On the Regularity of Flows with Ladyzhenskaya Shear-Dependent Viscosity and Slip or Nonslip Boundary Conditions

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To the memory of Olga Aleksandrovna Ladyzhenskaya

Abstract

Navier-Stokes equations with shear dependent viscosity under the classical nonslip boundary condition were introduced and studied in the 1960s by O. A. Ladyzhenskaya and, in the case of gradient dependent viscosity, by J.-L. Lions. A particular case is the well-known Smagorinsky turbulence model. This is nowadays a central subject of investigation. On the other hand, boundary conditions of slip type seems to be more realistic in some situations, in particular in numerical applications. They are a main research subject. The existence of weak solutions u to the above problems, with slip- (or nonslip-) type boundary conditions, is well-known in many cases. However, *regularity up to the boundary* still presents many open questions. In what follows we present some regularity results, in the stationary case, for weak solutions to this kind of problem; see Theorem 3.1 and Theorem 3.2. The evolution problem is studied in a forthcoming paper [6]. A cornerstone in our proof is the classical Nirenberg translation method; see [38].

1 Introduction

The Navier-Stokes system of equations with shear dependent viscosity has been studied in the last fifty years by a great number of researchers, not only in pure and applied mathematics, but also in engineering, physics, and biology. A typical model is the well-known *Ladyzhenskaya model*

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \cdot T(u, \pi) = f \\ \nabla \cdot u = 0 \end{cases} \tag{1.1}$$

where T denotes the stress tensor

$$T = -\pi I + \nu_T(u)\mathcal{D}u \,. \tag{1.2}$$

Here,

$$\mathcal{D}u = \nabla u + \nabla u^{\mathsf{T}} \,,$$

$$\nu_T(u) = \nu_0 + \nu_1 |\mathcal{D}u|^{p-2} \,, \tag{1.3}$$

and ν_0 and ν_1 are strictly positive constants. Note that (1.2) satisfies the Stokes principle; see [50] and [45, p. 231], where this physical principle is stated in a postulational form. For $p = n = 3$, the above model is the classical Smagorinsky model, introduced in reference [48] as a simple turbulence model; see [19] and references therein.

It is worth noting that, from the mathematical viewpoint, the crucial characteristic of models like (1.3) is the growth of the *convex* potential $|\mathcal{D}u|^p$ near ∞ (and, to a minor extent, near 0). In this sense, we prefer to show the main points by giving the proofs in the representative case (1.3), rather than risk hiding ideas and methods in a more general setting.

The first mathematical studies on the above kind of equations go back to O. A. Ladyzhenskaya in a series of remarkable contributions; see [22, 23, 24, 25]. Similar results were obtained by J.-L. Lions for models in which $\nabla u + \nabla u^{\mathsf{T}}$ is essentially replaced by ∇u; see [30] and [31, chap. 2, sec. 5]. More precisely,

$$T = -\pi I + \nu(u)\nabla u \tag{1.4}$$

where

$$\nu(u) = \nu_0 + \nu_1 |\nabla u|^{p-2}\,. \tag{1.5}$$

Essential existence, uniqueness, and regularity results for Ladyzhenskaya-type models under the nonslip boundary condition (1.10) can be found in [35] and references therein. The recent literature on this subject seems particularly wide. Hence, without any claim of completeness, we refer, for instance, to [1, 7, 9, 10, 11, 12, 13, 17, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 40, 41, 42, 44] and to the references given by these authors.

It should be emphasized that theoretical contributions (contrary to applied results) mostly concern the homogeneous boundary condition $u = 0$. However, many other boundary conditions are crucial in applications. In particular, the following nonhomogeneous slip-type boundary condition appears to be quite important in many fields:

$$\begin{cases} (u \cdot \underline{n})|_\Gamma = 0 \\ \beta u_\tau + \underline{\tau}(u)|_\Gamma = b(x) \end{cases} \tag{1.6}$$

where $\underline{n}$ is the unit outward normal to the domain's boundary Γ, $\beta \geq 0$ is a given constant, and $b(x)$ is a given tangential vector field. We denote by $\underline{t} = T \cdot \underline{n}$ the normal component of the tensor T, by $u_\tau = u - (u \cdot \underline{n})\underline{n}$ the tangential component of u, and by $\underline{\tau}$ the tangential component of $\underline{t}$,

$$\underline{\tau}(u) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}\,. \tag{1.7}$$

Straightforward calculations show that

$$\underline{\tau}(u) \cdot v = \nu_T(u) \sum_{i,k=1}^{n} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) n_k v_i \tag{1.8}$$

for each v tangential to the boundary. Hence, if $\Omega = \mathbb{R}^n_+$, then

$$\underline{\tau}(u) \cdot v = \nu_T(u) \sum_{j=1}^{n-1} \left(\frac{\partial u_j}{\partial x_n} + \frac{\partial u_n}{\partial x_j} \right) v_j \tag{1.9}$$

on $\Gamma = \mathbb{R}^n_+$, since $v_n = 0$. The first deep mathematical study of this type of boundary condition was done in the pioneering paper [49] by V. A. Solonnikov and V. E. Ščadilov.

Here we also take into account the nonslip boundary condition

$$u|_\Gamma = 0\,. \tag{1.10}$$

As shown in what follows, our proofs immediately apply to this problem by doing suitable simplifications.

For results and applications of boundary conditions like (1.6), see, for instance, [3, 4, 5, 8, 9, 14, 16, 20, 21, 28, 39, 43, 46, 49, 51] and references therein; see also [45, p. 240] for a discussion of this subject.

We are interested here in proving strong regularity, *up to the boundary*, of weak solutions. The really *new obstacles* that one faces arise due to the interaction between the nonlinear terms containing $\nabla u + \nabla u^{\mathsf{T}}$ and the boundary conditions. We concentrate our attention on this new point, by avoiding obstacles and situations that can be tackled by appealing to complex but known techniques. The classical obstacle to proving the regularity of the solutions is the presence of the convection term. However, as for proving regularity results for solutions to the classical Navier-Stokes equations, this term can be treated here as a "right-hand side." Concerning the evolution problem, it seems that the regularity of the derivative $\frac{\partial u}{\partial t}$ is not a substantial obstacle to proving the regularity of the solutions; see [6] and the remark below. Hence, we will concentrate our attention on the following stationary problem in $\mathbb{R}^n_+$:

$$\begin{cases} -\nu_0 \nabla \cdot (\nabla u + \nabla u^{\mathsf{T}}) \\ \quad - \nu_1 \nabla \cdot (|\nabla u + \nabla u^{\mathsf{T}}|^{p-2} (\nabla u + \nabla u^{\mathsf{T}})) + \nabla \pi = f \\ \nabla \cdot u = 0. \end{cases} \tag{1.11}$$

We have also obtained similar but stronger results for solutions to the simplest Lions model

$$\begin{cases} -\nu_0 \Delta u - \nu_1 \nabla \cdot (|\nabla u|^{p-2} \nabla u) + \nabla \pi = f(x) \\ \nabla \cdot u = 0. \end{cases} \tag{1.12}$$

We do not present these results here.

Remark 1.1. In reference [6], by heavily appealing to Theorems 3.1 and 3.2 below, we prove strong regularity results for solutions to the full Navier-Stokes evolution system (1.1) under the boundary conditions (1.6) or (1.10) and given initial data. For regular data, it is known that $u \in L^\infty(0, T; W^{1,p})$ and $\frac{\partial u}{\partial t} \in L^2(0, T; L^2)$ if

$p > 2$ is sufficient large. For the *Stokes system*, i.e., the system (1.1) without the convection term $u \cdot \nabla u$, the result holds for each $p \geq 2$.

In [6] we show, in addition, that

$$u \in L^2(0, T; W^{2,p'}) \tag{1.13}$$

where $p' = p/(p-1)$ and $p \in]\frac{4n}{n+2}, 4[$; in particular, $p \in]2 + \frac{2}{5}, 4[$ when $n = 3$. Moreover, for $n = 3$, we show that

$$u \in L^{4-p}(0, T; W^{2,l}) \tag{1.14}$$

where $l = \frac{3(4-p)}{5-p}$ and $p \in]2 + \frac{2}{5}, 3[$.

These results improve (and extend to slip boundary conditions) some of the fundamental results stated in [35] for solutions to the nonslip boundary condition (1.10) in the case $n = 3$. In this last reference it is proven (see theorem 1.17) that

$$u \in L^{\frac{2}{p-1}}(0, T; W^{2,\frac{6}{p+1}}) \tag{1.15}$$

for each $p \in]2 + \frac{1}{4}, 3[$. It is worth noting that, for $2 < p < 3$, one has $l > 6/(p+1)$ and $4 - p > 2/(p-1)$. On the other hand, for $p = 3$, (1.14) and (1.15) give $u \in L^1(0, T; W^{2,\frac{3}{2}})$, but (1.13) shows that $u \in L^2(0, T; W^{2,\frac{3}{2}})$. Moreover, (1.13) applies to $p > 3$.

By appealing to (1.15), we may show that (1.13) and (1.14) hold as well if $p \in [2 + \frac{1}{4}, 2 + \frac{2}{5}[$, at least for solutions to the boundary value problem (1.10) and $n = 3$.

It is significant that all the exponents that appear in equations (1.13), (1.14), and (1.15) are equal to 2 when $p = 2$. We point out that, for the Stokes system, all the above results hold for each $p \geq 2$.

2 Weak Solutions: Known Results and Notation

A formal integration by parts shows that

$$\frac{1}{2} \int_\Omega \nu_T(u) \mathcal{D}u \cdot \mathcal{D}v \, dx = -\int_\Omega [\nabla \cdot (\nu_T(u) \mathcal{D}u)] \cdot v \, dx + \int_\Gamma \underline{\tau}(u) \cdot v \, d\Gamma \tag{2.1}$$

for each divergence-free vector field v tangential to the boundary. For the time being Ω may be any sufficiently regular open set. It readily follows that (at least formally; see below for the functional framework) u is a solution to problem (1.11) and (1.6) for some π if and only if $u \in V$ satisfies (2.8) for all $v \in V$, where V denotes the set of all divergence-free "regular" vector fields tangential to the boundary. Note that $\underline{t} \cdot v = \underline{\tau}(u) \cdot v$, since the test functions v are tangential to the boundary. In the case of the boundary value problem (1.10), vector fields in V are assumed to vanish on the boundary and, in equation (2.8), the terms with β and b must be dropped.

