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Developable Surfaces as Generators of the "Isobaric Solutions" to the Euler Equations

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Abstract. The simplest solutions to the Euler equations (1.1) for which the pressure vanishes identically are those representing the motion of lines parallel to a fixed direction $\bar{r}$ moving in the same direction (each line with an independent, given, constant velocity). Are there many other solutions to this problem? If yes, is there a simple characterization of all the initial data (volume Ω occupied by the fluid at time $t = 0$ and initial velocity $u_0(x), x \in \Omega$) that gives rise to the general solutions? In this paper we show that the answer to both questions is positive. We prove, in particular, that there is a natural correspondence between solutions in R^2 of this problem and (Cartesian pieces of) developable surfaces in R^3. See Theorem 3.

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1. Introduction

The aim of the present paper is to investigate whether there is a simple characterization of the solutions to the Euler equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)\, u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

for which the pressure p vanishes identically or, more generally, the pressure depends only on time. At odds with the intuition, perhaps, we show that this problem admits many non-trivial solutions. Moreover, we give a simple and elegant geometrical characterization of all initial data $\{u^0, \Omega\}$ that generate these solutions. We call these solutions *isobaric solutions to the Euler incompressible equations.*

Obviously, the fluid is not assumed to be contained in a vessel. At an initial time $t = 0$ the fluid occupies a given volume Ω and has a given initial velocity u^0. The position Ω_t occupied by the fluid at any time $t \geq 0$ and its velocity field, at the same time, are unknowns that depend only on the initial data $\{u^0, \Omega\}$. Hence, we look for initial data $\{u^0, \Omega\}$ that generate (locally in time, at least) solutions to problem (1.1) with $p = 0$.

Assume that initial data $\{u^0, \Omega\}$ are given, where $\Omega \subset \mathbf{R}^n$, $n \geq 2$, is an open, connected set, locally situated on one side of its boundary and u^0 is an n-dimensional vector field defined in Ω. For each fixed $t \geq 0$ the solution $u(t, \cdot)$ to the above problem turns out to be defined in a domain Ω_t diffeomorphic to Ω. Since $\nabla \cdot u = 0$, the measure of Ω_t must not depend on t. Note that a regular solution of (1.1) with $p = 0$, if exists, is unique (even without the divergence-free condition).

Even though our results are valid in arbitrary dimension $n \geq 2$ (see Theorem 2), they assume a particularly attractive geometrical interpretation in the two-dimensional case. We shall therefore describe our main achievements for $n = 2$. To this end, denote by $\vec{e_1}$ and $\vec{e_2}$ the unit vectors in the x_1 and in the x_2 directions, respectively. As a starting point of our analysis we consider, in the whole plane $\mathbf{R}^2$, the following *trivial solutions* to our problem: each straight line in $\mathbf{R}^2$, parallel to a fixed direction $\overline{r}$, moves in this same direction (as a rigid body) with a given constant speed (that may change from line to line without any particular relation between the speeds of distinct lines). Clearly, $\Omega_t = \Omega$. Let us construct the same trivial solutions in a less simple way which, however, can be extended to the general case.

Assume that $\overline{r}$ is the x_1 axis and label each straight line r in $\mathbf{R}^2$, parallel to x_1, by its x_2 coordinate, say $r(x_2)$. Denote by $\psi(x_2)\vec{e_1}$ the velocity of the line $r(x_2)$ in the x_1 direction. Then, the above *trivial solutions* to our problem may be obtained also as follows. Let γ be the curve (in the $x_1 = 0$ plane) defined by the equation $x_3 = \phi(x_2)$, where ϕ is defined by $\phi'(x_2) = \psi(x_2)$. An additional constant has no effect here. Consider in $\mathbf{R}^3$ the cylinder $\mathcal{S}$ consisting of all the straight lines $l(x_2)$ parallel to the x_1 direction and intersecting the generatrix γ at the point $x_3 = \phi(x_2)$. Hence $\mathcal{S}$ is defined by an equation $x_3 = \Phi(x_1, x_2)$ where, in this particular case, $\Phi(x_1, x_2) = \phi(x_2)$. Clearly, $r(x_2)$ is the orthogonal projection of $l(x_2)$ into the $\{x_1, x_2\}$ plane. The planar motion described above consists just in impressing to each straight line $r(x_2)$ a velocity equal to a clockwise $\pi/2$ rotation of $\nabla\Phi = \phi'(x_2)\vec{e_2}$, i.e. equal to $\phi'(x_2)\vec{e_1}$.

