1$, such that $W^{\gamma,\,l}$ is continuously embedded in L^t. Hence, by (5.10),

$$\|u\|_{2,l} \preceq C\,\|u\|_{1+\gamma,\,l}\,. \tag{5.11}$$

By appealing to the compact embedding of $W^{1+\gamma,\,l}$ into $W^{2,\,l}$, one shows that to each positive real ϵ there corresponds a positive C_ϵ such that

$$\|u\|_{1+\gamma,\,l} \preceq C_\epsilon\, \|u\|_{1,p} + \epsilon\, \|u\|_{2,\,l}\,.$$

Consequently,

$$\|u\|_{2,\,l} \preceq C\; C_\epsilon\, \|u\|_{1,p} + C\, \epsilon\, \|u\|_{2,\,l}\,. \tag{5.12}$$

By fixing a sufficiently small ϵ, we obtain the desired a priori estimate for $\|u\|_{W^{2,\,l}}$.

References

[1] D. Apushkinskaya, M. Bildhauer and M. Fuchs, Steady states of anisitropic generalized Newtonian fluids, *J. Math. Fluid Mech.* **7** (2005), 261–297.

[2] M. Bildhauer, *Convex Variational Problems*, Lecture Notes in Mathematics **1818**, Springer-Verlag, Berlin, 2003.

[3] H. Beirão da Veiga, On the regularity of flows with Ladyzhenskaya shear dependent viscosity and slip and non-slip boundary conditions, *Comm. Pure Appl. Math.* **58** (2005), 552–577.

[4] H. Beirão da Veiga, Navier–Stokes equations with shear thikenning viscosity. Regularity up to the boundary, *J. Math. Fluid Mech.* **10** (2008).

[5] H. Beirão da Veiga, On the Ladyzhenskaya–Smagorinsky turbulence model of the Navier–Stokes equations in smooth domains. The regularity problem, *J. Eur. Math. Soc.*, in press.

[6] H. Beirão da Veiga, Turbulence models, p-fluid flows, and $W^{2,\,l}$ regularity of solutions (to appear).

[7] H. Beirão da Veiga, On the regularity of shear thinning flows in smooth domains (to appear).

[8] H. Beirão da Veiga, On non-Newtonian p-fluids. The pseudo-plastic case, *J. Math. Anal. Appl.*, in press.

[9] H.Beirão da Veiga, Generalized shear thinning models, and regularity (to appear).

[10] L. C. Berselli, On the $W^{2,q}$-regularity of incompressible fluids with shear-dependent viscosities: The shear-thinnig case, *J. Math. Fluid Mech.*, in press.

[11] M. Bildhauer and M. Fuchs, Variants of the Stokes problem: The case of anisitropic potentials, *J. Math. Fluid Mech.* **5** (2003), 364–402.

[12] M. Bildhauer, M. Fuchs and X. Zhong, On strong solutions of the differential equations modelling the steady flow of certain incompressible generalized Newtonian fluids, *Algebra i Analiz* **18** no. 2 (2006), 1–23.

[13] F. Crispo, Shear-thining viscous fluids in cylindrical domains. Regularity up to the boundary, *J. Math. Fluid Mech.*, in press.

[14] F. Crispo, Global regularity of a class of p-fluid flows in cylinders, *J. Math. Anal. Appl.*, in press.

[15] L. Diening, A. Prohl and M. Růžička, On time-discretizations for generalized Newtonian fluids, in: Birman et al. (eds.), *Nonlinear Problems in Mathematical Physics and Related Topics, II*, International Mathematical Series, Kluwer Academic/Plenum, New York, 2002.

[16] L. Diening, A. Prohl and M. Růžička, Semi-implicit Euler scheme for generalized Newtonian fluids, *SIAM Journal Numer. Anal.* **44** (2006), 1172–1190.

[17] L. Diening and M. Růžička, Strong solutions for generalized Newtonian fluids, *J. Math. Fluid Mech.* **7** (2005), 413–450.

[18] J. Frehse, J. Málek and M. Steinhauer, An existence result for fluids with shear dependent viscosity-steady flows, *Nonlinear Analysis, T M A* **30** (1997), 3041–3049.

[19] M. Fuchs, Regularity for a class of variational integrals motivated by nonlinear elasticity, *Asymp. Analysis* **9** (1994), 23–38.

[20] M. Fuchs and G. Seregin, *Variational Methods for Problems from Plasticity Theory and for Generalized Newtonian Fluids*, Lecture Notes in Mathematics **1749**, Springer-Verlag, Berlin, 2000.

[21] G. P. Galdi, Mathematical Problems in Classical and Non-Newtonian Fluid Mechanics, in: G. P. Galdi, A. M. Robertson, R. Rannacher, S. Turek, *Hemodynamical Flows: Modeling, Analysis and Simulation*, Oberwolfach Seminars, Birkhäuser-Verlag, 2007.

[22] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, second edition, Gordon and Breach, New York, 1969.

[23] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[24] J. Málek, J. Nečas and M. Růžička, On weak solutions to a class of non-Newtonian incompressilble fluids in bounded three-dimensional domains: the case $p \geq 2$, *Advances in Diff. Equations* **6** (2001), 257–302.

[25] J. Málek, M. Růžička and V. V. Shelukhin, Herschel–Bulkley fluids: Existence and regularity of steady flows, *Math. Models Methods Appl. Sci.* **15** (2005), 1845–1861.

[26] J. Nečas, *Équations aux Dérivées Partielles*, Presses de l'Université de Montréal, Montréal, 1965.

[27] C. Parés, Existence, uniqueness and regularity of solutions of the equations of a turbulence model for incompressible fluids, *Appl. Analysis* **43** (1992), 245–296.

[28] K. R. Rajagopal, *Mechanics of Non-Newtonian Fluids*, in: G.P. Galdi and J. Nečas (eds.), *Recent Developments in Theoretical Fluid Mechanics*, 129–162, Research Notes in Mathematics Series **291**, Longman, 1993.

[29] M. Růžička, A note on steady flow of fluids with shear dependent viscosity, *Nonlinear Analysis. TMA* **30** (1997), 3029–3039.

[30] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics **1748**, Springer-Verlag, Berlin–Heidelberg, 2000.

[31] M. Růžička, Modeling, mathematical and numerical analysis of electrorheological fluids, *Applications of Mathematics* **49** (2004), 565–609.

[32] J. Serrin, *Mathematical Principles of Classical Fluid Mechanics*, in: *Encyclopedia of Physics VIII*, 125–263, Springer-Verlag, Berlin, 1959.

[33] G. Stokes, *Trans. Cambridge Phil. Soc.* **8**, 287 (1845), 75–129.

H. Beirão da Veiga
Department of Applied Mathematics "U. Dini"
University of Pisa
Via Buonarrotti, 1/C
56100 Pisa
Italy
e-mail: bveiga@dma.unipi.it

(accepted: February 8, 2007; published Online First: March 14, 2008)