The existence of π as a distribution follows from well-known results by using divergence-free test functions $v \in C_0^\infty(\Omega)$ in equation (2.8).

The above considerations give rise to the definition of a weak solution described below.

Let us now introduce the functional setting used in the following and, in particular, fix the space V. If X is a Banach space, we denote by X' its strong dual space. We use the same notation for functional spaces and norms for both scalar and vector fields. The symbol $\|\cdot\|_p$ denotes the canonical norm in $L^p(\mathbb{R}^n_+)$, and $\|\cdot\|$ that in $L^2(\mathbb{R}^n_+)$. In general, "integer norms" as well as "integer Sobolev spaces" relate to $\mathbb{R}^n_+$, and "fractional norms" concern the boundary $\Gamma = \mathbb{R}^{n-1}$. For instance, $\|\cdot\|_{1/2} = \|\cdot\|_{1/2,\Gamma}$, and $H^{1/2} = H^{1/2}(\mathbb{R}^{n-1})$.

We define $D^1 := D^{1,2}(\mathbb{R}^n_+)$ as the completion of $C_0^\infty(\overline{\mathbb{R}^n_+})$ (or $C_0^k(\overline{\mathbb{R}^n_+})$, $k \geq 1$) with respect to the norm $\|\nabla v\|$. Moreover, D_0^1 is the completion of $C_0^\infty(\mathbb{R}^n_+)$ with respect to $\|\nabla v\|$. It is well-known (by Sobolev embedding theorems) that

$$D^1 = \{v : v \in L^r,\ \nabla v \in L^2\}, \tag{2.2}$$

where $\frac{1}{r} = \frac{1}{2} - \frac{1}{n}$. In particular, the norms $\|\nabla v\|$ and $\|\nabla v\| + \|v\|_{L^r}$ are equivalent in D^1 and in D_0^1. This can be shown by extending $C_0^k(\overline{\mathbb{R}^n_+})$ to $C_0^k(\mathbb{R}^n)$ by the well-known reflection method and then by applying the corresponding result in the whole space; see [29] and [18, theorems I.2 and I.4 and remark 1 on p. 234]. Though it is not used in what follows, one can show that

$$D^1 = \left\{ v : \nabla v \in L^2,\ \frac{v}{(1+|x|^2)^{1/2}} \in L^2 \right\};$$

see [18, theorem 1.2]. Clearly, the usual Sobolev spaces H_0^1 and H^1 are dense and strictly contained in D_0^1 and D^1, respectively. In particular, it follows that $L^{r'} \hookrightarrow (D^1)' \hookrightarrow (H^1)'$ and $L^{r'} \hookrightarrow (D_0^1)' \hookrightarrow H^{-1}$, where $r' = r/(r-1)$.

Since the restriction to a bounded set B of any function in D^1 belongs to the Sobolev space $H^1(B)$, it follows that its trace on the boundary $\mathbb{R}^{n-1}$ is (locally) well-defined as an element of $H^{1/2}$. Obviously, functions in D_0^1 have vanishing trace on $\mathbb{R}^{n-1}$. Trace spaces in $\mathbb{R}^{n-1}$ may be studied, in a convenient way, by resorting to the Fourier transform. The trace space of D^1 is denoted here by $D^{1/2} = D^{1/2}(\mathbb{R}^{n-1})$. Actually, it is the completion of $C_0^\infty(\mathbb{R}^{n-1})$ with respect to the norm induced in $\mathbb{R}^{n-1}$ by the norm $\|\nabla v\|$ in $C_0^\infty(\overline{\mathbb{R}^n_+})$. It consists of functions (distributions) ψ that have a "half derivative" in $L^2(\mathbb{R}^{n-1})$ (in the usual Fourier transform sense) and that, actually, belong to $L^s(\mathbb{R}^{n-1})$, where s is given by the Sobolev embedding exponent

$$\frac{1}{s} = \frac{1}{2} - \frac{1/2}{n-1}; \tag{2.3}$$

see [18, theorem II.3 and def. II.1] and [15] and references.

We set $D^{-1/2} = (D^{1/2})'$. Norms in $D^{1/2}$ and $D^{-1/2}$ are denoted, respectively, by $[\,\cdot\,]_{1/2}$ and $[\,\cdot\,]_{-1/2}$. Note that, by (2.3), one has $L^{s'} \hookrightarrow D^{-1/2}$ where $s' = 2(n-1)/n$.

It is worth noting that our main interest here is the local regularity up to the boundary. This leads us to avoid more complex functional frameworks, which have been introduced to deal with the behavior at infinity. For this kind of approach, see (without any claim of completeness) [2], [15, chap. II], [18, 47], and bibliography.

We define

$$D^1_\tau = \{v \in D^1 : v_n = 0 \text{ on } \Gamma\} \quad \text{and} \quad D^1_0 = \{v \in D^1 : v = 0 \text{ on } \Gamma\}\,.$$

V_2 denotes the space

$$V_2 = \{v \in D^1_\tau : \nabla \cdot v = 0 \text{ in } \mathbb{R}^n_+\} \tag{2.4}$$

if the boundary value problem under consideration is (1.6) and

$$V_2 = \left\{v \in D^1_0 : \nabla \cdot v = 0 \text{ in } \mathbb{R}^n_+\right\} \tag{2.5}$$

if the boundary value problem under consideration is (1.10). The above subspaces of D^1 are endowed with the norm $\|\nabla u\|$. Moreover, $[\,\cdot\,]_{-1}$ denotes the strong norm in the dual space $(V_2)'$.

We set

$$V = \{v \in V_2 : \|\mathcal{D}v\|_p < \infty\}$$

endowed with the norm

$$\|v\|_V = \|\nabla v\|_2 + \|\mathcal{D}v\|_p\,.$$

It should be remarked that, by appealing to inequalities of Korn's type, we can verify that $V = \{v \in V_2 : \|\nabla v\|_p < \infty\}$ and also that $\|\nabla v\|_2 + \|\mathcal{D}v\|_p$ and $\|\nabla v\|_2 + \|\nabla v\|_p$ are equivalent norms in V. However, this device is not necessary here and will not be used.

Convention. It is understood, once and for all, that when dealing with the boundary condition (1.10), all the terms containing β or b should be dropped from the equations.

Weak solutions exist under the assumptions

$$f \in (V_2)' \tag{2.6}$$

and, concerning the tangential vector field b,

$$b \in D^{-\frac{1}{2}}(\mathbb{R}^{n-1})\,. \tag{2.7}$$

Note that (2.6) holds if $f \in L^{r'}$, and (2.7) holds if $b \in L^{s'}(\mathbb{R}^{n-1})$.

DEFINITION 2.1 We say that u is a *weak solution* to problem (1.11) and (1.6) if $u \in V$ satisfies

$$\frac{1}{2}\int_\Omega \nu_T(u)\mathcal{D}u \cdot \mathcal{D}v\,dx + \beta\int_\Gamma u \cdot v\,d\Gamma = \int_\Omega f \cdot v\,dx + \int_\Gamma b \cdot v\,d\Gamma \tag{2.8}$$

for all $v \in V$.

If we consider the Dirichlet boundary value problem (1.10), this definition applies as well, by dropping in (2.8) the terms with β and b.

By defining $\langle Au, v\rangle$ for each pair $u, v \in V$ as the left-hand side of (2.8), the operator $A : V \to V'$ satisfies the assumptions in [31, theorems 2.1 and 2.2, chap. 2, sec. 2]. This shows existence and uniqueness of the weak solution.

By replacing v by u in equation (2.8), one gets

$$\nu_0\|\nabla u\|^2 + \nu_1\|\mathcal{D}u\|_p^p + \beta\|u\|_\Gamma^2 = \langle b, u\rangle_\Gamma + \langle f, u\rangle_\Omega \,, \tag{2.9}$$

where the symbols $\langle \cdot , \cdot \rangle$ denote "duality pairings" and the trace of u on the boundary is denoted simply by u. Note that the left-hand side of equation (2.9) is just $\langle Au, u\rangle$. This shows that assumption (2.3) in [31, theorem 2.1] holds.

From (2.9) there readily follows the basic estimate

$$\frac{\nu_0^2}{2}\|\nabla u\|^2 + \nu_0\nu_1\|\mathcal{D}u\|_p^p + \beta\|u\|_\Gamma^2 \le c_n\big([\,f\,]_{-1}^2 + [b]_{-1/2}^2\big)\,, \tag{2.10}$$

where the constant c_n depends only on n. In proving (2.10), we use the estimate $[u]_{1/2} \le \|\nabla u\|$.

Remark 2.2. We remark that

$$\|\nabla u\|_p \le c_{n,p}\|\mathcal{D}u\|_p \quad \text{and} \quad \|\nabla u\|_{p,R} \le c_{n,p}\|\mathcal{D}u\|_{p,R}\,.$$

However, we point out that we will not appeal to these estimates.

By restriction of (2.8) to divergence-free test functions v with compact support in $\mathbb{R}^n_+$ and by (2.1), there follows the existence of a distribution π (determined up to a constant) such that

$$\nabla\pi = -\nabla\cdot[\nu_0\nabla u + \nu_1|\mathcal{D}u|^{p-2}\mathcal{D}u] + f \equiv \nabla\cdot(U_1 + U_2 + K)\,, \tag{2.11}$$

where, for convenience, we represent f in the form

$$\langle f, w\rangle = \int K\cdot\nabla w\,dx = \int\sum_{i,j=1}^n K_j^i\frac{\partial w_i}{\partial x_j}\,dx \tag{2.12}$$

for all $w \in D^{1,2}(\mathbb{R}^n_+)$ where $K \in L^2(\mathbb{R}^n_+)$. Actually, this representation holds with $K_j^i = \frac{\partial g_i}{\partial x_j}$ and $g \in D^{1,2}(\mathbb{R}^n_+)$. Moreover,

$$[\,f\,]_{-1} = \|K\|\,. \tag{2.13}$$

Equation (2.11) shows that the first equation in (1.11) holds in the distributional sense.