A little more general "trivial solution" is obtained by imparting to each of the above moving lines $r(x_2)$ an additional velocity $c\vec{e_2}$ in the x_2 direction, where c is a given constant. In terms of the above construction, the addition of this velocity can be easily treated just by giving to the previous cylinder $\mathcal{S}$ a slope in the x_1 direction equal to the desired additional speed. In fact, if the equation of the modified cylinder is given by $x_3 = \phi(x_2)$ - cx_1, its gradient is just $\nabla\Phi = \phi'(x_2)\vec{e_2} + c\vec{e_1}$. A clockwise $\pi/2$ rotation gives now the desired velocity $\phi'(x_2)\vec{e_1} + c\vec{e_2}$.

Clearly, we may consider only a regular, *Cartesian*, portion S of the cylinder $\mathcal{S}$ and take as initial domain Ω just the projection of S into $\mathbf{R}^2$.

In the sequel we show that the general solution to our problem is obtained just by replacing, in the above construction, the cylinder $\mathcal{S}$ by any other (piece of) Cartesian developable surface in R^3 (more precisely, surface with vanishing Gaussian curvature).

For the readers convenience we briefly recall here some results on the classical theory of curves and surfaces. For more details we refer to any classical treatise on Differential Geometry (for a particularly elementary description of the basic properties see [11], Chapter 11).

In general, a *ruled surface* ([11], 8.4) is a surface generated by the motion of a line in the space. Each position of the moving line defines a *ruling*. *Developable surfaces* are ruled surfaces for which the tangent plane is constant along each ruling (see [11], 11.25). Cylinders and cones are the simplest, trivial developable surfaces. Cylinders are surfaces generated by a line l as it moves, parallel to itself, along a curve γ. Cones are generated by a line l which pass through a fixed point P and moves along a curve γ. Non trivial developable surfaces are obtained from any generical twisted (non-planar) curve γ in R^3 as the geometrical locus of the tangent lines to this curve ([11], 11.26). *Developable regular surfaces* are also characterized by having vanishing *Gaussian curvature* K ([11], 11.27). If we are considering a Cartesian surface $x_3 = \Phi(x_1, x_2)$, this means that

$$\frac{\partial^2 \Phi}{\partial x_1^2}\frac{\partial^2 \Phi}{\partial x_2^2} - \left(\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}\right)^2 = 0, \qquad \forall\, (x_1, x_2) \in \Omega. \tag{1.2}$$

Next we use the construction of the above particular solution u as a starting point to the construction of the general solutions. More precisely, we will replace the above cylinder by an arbitrary (Cartesian) developable surface $x_3 = \Phi(x_1, x_2)$.

By taking into account that the lines $l(x_2)$ are just the cylinder rulings, our construction is equivalent to consider the projection $r(x_2)$ of each ruling $l(x_2)$ into the plane of the motion $\mathbf{R}^2$ and then imparting to it a constant velocity equal to a clockwise $(\pi/2)$-rotation of $\nabla\Phi$. Later in this paper, we will show that this construction still gives rise to solutions of our problem if the cylinder $\mathcal{S}$ is replaced by any regular, *Cartesian*, portion of a developable surface S. Conversely, each solution of problem (1.1), for which $p = 0$, can be obtained in that way. (Clearly, in the general case, the rulings may be lines, half lines or even linear segments.) Consequently, for planar motions, we show that there is a one-to-one correspondence between initial data $\{u^0, \Omega\}$ generating solutions to the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)\, u = 0 \quad \text{in } \Omega_t, \\ \nabla \cdot u = 0, \\ u(x, 0) = u^0(x) \quad \text{in } \Omega, \end{cases} \tag{1.3}$$

and Cartesian regular surfaces $x_3 = \Phi(x_1, x_2)$ with vanishing Gaussian curvature. This shows that there are many non-trivial solutions to the above problem (1.3), even for planar motions. In addition, it seems interesting that the above solutions are characterized by such a simple geometric property.

Summarizing. Let $x_3 = \Phi(x_1, x_2)$, where $x \equiv (x_1, x_2) \in \Omega$, describes a regular Cartesian portion S of a developable surface $\mathcal{S}$ in $\mathbf{R}^3$. Hence the gradient $\nabla\Phi$ is constant along each P-ruling (for convenience, we call P-rulings the orthogonal projections of the rulings into the (x_1, x_2)-plane, i.e. into Ω). The solution u to the problem (1.3), generated by Φ, is constructed as follows. The initial velocity $u^0(x_1, x_2)$ is defined in Ω by setting

$$u_1^0 = \frac{\partial \Phi}{\partial x_2}, \qquad u_2^0 = -\frac{\partial \Phi}{\partial x_1}, \tag{1.4}$$

i.e., the initial velocity is a clockwise $\pi/2$ rotation of $\nabla\Phi$. Moreover, the motion, for $t \geq 0$, is obtained by impressing to each P-ruling the above velocity u^0. Clearly, the magnitude and direction of the velocity change from P-ruling to P-ruling.