Let us make some remarks concerning the pressure. Note that K and $U_1 = -\nu_0\nabla u$ belong to L^2 and $U_2 = -\nu_1|\mathcal{D}u|^{p-2}\mathcal{D}u$ belongs to $L^{p'}$. In fact, from (2.10) and (2.13) it follows that

$$\|U_1\|^2 + \nu_0\nu_1^{1-p'}\|U_2\|_{p'}^{p'} + \nu_0\beta\|u\|_\Gamma^2 \le c_n\big([\,f\,]_{-1}^2 + [b]_{-1/2}^2\big)\,. \tag{2.14}$$

On the other hand, it is well-known that if

$$\nabla\pi = \nabla\cdot U$$

for some $U \in L^\alpha(B_R^+)$, then

$$\|\pi\|_{L^\alpha_\#(B_R^+)} \leq c\|U\|_{L^\alpha(B_R^+)}\,, \tag{2.15}$$

where $L^\alpha_\# = L^\alpha/\mathbb{R}$ and

$$B_R^+ = \{x : |x| < R,\ x_n > 0\}\,,$$

hence

$$\pi \in L^{p'}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})\,.$$

On the other hand, one has, for $p_0 < p_1$,

$$\|\cdot\|_{L^{p_0}(B_R^+)} \leq |B_R^+|^{\frac{1}{p_0}-\frac{1}{p_1}}\,\|\cdot\|_{L^{p_1}(B_R^+)}\,, \tag{2.16}$$

where $|B_R^+|$ denotes the Lebesgue measure of B_R^+.

It readily follows from (2.11), (2.14), and (2.15) (with $\alpha = p'$) that

$$\|\pi\|_{L^{p'}_\#(B_R^+)} \leq c|B_R^+|^{\frac{1}{p'}-\frac{1}{2}}\big([f]_{-1} + [b]_{-\frac{1}{2}}\big) + c\big([f]_{-1} + [b]_{-\frac{1}{2}}\big)^{\frac{1}{p'}-\frac{1}{2}}\,.$$

A difficulty similar to the one above (the need to localize the estimates, due to the fact that the canonical inclusion $L^{p_1} \hookrightarrow L^{p_0}$, $p_0 < p_1$, fails near infinity) will occur as well in studying the L^α regularity of $\nabla\pi$. In this case, however, this difficulty will propagate from the gradient of the pressure to the second-order derivatives of u that are not included in $D^2_* u$.

We end this section by introducing some more notation. We denote by $D^2 u$ the set of all the second derivatives of u. The meaning of expressions like $\|D^2 u\|$ is clear. The symbol $D^2_* u$ may denote any of the second-order derivatives $\partial^2 u_j/\partial x_i \partial x_k$ except for the derivatives $\partial^2 u_j/\partial x_n^2$, if $j < n$. Moreover,

$$|D^2_* u|^2 := \left|\frac{\partial^2 u_n}{\partial x_n^2}\right|^2 + \sum_{\substack{i,j,k=1\\(i,k)\neq(n,n)}}^{n} \left|\frac{\partial^2 u_j}{\partial x_i \partial x_k}\right|^2\,.$$

Similarly, ∇^* may denote any first-order partial derivative except for $\partial/\partial x_n$.

We set

$$\|\cdot\|_{\alpha,R} = \|\cdot\|_{L^\alpha(B_R^+)}\,.$$

Finally,

$$\|f,b\|^2 = \|f\|^2 + [b]^2_{1/2} \quad \text{and} \quad [f,b]^2 = [f]^2_{-1} + [b]^2_{-1/2}\,.$$

3 Results

Now we state the two main theorems. We set, for each $q > 1$,

$$\begin{aligned}\mathcal{K}_q = {} & c_n R\big(|B_R^+|^{\frac{1}{q}-\frac{1}{2}}[f,b] + \nu_1|B_R^+|^{\frac{1}{q}-\frac{1}{p'}}\|\mathcal{D}u\|_p^{p-1}\big)\\ & + c_n\left(|B_R^+|^{\frac{1}{r}-\frac{1}{2}} + (p-1)\left(\frac{\nu_1}{\nu_0}\right)^{\frac{1}{2}}|B_R^+|^{\frac{1}{q}-\frac{1}{p'}}\|\mathcal{D}u\|_p^{\frac{p-2}{2}}\right)\|f,b\|\,.\end{aligned} \tag{3.1}$$

For convenience we set $\mathcal{K} = \mathcal{K}_{p'}$.

THEOREM 3.1 *Assume that* $2 < p$ *and that*

$$\begin{cases} f \in L^2(\mathbb{R}^n_+) \\ b \in D^{\frac{1}{2}}(\mathbb{R}^{n-1}). \end{cases} \tag{3.2}$$

Let u, π *be the weak solution to problem* (1.11) *under boundary condition* (1.6) *or* (1.10)*. Then the derivatives* $D^2_* u$ *belong to* $L^2(\mathbb{R}^n_+)$ *and satisfy the estimate*

$$\nu_0 \| D^2_* u \| + (\nu_0 \nu_1)^{\frac{1}{2}} \left\| |\mathcal{D}u|^{\frac{p-2}{2}} \nabla^* \mathcal{D}u \right\| \le c_n \| f, b \| . \tag{3.3}$$

On the other hand,

$$D^2 u,\, |\mathcal{D}u|^{p-2} \nabla^* \mathcal{D}u,\, \nabla^* \pi \in L^{p'}_{\text{loc}}(\overline{\mathbb{R}^n_+})$$

where

$$p' = \frac{p}{p-1}\,. \tag{3.4}$$

In particular, if $p < \frac{n}{n-2}$*, then* $u \in C^{0,\alpha}_{\text{loc}}(\overline{\mathbb{R}^n_+})$ *where* $\alpha = \frac{n-(n-2)p}{p}$.

More precisely, for each $R > 0$,

$$\frac{1}{p-1} \| \nabla^* \pi \|_{p',R} + \nu_0 \| D^2 u \|_{p',R} + \nu_1 \| |\mathcal{D}u|^{p-2} \nabla^* \mathcal{D}u \|_{p',R} \le \mathcal{K}\,, \tag{3.5}$$

where

$$\begin{aligned} \mathcal{K} = {} & c_n R \big(|B^+_R|^{\frac{1}{p'} - \frac{1}{2}} [f, b] + \nu_1 \| \mathcal{D}u \|_p^{p-1} \big) \\ & + c_n \left(|B^+_R|^{\frac{1}{p'} - \frac{1}{2}} + (p-1) \left(\frac{\nu_1}{\nu_0} \right)^{\frac{1}{2}} \| \mathcal{D}u \|_p^{\frac{p-2}{2}} \right) \| f, b \| \end{aligned} \tag{3.6}$$

and $\| \mathcal{D}u \|_p$ *satisfies estimate* (2.10)*. Moreover,* $\frac{\partial \pi}{\partial x_n}$ *satisfies the estimate*

$$\left| \frac{\partial \pi}{\partial x_n} \right| \le c_{n,p} \big[(\nu_0 + \nu_1 |\mathcal{D}u(x)|^{p-2}) |D^2_* u(x)| + |\nabla^* \pi| + |f| \big] \quad \text{a.e. in } \mathbb{R}^n_+\,. \tag{3.7}$$

In particular, if $p < 4$,

$$\frac{\partial \pi}{\partial x_n} \in L^{\overline{p}}_{\text{loc}}(\overline{\mathbb{R}^n_+})\,,$$

where

$$\overline{p} = \frac{2p}{3p-4}\,, \tag{3.8}$$

and, for each $R > 0$,

$$\begin{aligned} \left\| \frac{\partial \pi}{\partial x_n} \right\|_{\overline{p},R} \le {} & c_{n,p} R \big(|B^+_R|^{\frac{1}{\overline{p}} - \frac{1}{2}} [f, b] + \nu_1 |B^+_R|^{\frac{1}{\overline{p}} - \frac{1}{p'}} \| \mathcal{D}u \|_p^{p-1} \big) \\ & + c_{n,p} \left(|B^+_R|^{\frac{1}{\overline{p}} - \frac{1}{2}} + \frac{\nu_1}{\nu_0} \| \mathcal{D}u \|_p^{p-2} \right) \| f, b \| . \end{aligned} \tag{3.9}$$

In particular, $\nabla \pi \in L^{\overline{p}}_{\text{loc}}(\overline{\mathbb{R}^n_+})$; *see also* (3.14).

If $2 < p < 2 + \frac{2}{n-1}$, the second part of the above theorem may be improved. Merely for convenience we prove this result for $n = 3$ and leave to the interested reader the straightforward extension to higher dimensions. For brevity, assume that $\nu_0 = \nu_1 = 1$.

THEOREM 3.2 *Assume that $n = 3$, $\nu_0 = \nu_1 = 1$, and*

$$2 \leq p \leq 3 .$$

Let f, b, u, and π be as in Theorem 3.1. Then, in addition to the results stated in this last theorem, one has

$$D^2 u,\, |\mathcal{D}u|^{p-2} \nabla^* \mathcal{D}u,\, \nabla^* \pi \in L^l_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$$

where

$$l = \frac{3(4-p)}{5-p} . \tag{3.10}$$

In particular, $u \in C^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$ where $\alpha = \frac{3-p}{4-p}$.

More precisely, for each $R > 0$,

$$\|\nabla^* \pi\|_{l,R} + \|D^2 u\|_{l,R} + \| |\mathcal{D}u|^{p-2} \nabla^* \mathcal{D}u\|_{l,R} \leq \mathcal{K}_l + c_p \|f, b\|^{\frac{2}{4-p}} . \tag{3.11}$$

Finally,

$$\frac{\partial \pi}{\partial x_n} \in L^m_{\mathrm{loc}}(\overline{\mathbb{R}^n_+}) \tag{3.12}$$

where

$$m = \frac{6(4-p)}{8-p} . \tag{3.13}$$

In particular,

$$\nabla \pi \in L^m_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$$

and

$$\|\nabla \pi\|_{m,R} \leq C_R \big(\|\mathcal{D}u\|_p + \|\mathcal{D}u\|_p^{\frac{p(p-1)}{2}} + \|f, b\| + \|f, b\|^{\frac{p}{4-p}} \big) \tag{3.14}$$

where C_R depends on $|B^+_R|$ and on the various exponents.

Remark 3.3. Concerning (3.14), a more precise estimate is easily obtained by following its proof. This is left to the interested reader. Note that $m > p'$ if $p > 2 + \frac{2}{5}$.

COROLLARY 3.4 *Under the hypotheses of Theorem 3.2 one has*

$$u \in W^{1,l^*}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+}) ;$$

moreover,

$$\|\nabla u\|_{l^*,R} \leq c \|D^2 u\|_{l,R} + c|B^+_R|^{\frac{1}{l^*} - \frac{1}{p}} \|\nabla u\|_p \tag{3.15}$$

where

$$l^* = 3(4-p) .$$

PROOF: The proof of (3.15) follows by appealing to (5.15) with $g = \nabla u$ and $s = l$. Note that $l^* > p$ for $2 < p < 3$.

The linear case, $p = 2$, is well studied and will not be considered in the proofs that follow; see [4, 5, 49]. Nevertheless, it is significant that, when $p = 2$, the statements and estimates established in Theorems 3.1 and 3.2 coincide with the classical results. Note that (3.11) improves (3.9) since $p' < l$ if $2 < p < 3$. For $p = 2$ one has $p' = l = 2$, and for $p = 3$ one has $p' = s = \frac{3}{2}$. □

To end this section, we recall a well-known result that is a main tool in our proofs.

LEMMA 3.5 *Let $g(x)$ be a scalar field defined in B_R^+ such that*

$$g = \nabla \cdot w_0 \quad \textit{and} \quad \nabla g = \nabla \cdot W$$

where w_0 and W belong to $L^\alpha(B_R^+)$ for some $\alpha > 1$. Then

$$\|g\|_{L^\alpha(B_R^+)} \le c_n\big(R\|w_0\|_{L^\alpha(B_R^+)} + \|W\|_{L^\alpha(B_R^+)}\big) \tag{3.16}$$

where c_n is independent of R.

The lack of dependence on R follows by a scaling argument. It is worth noting that the constant c may be chosen independently of α provided that $1 < \alpha_1 \le \alpha \le \alpha_2 < \infty$. In this case $c = c(\alpha_1, \alpha_2)$. The above result (for a bounded domain with a Lipschitz-continuous boundary) is proven in reference [37].

4 Main Estimates: Proof of Theorem 3.1

Roughly speaking, inequality (3.3) shows that *tangential* derivatives are square integrable. The proof of this main estimate appeals to Nirenberg's translation method; see [38].

LEMMA 4.1 *Under the assumptions of Theorem* 3.1 *the derivatives $D_*^2 u$ satisfy inequality* (3.3)*.*

PROOF: Let u be a weak solution, i.e., $u \in V$ is a solution to the problem

$$\frac{\nu_0}{2}\int \mathcal{D}u \cdot \mathcal{D}v\,dx + \frac{\nu_1}{2}\int |\mathcal{D}u|^{p-2}\mathcal{D}u \cdot \mathcal{D}v\,dx + \beta\int_\Gamma u \cdot v\,d\Gamma = \int f \cdot v\,dx + \int_\Gamma b \cdot v\,d\Gamma \quad \text{for each } v \in V. \tag{4.1}$$

For arbitrary scalar or vector fields v we set

$$\tau_h v(x) = v(x_1, \dots, x_{k-1}, x_k + h, x_{k+1}, \dots, x_n)\,,$$

where $h \in \mathbb{R}$ and k, $k \neq n$, is assumed to be fixed. We also set

$$v^h = \tau_h v\,, \quad \Delta_h v = \frac{v^h - v}{h}\,.$$

Note that the above translations are done in the tangential directions.

By writing (4.1) with v replaced by v^h and by replacing, in the integrals on the left-hand side, the variable x_k by $x_k - h$, one easily shows that

$$\frac{\nu_0}{2}\int \mathcal{D}u^{-h}\cdot \mathcal{D}v\,dx + \frac{\nu_1}{2}\int |\mathcal{D}u^{-h}|^{p-2}\mathcal{D}u^{-h}\cdot \mathcal{D}v\,dx + \beta\int_\Gamma u^{-h}\cdot v\,d\Gamma = \int f\cdot v^h\,dx + \int_\Gamma b\cdot v^h\,d\Gamma\,. \tag{4.2}$$

Taking the difference between equations (4.1) and (4.2), respecting the left and right sides, one gets

$$\begin{aligned}&\frac{\nu_0}{2}\int (\mathcal{D}u - \mathcal{D}u^{-h})\cdot \mathcal{D}v\,dx\\ &\quad+\frac{\nu_1}{2}\int (|\mathcal{D}u|^{p-2}\mathcal{D}u - |\mathcal{D}u^{-h}|^{p-2}\mathcal{D}u^{-h})\cdot \mathcal{D}v\,dx\\ &\quad+\beta\int_\Gamma (u-u^{-h})\cdot v\,d\Gamma\\ &\quad\;= \int f\cdot (v-v^h)dx + \int_\Gamma b\cdot (v-v^h)\,d\Gamma\,.\end{aligned} \tag{4.3}$$

By setting $v = u - u^{-h}$ in equation (4.3) and by taking into account the estimate

$$\left|\int f\cdot (v-v^h)\,dx\right| \leq |h|\|f\|\left\|\frac{v-v^h}{h}\right\| \leq |h|\|f\|\,\|\nabla v\| \tag{4.4}$$

and the inequality (see the proof below)

$$\left|\int_\Gamma b\cdot (v-v^h)\,d\Gamma\right| \leq c_n|h|[b]_{\frac12}\|\nabla v\|\,, \tag{4.5}$$

it follows that

$$\begin{aligned}&\frac{\nu_0}{2}\int |\mathcal{D}u - \mathcal{D}u^{-h}|^2\,dx\\ &\quad+\frac{\nu_1}{2}\int (|\mathcal{D}u|^{p-2}\mathcal{D}u - |\mathcal{D}u^{-h}|^{p-2}\mathcal{D}u^{-h})\cdot (\mathcal{D}u - \mathcal{D}u^{-h})dx\\ &\quad+\beta\int_\Gamma |u-u^{-h}|^2\,d\Gamma\\ &\quad\leq c_n|h|\big(\|f\| + [b]_{1/2}\big)\|\nabla(u-u^{-h})\|\,.\end{aligned} \tag{4.6}$$

On the other hand, an inequality of Korn's type (see, for instance, [5]) shows that

$$\int |\mathcal{D}(u - u^{-h})|^2\, dx = 2 \int |\nabla (u - u^{-h})|^2\, dx\,.$$

Since the second term on the left-hand side of (4.6) is nonnegative, it follows (after dividing by h^2) that $D^2_* u \in L^2(\mathbb{R}^n_+)$; moreover,

$$\nu_0 \| D^2_* u \| \equiv \nu_0 \left(\left\| \frac{\partial^2 u_n}{\partial x_n^2} \right\| + \sum_{\substack{i,j,k=1 \\ (i,k)\neq(n,n)}}^{n} \left\| \frac{\partial^2 u_j}{\partial x_i \partial x_k} \right\| \right) \leq c_n \| f, b \|\,, \tag{4.7}$$

where, from now on, the symbol D^2_* denotes any of the second derivatives $\partial^2 u_j / \partial x_i \partial x_k$ except for the derivatives $\partial^2 u_j/\partial x_n^2$ when $j < n$. The inclusion of the derivative $\partial^2 u_n / \partial x_n^2$ in the above estimate follows by differentiation with respect to x_n of the equation $\nabla \cdot u = 0$. □

PROOF OF (4.5): One has

$$\int_\Gamma b \cdot (v - v_h) d\Gamma = h \int_{\mathbb{R}^{n-1}} \frac{\hat{b} - \hat{b}_{-h}}{h} \cdot \hat{v}\, d\xi$$

where $\widehat{\phi}(\xi)$ denotes the Fourier transform of ϕ in $\mathbb{R}^{n-1}$. Since

$$\widehat{\tau_{-h}\phi}(\xi) = e^{-2\pi i \xi_k h} \widehat{\phi}(\xi)\,,$$

straightforward manipulations show that

$$\left| \int_\Gamma b \cdot (v - v_h)\, d\Gamma \right| \leq |h| \left(\int_{\mathbb{R}^{n-1}} |\hat{b}(\xi)|^2 \frac{|\exp(-2\pi i \xi_k h) - 1|^2}{h^2 |\xi|}\, d\xi \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^{n-1}} |\hat{v}(\xi)|^2 |\xi|\, d\xi \right)^{\frac{1}{2}}.$$

Since $|(e^{i\theta} - 1)/\theta| \leq 1$, it readily follows that the right-hand side of the last equation is bounded by $2\pi |h| [v]_{1/2} [b]_{1/2}$. Since $[v]_{1/2} \leq c_n \| \nabla v \|$, inequality (4.5) follows. □

The following Taylor expansion is a main tool in what comes later:

LEMMA 4.2 *Let U and V be two arbitrary vectors in $\mathbb{R}^N$, $N \geq 1$. Then there are reals α and β, $0 < \alpha, \beta < 1$, that depend on the pair (U, V) such that*

$$p(|U|^{p-2}U - |V|^{p-2}V) \cdot (U - V) = \frac{1}{2} (U - V) [H(\overline{U}) + H(\overline{V})] (U - V)^{\mathsf{T}} \tag{4.8}$$

where

$$\overline{U} = \alpha U + (1 - \alpha) V\,, \quad \overline{V} = \beta U + (1 - \beta) V\,,$$

and the $N \times N$ matrix field $H(W)$ satisfies

$$\xi H(W)\xi^{\mathsf{T}} = p|W|^{p-2}|\xi|^2 + p(p-2)|W|^{p-4}(W \cdot \xi)^2 \tag{4.9}$$

for all $W, \xi \in \mathbb{R}^N$.

PROOF: Consider the real function $\psi(U) = |U|^p$. One has

$$\frac{\partial \psi}{\partial U_i} = p|U|^{p-2}U_i$$

and

$$H_{i,j}(U) := \frac{\partial^2 \psi}{\partial U_i \partial U_j} = p|U|^{p-2}\delta_{ij} + p(p-2)|U|^{p-4}U_i U_j$$

where H is the Hessian matrix. By Taylor's formula

$$\psi(U) = \psi(V) + \nabla\psi(V)\cdot(U-V) + \frac{1}{2}(U-V)H(\overline{V})(U-V)^{\mathsf{T}}.$$

By interchanging U and V in the above formulae and by adding the respective sides in the two equations, one obtains the symmetrized form of Taylor expansion (4.8). □

Note that

$$p|W|^{p-2}|\xi|^2 \le \xi H(W)\xi^{\mathsf{T}} \le p(p-1)|W|^{p-2}|\xi|^2 .$$

It is worth noting that the particular form of the convex function $\psi(U)$ is not essential here or later.

LEMMA 4.3 *The vector field u satisfies estimate* (3.3).

PROOF: The first part of the estimate was already proven in the previous lemma. Setting $U = \mathcal{D}u$ and $V = \mathcal{D}u^{-h}$ in equation (4.8) and taking into account (4.9), it follows that

$$\begin{aligned}
&(|\mathcal{D}u|^{p-2}\mathcal{D}u - |\mathcal{D}u^{-h}|^{p-2}\mathcal{D}u^{-h})\cdot(\mathcal{D}u - \mathcal{D}u^{-h}) \\
&\quad = \frac{1}{2}(|\tilde{U}|^{p-2} + |\tilde{V}|^{p-2})|\mathcal{D}u - \mathcal{D}u^{-h}|^2 \\
&\qquad + \frac{p-2}{2}|\tilde{V}|^{p-4}(\tilde{V}\cdot(\mathcal{D}u - \mathcal{D}u^{-h}))^2 \\
&\qquad + \frac{p-2}{2}|\tilde{U}|^{p-4}(\tilde{U}\cdot(\mathcal{D}u - \mathcal{D}u^{-h}))^2 \quad \text{a.e. in } \mathbb{R}^n_+ ,
\end{aligned} \tag{4.10}$$

where

$$\begin{cases} \tilde{U} = \alpha\mathcal{D}u + (1-\alpha)\mathcal{D}u^{-h} \\ \tilde{V} = \beta\mathcal{D}u + (1-\beta)\mathcal{D}u^{-h}. \end{cases} \tag{4.11}$$

The reals $\alpha = \alpha(x)$ and $\beta = \beta(x)$ take values between 0 and 1 and depend on the point $x \in \mathbb{R}^n_+$. Clearly,

$$\begin{cases} |\tilde{U} - \mathcal{D}u| \le |\mathcal{D}u - \mathcal{D}u^{-h}| \\ |\tilde{V} - \mathcal{D}u| \le |\mathcal{D}u - \mathcal{D}u^{-h}| \end{cases} \qquad \text{a.e. in } \mathbb{R}^n_+ \,. \tag{4.12}$$

Next, by using (4.7) to estimate the right-hand side of (4.6), dividing this last equation by $|h|^2$, and using (4.10), it follows that

$$\begin{aligned} &\nu_0 \int \left| \mathcal{D} \frac{u - u^{-h}}{h} \right|^2 dx \\ &\quad + \nu_1 \int \left\{ (|\tilde{U}|^{p-2} + |\tilde{V}|^{p-2}) \left| \mathcal{D} \frac{u - u^{-h}}{h} \right|^2 \right. \\ &\qquad + (p-2)|\tilde{U}|^{p-4} \left(\tilde{U} \cdot \left(\mathcal{D} \frac{u - u^{-h}}{h} \right) \right)^2 \\ &\qquad \left. + (p-2)|\tilde{V}|^{p-4} \left(\tilde{V} \cdot \left(\mathcal{D} \frac{u - u^{-h}}{h} \right) \right)^2 \right\} dx \\ &\quad \le c_n \nu_0^{-1} \| f, b \|^2 \,. \end{aligned} \tag{4.13}$$

Next we pass to the limit in (4.13) as $h \to 0$. From (4.12) it follows that $\tilde{U} \to \mathcal{D}u$ and $\tilde{V} \to \mathcal{D}u$ almost everywhere in $\mathbb{R}^n_+$. On the other hand, due to (4.7), we know that

$$\nabla \frac{u - u^{-h}}{h} \to \nabla \frac{\partial u}{\partial x_k} \quad \text{a.e. in } \mathbb{R}^n_+ \,.$$

In particular, the same property holds by replacing ∇ by $\mathcal{D}$. The above considerations, together with the nonnegativity of all the integrands that appear on the left-hand side of inequality (4.13), allow us to pass to the limit by using Fatou's lemma. This yields

$$\begin{aligned} &\nu_0 \int \left| \mathcal{D} \frac{\partial u}{\partial x_k} \right|^2 dx \\ &\quad + \nu_1 \int \left\{ |\mathcal{D}u|^{p-2} \left| \mathcal{D} \frac{\partial u}{\partial x_k} \right|^2 + (p-2)|\mathcal{D}u|^{p-4} \left(\mathcal{D}u \cdot \mathcal{D} \frac{\partial u}{\partial x_k} \right)^2 \right\} dx \\ &\quad \le c_n \nu_0^{-1} \big(\| f \|^2 + [b]^2_{1/2} \big) \end{aligned} \tag{4.14}$$

for each index k, $k \neq n$. Hence,

$$\nu_0 \| D^2_* u \|^2 + \nu_1 \sum_{k=1}^{n-1} \left\| |\mathcal{D}u|^{\frac{p-2}{2}} \mathcal{D} \frac{\partial u}{\partial x_k} \right\| \le c_n \nu_0^{-1} \big(\| f \|^2 + [b]^2_{1/2} \big) \,. \tag{4.15}$$

The proof of estimate (3.3) is accomplished. □

The next step is to prove estimate (3.9) for $\nabla^* \pi$.

LEMMA 4.4 *For each $k \neq n$, the terms $|\mathcal{D}u|^{p-2}\mathcal{D}\frac{\partial u}{\partial x_k}$ and the derivatives $\frac{\partial \pi}{\partial x_k}$ satisfy estimate* (3.5). *In particular,*

$$\left\| \frac{\partial \pi}{\partial x_k} \right\|_{p',R} \leq \mathcal{K}\,. \tag{4.16}$$

PROOF: Straightforward calculations show that

$$\frac{\partial}{\partial x_k}(|\mathcal{D}u|^{p-2}\mathcal{D}u) = |\mathcal{D}u|^{p-2}\mathcal{D}\frac{\partial u}{\partial x_k} + (p-2)|\mathcal{D}u|^{p-4}\left(\mathcal{D}u \cdot \mathcal{D}\frac{\partial u}{\partial x_k}\right)\mathcal{D}u\,. \tag{4.17}$$

On the other hand, by differentiation of equation (1.11) with respect to x_k, $k \neq n$, it follows

$$\begin{aligned}\nabla\frac{\partial \pi}{\partial x_k} &= \nabla\cdot\left[-\nu_0\mathcal{D}\frac{\partial u}{\partial x_k}\right] + \nabla\cdot\left[-\nu_1\frac{\partial}{\partial x_k}(|\mathcal{D}u|^{p-2}\mathcal{D}u)\right] + \nabla\cdot G\\ &\equiv \nabla\cdot[U_3 + U_4 + G]\,,\end{aligned} \tag{4.18}$$

where, for uniformity of notation, we introduce $G_{ij} = \delta_{kj} f_i$. Hence $\nabla\cdot G = \frac{\partial f}{\partial x_k}$; moreover, $\|G\| = \|f\|$.

Next we estimate suitable norms of the terms inside square brackets that appear on the right-hand side of equation (4.18). By (4.7),

$$\|U_3\| \equiv \left\|\nu_0\mathcal{D}\frac{\partial u}{\partial x_k}\right\| \leq c_n\|f, b\|\,. \tag{4.19}$$

On the other hand, by using (4.17), one shows that

$$\left|\frac{\partial}{\partial x_k}(|\mathcal{D}u|^{p-2}\mathcal{D}u)\right| \leq c_n(p-1)|\mathcal{D}u|^{p-2}\left|\mathcal{D}\frac{\partial u}{\partial x_k}\right| \quad \text{a.e. in } \mathbb{R}^n_+\,. \tag{4.20}$$

Moreover, by Hölder's inequality,

$$\left\||\mathcal{D}u|^{p-2}\mathcal{D}\frac{\partial u}{\partial x_k}\right\|_{p'} \leq \|\mathcal{D}u\|_p^{\frac{p-2}{2}}\left\||\mathcal{D}u|^{\frac{p-2}{2}}\mathcal{D}\frac{\partial u}{\partial x_k}\right\|\,. \tag{4.21}$$

Hence, by (4.15), it follows that

$$\left\||\mathcal{D}u|^{p-2}\mathcal{D}\frac{\partial u}{\partial x_k}\right\|_{p'} \leq c_n\left(\frac{1}{\nu_0\nu_1}\right)^{\frac{1}{2}}\|\mathcal{D}u\|_p^{(p-2)/2}\|f, b\|\,. \tag{4.22}$$

This proves the first statement in the lemma. Furthermore,

$$\begin{aligned}\|U_4\|_{p'} &\equiv \left\|\nu_1\frac{\partial}{\partial x_k}(|\mathcal{D}u|^{p-2}\mathcal{D}u)\right\|_{p'}\\ &\leq c_n(p-1)\left(\frac{\nu_1}{\nu_0}\right)^{\frac{1}{2}}\|\mathcal{D}u\|_p^{(p-2)/2}\|f, b\|\,.\end{aligned} \tag{4.23}$$

Recall that $\|\mathcal{D}u\|_p$ is bounded; see (2.10). From (2.11) and (4.18), and by using (3.16) and (2.16) with $g = \frac{\partial \pi}{\partial x_k}$ and $\alpha = p_0 = p'$, $p_1 = 2$, it follows that

$$\left\| \frac{\partial \pi}{\partial x_k} \right\|_{p',R} \leq c_n R\Big(|B_R^+|^{\frac{1}{p'}-\frac{1}{2}}(\|U_1\| + \|K\|) + \|U_2\|_{p',R}\Big) + c_n\Big(|B_R^+|^{\frac{1}{p'}-\frac{1}{2}}(\|U_3\| + \|G\|) + \|U_4\|_{p',R}\Big)\,. \tag{4.24}$$

Next, by (4.19) and (4.23), (4.16) follows. □

LEMMA 4.5 *The derivatives* $\frac{\partial^2 u_j}{\partial x_n^2}$, $j \neq n$, *satisfy estimate* (3.5).

PROOF: By using (4.17), the j^{th} equation (1.11) may be written in the form

$$\begin{aligned} &- \nu_0 \sum_{k=1}^{n} \frac{\partial^2 u_j}{\partial x_k^2} - \nu_1 |\mathcal{D}u|^{p-2} \sum_{k=1}^{n} \left(\frac{\partial^2 u_j}{\partial x_k^2} + \frac{\partial^2 u_k}{\partial x_j \partial x_k} \right) \\ &- (p-2)\nu_1 |\mathcal{D}u|^{p-4} \sum_{l,m,k=1}^{n} \mathcal{D}_{lm}\mathcal{D}_{jk} \left(\frac{\partial^2 u_l}{\partial x_m \partial x_k} + \frac{\partial^2 u_m}{\partial x_l \partial x_k} \right) + \frac{\partial \pi}{\partial x_j} = f_j\,, \end{aligned} \tag{4.25}$$

where $\mathcal{D}_{ij} = (\mathcal{D}u)_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ and $1 \leq j \leq n$. Let us write the first $n-1$ equations (4.25) as follows:

$$\nu_0 \frac{\partial^2 u_j}{\partial x_n^2} + \nu_1 |\mathcal{D}u|^{p-2} \frac{\partial^2 u_j}{\partial x_n^2} + 2(p-2)\nu_1 |\mathcal{D}u|^{p-4} \mathcal{D}_{jn} \sum_{l=1}^{n-1} \mathcal{D}_{ln} \frac{\partial^2 u_l}{\partial x_n^2} = F_j(x) + \frac{\partial \pi}{\partial x_j} - f_j\,, \tag{4.26}$$

where the $F_j(x)$, $j \neq n$, are given by

$$\begin{aligned} F_j(x) :=& \\ &- \nu_0 \sum_{k=1}^{n-1} \frac{\partial^2 u_j}{\partial x_k^2} - \nu_1 |\mathcal{D}u|^{p-2} \sum_{k=1}^{n-1} \frac{\partial^2 u_j}{\partial x_k^2} - \nu_1 |\mathcal{D}u|^{p-2} \sum_{k=1}^{n-1} \frac{\partial^2 u_k}{\partial x_j \partial x_k} \\ &- 2(p-2)\nu_1 |\mathcal{D}u|^{p-4} \left\{ \mathcal{D}_{nn}\mathcal{D}_{jn} \frac{\partial^2 u_n}{\partial x_n^2} + \sum_{\substack{l,m,k=1 \\ (m,k)\neq(n,n)}}^{n} \mathcal{D}_{lm}\mathcal{D}_{jk} \frac{\partial^2 u_l}{\partial x_m \partial x_k} \right\}. \end{aligned} \tag{4.27}$$

In what follows, equation (4.26), $1 \leq j \leq n-1$, will be treated as an $(n-1) \times (n-1)$ linear system in the unknowns $\partial^2 u_j / \partial x_n^2$, $j \neq n$. Note that, with an obviously simplified notation, the measurable functions F_j satisfy

$$|F_j(x)| \leq c_n \left(\nu_0 + (p-1)\nu_1 |\mathcal{D}u(x)|^{p-2}\right) |D_*^2 u(x)| \quad \text{a.e. in } \mathbb{R}_+^n. \tag{4.28}$$

We denote by $\widetilde{F}_j$ the right-hand sides

$$\widetilde{F}_j(x) := F_j(x) + \frac{\partial \pi}{\partial x_j} - f_j \tag{4.29}$$

that appear in the above $(n-1)\times(n-1)$ system (4.26).

Let us show that the $(n-1)\times(n-1)$ system (4.26) can be solved for the unknowns $\partial^2 u_j/\partial x_n^2$, $j\neq n$, for almost all $x\in\mathbb{R}^n_+$. The elements a_{jl} of the matrix system A are given by

$$a_{jl} = \big(\nu_0+\nu_1|\mathcal{D}u|^{p-2}\big)\delta_{jl} + 2(p-2)\nu_1|\mathcal{D}u|^{p-4}\mathcal{D}_{ln}\mathcal{D}_{jn}$$

for $j,l\neq n$. Note that $a_{jl}=a_{lj}$. One easily shows that

$$\sum_{j,l=1}^{n-1} a_{jl}\xi_j\xi_l = (\nu_0+\nu_1|\mathcal{D}u|^{p-2})|\xi|^2 + 2(p-2)\nu_1|\mathcal{D}u|^{p-4}[(\mathcal{D}u)\cdot\xi]_n^2\,.$$

Hence the matrix A is symmetric and positive definite. Moreover, the above identity shows that all the eigenvalues are larger than or equal to $\nu_0+\nu_1|\mathcal{D}u|^{p-2}$. Hence,

$$\det A \geq \big(\nu_0+\nu_1|\mathcal{D}u|^{p-2}\big)^{n-1}\,.$$

Next, by setting $\xi_l=\partial^2 u_l/\partial x_n^2$, we get from (4.26), i.e., from

$$\sum_{l=1}^{n-1} a_{jl}\xi_l = \widetilde{F}_j\,, \tag{4.30}$$

that

$$\sum_{l,j=1}^{n-1} a_{jl}\xi_l\xi_j = \sum_{j=1}^{n-1}\widetilde{F}_j\xi_j\,. \tag{4.31}$$

Consequently, $(\nu_0+\nu_1|\mathcal{D}u|^{p-2})|\xi|^2\leq|\widetilde{F}||\xi|$, which shows that

$$\big(\nu_0+\nu_1|\mathcal{D}u|^{p-2}\big)\sum_{l=1}^{n-1}\left|\frac{\partial^2 u_l}{\partial x_n^2}\right| \leq |\widetilde{F}| := \Big(\sum_{j=1}^{n-1}|\widetilde{F}_j|^2\Big)^{\frac{1}{2}} \quad \text{a.e. in } \mathbb{R}^n_+\,. \tag{4.32}$$

In particular,

$$\nu_0\sum_{l=1}^{n-1}\left|\frac{\partial^2 u_l}{\partial x_n^2}\right| \leq c_n(p-1)\nu_0|D^2_* u(x)| + c_n(|\nabla^*\pi|+|f|) \quad \text{a.e. in } \mathbb{R}^n_+\,. \tag{4.33}$$

There readily follows, by appealing to (4.16) and(4.7), that

$$\nu_0\sum_{l=1}^{n-1}\left\|\frac{\partial^2 u_l}{\partial x_n^2}\right\|_{p',R} \leq \mathcal{K}\,. \tag{4.34}$$

□

LEMMA 4.6 *Estimate* (3.9) *holds.*

PROOF: We note that, by Hölder's inequality, it readily follows that

$$\||\mathcal{D}u|^{p-2}D^2_* u\|_{\overline{p}} \leq \|\mathcal{D}u\|_p^{p-2}\|D^2_* u\|\,, \tag{4.35}$$

where $\overline{p}$ is given by (3.8). Hence, by (4.7), one gets

$$\| |\mathcal{D}u|^{p-2} D_*^2 u \|_{\overline{p}} \leq \|\mathcal{D}u\|_p^{p-2} c_n \nu_0^{-1} \| f, b\| \,. \tag{4.36}$$

From equation (4.25) written for $j = n$, we get an expression for $\frac{\partial \pi}{\partial x_n}$ in terms of functions already estimated. More precisely,

$$\begin{aligned} \left| \frac{\partial \pi}{\partial x_n} \right| &\leq c_n \left(\nu_0 + (p-1)\nu_1 |\mathcal{D}u(x)|^{p-2} \right) |D_*^2 u(x)| \\ &\quad + c_n (p-2) \nu_1 |\mathcal{D}u(x)|^{p-2} \sum_{l=1}^{n-1} \left| \frac{\partial^2 u_l}{\partial x_n^2} \right| + | f_n(x)| \quad \text{a.e. in } \mathbb{R}^n_+ \,. \end{aligned} \tag{4.37}$$

By appealing to (4.28), (4.29), and (4.32), we prove (3.7). Hence, by inequalities (2.16) and (4.36),

$$\begin{aligned} \left\| \frac{\partial \pi}{\partial x_n} \right\|_{\overline{p},R} &\leq c_{n,p} \left(\nu_0 \| D_*^2 u\| + \| f \| \right) |B_R^+|^{\frac{1}{\overline{p}} - \frac{1}{2}} \\ &\quad + c_{n,p} \frac{\nu_1}{\nu_0} \|\mathcal{D}u\|_p^{p-2} \| f, b\| + c_{n,p} \|\nabla^* \pi\|_{p',R} |B_R^+|^{\frac{1}{\overline{p}} - \frac{1}{p'}} \,. \end{aligned} \tag{4.38}$$

By appealing to (3.3) and (4.16) one proves (3.9). Note that

$$\left(\frac{\nu_1}{\nu_0} \right)^{\frac{1}{2}} \|\mathcal{D}u\|_p^{\frac{p-2}{2}} |B_R^+|^{\frac{1}{\overline{p}} - \frac{1}{p'}}$$

is bounded by the last term on the left-hand side of (3.9). □

PROOF OF THEOREM 3.1: Estimate (3.3) is just (4.15). Estimate (3.5) follows from (4.16) and (4.34). The inclusion of the derivatives $D_*^2 u$ on the left-hand side of (3.5) follows from (4.36) and from (2.16) and (4.7). □

5 Proof of Theorem 3.2

The above result may be improved if $2 < p \leq 2 + \frac{2}{n-1}$. Merely for convenience we will assume that $n = 3$. Hence, in the following we assume that $2 \leq p \leq 3$. Note that p is fixed, once and for all.

LEMMA 5.1 *Assume that* (3.2) *holds and let* (u, π) *be the corresponding solution to problem* (2.8) *under one of the boundary conditions* (1.6) *or* (1.10). $R > 0$ *is arbitrary but fixed. Assume that*

$$\mathcal{D}u \in L^q(B_R^+) \tag{5.1}$$

where

$$3 \leq q \leq 6 \,.$$

Then, besides (3.3)*, one has*

$$D^2 u, \, |\mathcal{D}u|^{p-2} \nabla^* \mathcal{D}u, \, \nabla^* \pi \in L^r(B_R^+) \tag{5.2}$$

where

$$\frac{1}{r} = \frac{p-2}{2q} + \frac{1}{2}\,. \tag{5.3}$$

More precisely,

$$\|\nabla^* \pi\|_{r,R} + \|D^2 u\|_{r,R} + \nu_1 \| |\mathcal{D}u|^{p-2} \nabla^* \mathcal{D}u\|_{r,R} \leq \mathcal{K}_r \tag{5.4}$$

where

$$\begin{aligned} \mathcal{K}_r &= c_n R\big(|B_R^+|^{\frac{1}{r}-\frac{1}{2}}[f,b] + \nu_1 |B_R^+|^{\frac{1}{r}-\frac{1}{p'}} \|\mathcal{D}u\|_p^{p-1}\big) \\ &\quad + c_n \left(|B_R^+|^{\frac{1}{r}-\frac{1}{2}} + \frac{\nu_1}{\nu_0} \|\mathcal{D}u\|_{q,R}^{\frac{p-2}{2}} \right) \|f,b\|\,. \end{aligned} \tag{5.5}$$

Note that $2 < p \leq 3$, $3 \leq q \leq 6$, and $2 \leq r \leq 3$. The lack of dependence of the constants c_n on p, q, and r follows from this fact, since the constants that appear in the embedding theorems used in what follows, as well as in (2.15), are uniformly bounded from above if the exponents in the Lebesgue spaces lie away from 1 and from ∞.

PROOF: The proof follows step by step that of Theorem 3.1. The proof remains unchanged until the end of the proof of Lemma 4.3. The main point is that now, in the proof of Lemma 4.4, assumption (5.1), together with Hölder's inequality, allows us to replace estimate (4.21) by

$$\left\| |\mathcal{D}u|^{p-2} \mathcal{D} \frac{\partial u}{\partial x_k} \right\|_{r,R} \leq \|\mathcal{D}u\|_{q,R}^{\frac{p-2}{2}} \left\| |\mathcal{D}u|^{\frac{p-2}{2}} \mathcal{D} \frac{\partial u}{\partial x_k} \right\|\,. \tag{5.6}$$

Hence, we start from the beginning of the proof of Lemma 4.4 by doing the above substitution, and we follow, step by step, the proofs given in the previous section (roughly speaking, we replace $L^{p'}$ norms by L^r norms and L^p norms by L^q norms).

More precisely, start from (5.6) instead of (4.21). Then replace (by following, in an obvious way, the corresponding proofs) equations (4.22), (4.23), (4.16), and (4.34) by, respectively,

$$\left\| |\mathcal{D}u|^{p-2} \mathcal{D} \frac{\partial u}{\partial x_k} \right\|_{r,R} \leq c_n \frac{1}{\nu_0} \|\mathcal{D}u\|_{q,R}^{(p-2)/2} \|f,b\|\,, \tag{5.7}$$

$$\|U_4\|_{r,R} \equiv \left\| \nu_1 \frac{\partial}{\partial x_k} (|\mathcal{D}u|^{p-2} \mathcal{D}u) \right\|_{r,R} \leq c_n (p-1) \frac{\nu_1}{\nu_0} \|\mathcal{D}u\|_{q,R}^{(p-2)/2} \|f,b\|\,, \tag{5.8}$$

$$\left\| \frac{\partial \pi}{\partial x_k} \right\|_{r,R} \leq \mathcal{K}_r\,, \tag{5.9}$$

$$\nu_0 \sum_{l=1}^{n-1} \left\| \frac{\partial^2 u_l}{\partial x_n^2} \right\|_{r,R} \leq \mathcal{K}_r\,. \tag{5.10}$$

Estimate (5.4) follows by appealing to (5.9) and (5.10). The proof of Lemma 5.1 is accomplished. □

Next we show the following result (for the reader's convenience, we set $\nu_0 = \nu_1 = 1$):

COROLLARY 5.2 *Assume that* (3.2) *holds and let* (u, π) *be a weak solution to problem* (1.11) *under one of the boundary conditions* (1.6) *or* (1.10). *In addition, assume that, for some* $R > 0$,

$$D^2 u \in L^s(B_R^+) \tag{5.11}$$

where

$$\frac{3}{2} \le s \le 3\,. \tag{5.12}$$

Then, besides (3.3), *one has*

$$D^2 u,\ \nabla^* \pi,\ |\mathcal{D}u|^{p-2} \nabla^* \mathcal{D}u \in L^r(B_R^+)$$

where

$$\frac{1}{r} = \frac{(p-2)(3-s)}{6s} + \frac{1}{2}\,. \tag{5.13}$$

More precisely,

$$\begin{aligned} &\|\nabla^* \pi\|_{r,R} + \|D^2 u\|_{r,R} + \||\mathcal{D}u|^{p-2} \nabla^* \mathcal{D}u\|_{r,R} \\ &\quad \le cR\big(|B_R^+|^{\frac{1}{r}-\frac{1}{2}} [f, b] + |B_R^+|^{\frac{1}{r}-\frac{1}{p'}} \|\mathcal{D}u\|_p^{p-1}\big) \\ &\qquad + c\big(|B_R^+|^{\frac{1}{r}-\frac{1}{2}} + |B_R^+|^{\frac{1}{r}-\frac{1}{p'}} \|\nabla u\|_p^{(p-2)/2} + \|D^2 u\|_{s,R}^{(p-2)/2}\big) \\ &\qquad \times (\|f\| + [b]_{1/2})\,, \end{aligned} \tag{5.14}$$

where the constant c is independent of p, s, r, *and* R.

PROOF: We start by noting that

$$\|g\|_{q,R} \le c\|\nabla g\|_{s,R} + c|B_R^+|^{\frac{1}{q}-\frac{1}{p}} \|g\|_{p,R} \tag{5.15}$$

where

$$\frac{1}{q} = \frac{1}{s} - \frac{1}{3}\,, \tag{5.16}$$

and the constant c is independent of R. In fact, by a Sobolev embedding theorem, $W^{1,s}(B_1^+)$ is continuously embedded in $L^q(B_1^+)$. Clearly, $\|\nabla g\|_s + \|g\|_p$ is a norm in $W^{1,s}(B_1^+)$, equivalent to the canonical one. Hence the above estimate holds for $R = 1$. The result for an arbitrary R follows by a scaling argument. By applying (5.15) to ∇u, we prove that (5.1) holds, where q is given by (5.16). Estimate (5.14) follows from (5.4). □

PROOF OF THEOREM 3.2: Define $r = \phi_p(s)$ by (5.13) and (5.16), i.e.,

$$r = \phi_p(s) := \frac{6s}{(5-p)s + 3(p-2)}$$

where $p \in [2,3]$. Note that s and r belong, at most, to the interval $[\frac{3}{2}, 3]$. The particular ranges of the parameters q, r, and s are not significant here. We merely want to note that these ranges remain away from 1 and from ∞. Note that, for $p > 2$, the function $\phi_p(s)$ is strictly increasing.

In what follows we define

$$r_1 = p', \quad r_{n+1} = \phi_p(r_n), \tag{5.17}$$

for each positive integer n. We observe that

$$\lim_{n\to\infty} r_n = l := 2 - \frac{p-2}{5-p}.$$

Next we define, for each positive integer n,

$$a_n = \|\nabla^*\pi\|_{r_n,R} + \|D^2 u\|_{r_n,R} + \||\mathcal{D}u|^{p-2}\nabla^*\mathcal{D}u\|_{r_n,R}. \tag{5.18}$$

Note that, by (3.5), it follows that

$$a_1 \le cR\big(|B_R^+|^{\frac{1}{p'}-\frac{1}{2}}[f,b] + \|\mathcal{D}u\|_p^{p-1}\big)$$
$$+ c\big(|B_R^+|^{\frac{1}{p'}-\frac{1}{2}} + \|\mathcal{D}u\|_p^{(p-2)/2}\big)(\|f\| + [b]_{1/2}).$$

From (5.14) one gets

$$a_{n+1} \le (C_0 + CE)B_n + Ea_n^\beta$$

where $E = c(\|f\| + [b]_{1/2})$, $B_n = |B_R^+|^{\frac{1}{r_{n+1}}}$, $\beta = \frac{p-2}{2}$ (hence $0 < \beta \le \frac{1}{2}$),

$$C_0 = cR\big(|B_R^+|^{-\frac{1}{2}}[f,b] + |B_R^+|^{-\frac{1}{p'}}\|\mathcal{D}u\|_p^{p-1}\big),$$

and

$$C = |B_R^+|^{-\frac{1}{2}} + |B_R^+|^{-\frac{1}{p'}}\|\mathcal{D}u\|_p^\beta.$$

$R > 0$ is arbitrary but fixed. Moreover, $B_n \to B := |B_R^+|^{1/l}$ as $n \to \infty$.

Set $b_1 = a_1$ and $b_{n+1} = (C_0 + CE)B_n + Eb_n^\beta$. Clearly, $a_n \le b_n$ for each n. On the other hand, it is easily shown that $\lim b_n = b$ where b is the solution of the equation

$$b = (C_0 + CE)B + Eb^\beta.$$

It is easily shown that

$$b \le 2(C_0 + CE)B + 4E^{\frac{1}{1-\beta}}.$$

Clearly, $\limsup a_n \le b$. By taking into account the definition of the a_n's and well-known properties of Lebesgue spaces L^α, it readily follows that

$$\|\nabla^*\pi\|_{L^l(B_R^+)} + \|D^2 u\|_{L^l(B_R^+)} \le 2(C_0 + CE)B + 4E^{\frac{1}{1-\beta}}.$$

This proves (3.11). Alternatively, we may set $b_{n+1} = (C_0 + CE)(B + \epsilon) + Eb_n^\beta$, with $\epsilon > 0$, show that the above inequality holds with B replaced by $B + \epsilon$, and let $\epsilon \to 0$.

Finally, we prove (3.14). By Hölder's inequality it follows that

$$\| |\mathcal{D}u|^{p-2} D_*^2 u \|_{m,R} \le \|\mathcal{D}u\|_{l^*,R}^{p-2} \|D_*^2 u\| \,.$$

By appealing to (3.3), (3.11), and (3.15), it readily follows that

$$\| |\mathcal{D}u|^{p-2} D_*^2 u \|_{m,R} \le C_R \big(\mathcal{K}_l + \| f, b\|^{\frac{2}{4-p}} + \|\nabla u\|_p\big)^{p-2} \| f, b\| \,. \tag{5.19}$$

Hence, from this last inequality together with (3.7) and (3.11), one easily gets

$$\begin{aligned}\left\| \frac{\partial \pi}{\partial x_n} \right\|_{m,R} &\le C_R \big(\mathcal{K}_l + \| f, b\|^{\frac{2}{4-p}} + \| f, b\|\big) \\ &\quad + C_R \big(\mathcal{K}_l + \| f, b\|^{\frac{2}{4-p}} + \|\nabla u\|_p\big)^{p-2} \| f, b\| \,.\end{aligned}$$

Note that $\|\nabla^* \pi\|_{m,R} \le \|\nabla^* \pi\|_{l,R}$, since $m \le l$. Finally, by taking into account the expression of $\mathcal{K}_l$ and by appealing to Young's inequalities, one proves (3.14). Note that, in particular,

$$\|\mathcal{D}u\|_p^{(p-2)/2} \| f\| \le c\big(\|\mathcal{D}u\|_p + \| f\|^{\frac{2}{4-p}}\big) \,.$$

□

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Bibliography

[1] Amann, H. Stability of the rest state of a viscous incompressible fluid. *Arch. Rational Mech. Anal*. **126** (1994), no. 3, 231–242.

[2] Avantaggiati, A. Spazi di Sobolev con peso ed alcune applicazioni. *Boll. Un. Mat. Ital. (5)* **13A** (1976), no. 1, 1–52.

[3] Beavers, G. S.; Joseph, D. D. Boundary conditions at a naturally permeable wall. I. *J. Fluid Mech.* **30** (1967), 197–207.

[4] Beirão da Veiga, H. Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip type boundary conditions. *Adv. Differential Equations* **9** (2004), no.9-10, 1079–1114.

[5] Beirão da Veiga, H. Regularity of solutions to a nonhomogeneous boundary value problem for general Stokes systems in $\mathbb{R}^n_+$. *Math. Ann.*, in press.

[6] Beirão da Veiga, H. On the regularity of flows with Ladyzhenskaya shear dependent viscosity and slip and non-slip boundary conditions. Part II: The evolution problem. In preparation.

[7] Bellout, H.; Bloom, F.; Nečas, J. Young measure-valued solutions for non-Newtonian incompressible fluids. *Comm. Partial Differential Equations* **19** (1994), no. 11-12, 1763–1803.

[8] Conca, C. On the application of the homogenization theory to a class of problems arising in fluid mechanics. *J. Math. Pures Appl. (9)* **64** (1985), no. 1, 31–75.

[9] Consiglieri, L. Weak solutions for a class of non-Newtonian fluids with energy transfer. *J. Math. Fluid Mech*. **2** (2000), no. 3, 267–293.

[10] Diening, L.; Prohl, A.; Růžička, M. On time-discretizations for generalized Newtonian fluids. *Nonlinear problems in mathematical physics and related topics, II*, 89–118. International Mathematical Series (New York), 2. Kluwer/Plenum, New York, 2002.

[11] Frehse, J.; Málek, J. Problems due to the no-slip boundary in incompressible fluid dynamics. *Geometric analysis and nonlinear partial differential equations*, 559–571. Springer, Berlin, 2003.

[12] Frehse, J.; Málek, J.; Steinhauer, M. An existence result for fluids with shear dependent viscosity—steady flows. *Nonlinear Anal*. **30** (1997), 3041–3049.

[13] Fuchs, M.; Seregin, G. Some remarks on non-Newtonian fluids including nonconvex perturbations of the Bingham and Powell-Eyring model for viscoplastic fluids. *Math. Models Methods Appl. Sci*. **7** (1997), no. 3, 405–433.

[14] Fujita, H. Remarks on the Stokes flow under slip and leak boundary conditions of friction type. *Topics in mathematical fluid mechanics (Napoli, 2000)*, 73–94. Quaderni di Matematica, 10. Napoli, 2002.

[15] Galdi, G. P. *An introduction to the mathematical theory of the Navier-Stokes equations*. I. Linearized steady problems. 2nd ed. Springer Tracts in Natural Philosophy, 38. Springer, Berlin–New York, 1998.

[16] Galdi, G. P.; Layton, W. J. Approximation of the larger eddies in fluid motions. II. A model for space-filtered flow. *Math. Models Methods Appl. Sci*. **10** (2000), no. 3, 343–350.

[17] Guo, B.; Zhu, P. Partial regularity of suitable weak solutions to the system of the incompressible non-Newtonian fluids. *J. Differential Equations* **178** (2002), no. 2, 281–297.

[18] Hanouzet, B. Espaces de Sobolev avec poids application au problème de Dirichlet dans un demi espace. *Rend. Sem. Mat. Univ. Padova* **46** (1971), 227–272.

[19] Hughes, T. J. R.; Mazzei, L.; Oberai, A. A.; Wray, A. The multiscale formulation of large eddy simulation: decay of homogeneous isotropic turbulence. *Phys. Fluids* **13** (2001), no. 2, 505–512.

[20] John, V. Slip with friction and penetration with resistance boundary conditions for the Navier-Stokes equations—numerical tests and aspects of the implementation. *J. Comput. Appl. Math*. **147** (2002), no. 2, 287–300.

[21] John, V. *Large eddy simulation of turbulent incompressible flows*. Analytical and numerical results for a class of LES models. Lecture Notes in Computational Science and Engineering, 34. Springer, Berlin, 2004.

[22] Ladyhzenskaya, O. A. Sur de nouvelles équations dans la dynamique des fluides visqueux et leurs résolution globale. *Trudy Mat. Inst. Steklov.* **102** (1967), 85–104.

[23] Ladyzhenskaya, O. A. On nonlinear problems of continuum mechanics. *Proc. Int. Congr. Math. (Moscow, 1966)*, 560–573. Nauka, Moscow, 1968; translation in *Amer. Math. Soc. Transl. (2)* **70**, 1968.

[24] Ladyzhenskaya, O. A. Sur des modifications des équations de Navier-Stokes pour des grand gradients de vitesses. *Sem. Math. V. A. Steklov* **7** (1968), 126–154.

[25] Ladyzhenskaya, O. A. *The mathematical theory of viscous incompressible flow*. 2nd ed. Mathematics and Its Applications, 2. Gordon and Breach, New York–London–Paris, 1969.

[26] Ladyzhenskaya, O. A.; Seregin, G. A. On the smoothness of systems describing flows of generalized Newtonian fluids, and on the estimation of the dimensions of their attractors. *Izv. Ross. Akad. Nauk Ser. Mat*. **62** (1998), no. 1, 59–122; translation in *Izv. Math*. **62** (1998), no. 1, 55–113.

[27] Ladyzhenskaya, O. A.; Seregin, G. A. On the regularity of solutions of two-dimensional equations of the dynamics of fluids with nonlinear viscosity. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **259** (1999), *Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 30*, 145–166, 298.

[28] Liakos, A. Discretization of the Navier-Stokes equations with slip boundary condition. *Numer. Methods Partial Differential Equations* **17** (2001), no. 1, 26–42.

[29] Lions, J.-L. *Problèmes aux limites dans les équations aux dérivées partielles.* 2nd ed. Séminaire de Mathématiques Supérieures, 1 (Été, 1962). University of Montreal, Montreal, 1965.

[30] Lions, J.-L. Sur certaines équations paraboliques non linéaires. *Bull. Soc. Math. France* **93** (1965), 155–175.

[31] Lions, J.-L. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.

[32] Málek, J.; Nečas, J.; Rajagopal, K. R. Global analysis of the flows of fluids with pressure-dependent viscosities. *Arch. Ration. Mech. Anal*. **165** (2002), no. 3, 243–269.

[33] Málek, J.; Nečas, J.; Rokyta, M.; Růžička, M. Weak and measure-valued solutions to evolutionary PDEs. Applied Mathematics and Mathematical Computation, 13. Chapman and Hall, London, 1996.

[34] Málek, J.; Nečas, J.; Růžička, M. On the non-Newtonian incompressible fluids. *Math. Models Methods Appl. Sci*. **3** (1993), no. 1, 35–63.

[35] Málek, J.; Nečas, J.; Růžička, M. On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case $p \geq 2$. *Adv. Differential Equations* **6** (2001), no. 3, 257–302.

[36] Málek, J.; Rajagopal, K. R.; Růžička, M. Existence and regularity of solutions and the stability of the rest state for fluids with shear dependent viscosity. *Math. Models Methods Appl. Sci*. **5** (1995), no. 6, 789–812.

[37] Nečas, J. *Équations aux dérivées partielles*. University of Montreal, Montreal, 1965.

[38] Nirenberg, L. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa (3)* **13** (1959), 115–162.

[39] Parés, C. Existence, uniqueness and regularity of solution of the equations of a turbulence model for incompressible fluids. *Appl. Anal*. **43** (1992), no. 3-4, 245–296.

[40] Pokorný, M. Cauchy problem for the non-Newtonian viscous incompressible fluid. *Appl. Math*. **41** (1996), no. 3, 169–201.

[41] Růžička, M. A note on steady flow of fluids with shear dependent viscosity. *Nonlinear Anal*. **30** (1997), no. 5, 3029–3039.

[42] Růžička, M. *Electrorheological fluids: modeling and mathematical theory*. Lecture Notes in Mathematics, 1748. Springer, Berlin, 2000.

[43] Saito, H; Scriven, L. E. Study of the coating flow by the finite element method. *J. Comput. Phys.* **42** (1981), 53–76.

[44] Seregin, G. A. Interior regularity for solutions to the modified Navier-Stokes equations. *J. Math. Fluid Mech*. **1** (1999), no. 3, 235–281.

[45] Serrin, J. Mathematical principles of classical fluid mechanics. *Handbuch der Physik (herausgegeben von S. Flügge),* 8/1. Strömungsmechanik I (Mitherausgeber C. Truesdell), 125–263. Springer, Berlin-Göttingen-Heidelberg, 1959.

[46] Silliman, W. J.; Scriven, L. E. Separating flow near a static contact line: slip at a wall and shape of a free surface. *J. Comput. Phys.* **34** (1980), no. 3, 287–313.

[47] Simader, C. G.; Sohr, H. *The Dirichlet problem for the Laplacian in bounded and unbounded domains*. A new approach to weak, strong and $(2+k)$-solutions in Sobolev-type spaces. Pitman Research Notes in Mathematics Series, 360. Longman, Harlow, 1996.

[48] Smagorinsky, J. S. General circulation experiments with the primitive equations. I. The basic experiment. *Mon. Weather Rev.* **91** (1963), no. 3, 99–164.

[49] Solonnikov, V. A.; Ščadilov, V. E. A certain boundary value problem for the stationary system of Navier-Stokes equations. Boundary Value Problems of Mathematical Physics, 8. *Trudy*

Mat. Inst. Steklov. **125** (1973), 196–210; translation in *Proc. Steklov Inst. Math.* **125** (1973), 186–199.

[50] Stokes, G. *Trans. Cambridge Phil. Soc.*, 8, 287, (1845), 75–129.

[51] Verfürth, R. Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition. *Numer. Math.* **50** (1987), no. 6, 697–721.

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