Remark 1. If $\mathcal{S}$ is a cone, its vertex P is a singular point. Hence a regular Cartesian portion S of $\mathcal{S}$ does not contain the vertex P. A similar situation holds in the case of surfaces generated by the tangent lines to a twisted curve γ since points in γ are singular (cusp points). Moreover in a neighborhood of γ the surface $\mathcal{S}$ is two-folded. Consequently a regular Cartesian portion S of $\mathcal{S}$ must be part of one of the two folds (generated respectively by the "forward" or by the "backward" half-rulings). Clearly, if (for instance) Ω is bounded, the intersection of each P-ruling with Ω is a linear segment. For convenience, as already said, we call these segments (or half-lines) P-rulings. Each of these P-rulings moves, as explained above, with a characteristic velocity given by (1.4). If two of these moving P-rulings intersect, the regular solution $u(x,t)$ blows up (there is a shock). As the reader easy verifies (by constructing simple examples) the solution $u(x,t)$ may be local or global in time, depending not only on the ruled surface $\mathcal{S}$, but also on the chosen Cartesian portion S. It is worth noticing that it is not possible to obtain a shock inside a *fixed* domain

$$\Omega^0 \subset \bigcup_{0 \leq t \leq t_0} \Omega_t$$

at a time $t_0 > 0$ if the boundary values on $\partial\Omega^0$ are regular up to time t_0. In fact, if two moving segments collide inside Ω^0 they must collide (at a previous time) on a boundary point of Ω^0.

Finally we would like to give some bibliography concerning the Euler equations, with a particular regard to the pioneering papers on the subject. However, the list of references is far from being exhaustive.

As far as we know, the first mathematical article on the Euler equations is that of Lichtenstein [10]. This author considers the problem in the whole space R^3 and assumes that the initial data are smooth and compactly supported. Under these assumptions he proved the existence of a unique local (in time) solution. Other classical results in this direction are due to Gyunter (see [7] for a list of references). Further classical, fundamental contributions are those of Wolibner [15], Hölder [6], Leray [9], and Shaeffer [12]. In particular, these authors show the existence of a global solution in Hölder spaces. For more recent papers, we refer to [7], [8], [5], [1], [4], [13], [14], [3], [2].

2. Results and proofs

We denote by $C^1(\overline{\Omega})$ the set of continuously differentiable vector fields w in $\overline{\Omega}$, $D_x w$ denotes the matrix whose i-row and j-column element is $\partial w^i/\partial x_j$ (the Jacobian matrix), and $[w]$ denotes the Lipschitz norm of the vector field w on $\overline{\Omega}$. We shall indicate the Lagrangian coordinates by (t, α). Define

$$T = \frac{1}{[u^0]},$$

and, for each $\alpha \in \Omega$, set

$$\begin{cases} x(t,\alpha) = \alpha + t\, u^0(\alpha), \\ u(t, x(t,\alpha)) = u^0(\alpha). \end{cases} \tag{2.1}$$

It readily follows that

$$\frac{dx(t,\alpha)}{dt} = u(t, x(t,\alpha))$$

and that $x(0,\alpha) = \alpha$. There is a shock of two characteristics $x(t,\alpha)$ and $x(t,\alpha')$ at time t if

$$\alpha + t\, u^0(\alpha) = \alpha' + t\, u^0(\alpha')$$

with $\alpha \neq \alpha'$. Since $u^0 \in C^1(\overline{\Omega})$ this would imply that

$$|\alpha - \alpha'| = t|u^0(\alpha) - u^0(\alpha')| \leq t\,[u^0]\,|\alpha - \alpha'|.$$

Hence there are no shocks for $t < T$. Consequently there is a (unique) local regular solution $u(t,x)$ of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)\, u = 0 \quad \text{in } \Lambda_T, \\ u(x,0) = u^0(x) \quad \text{in } \Omega, \end{cases} \tag{2.2}$$

given by $u(t, \alpha + t\, u^0(\alpha)) = u^0(\alpha)$, where

$$\Lambda_T = \Big\{ (t, \alpha + t\, u^0(\alpha)) : \quad (t,\alpha) \in [0, T[\times \Omega) \Big\}.$$

The region occupied by the fluid at time t is

$$\Omega_t = \Big\{ x = \alpha + t\, u^0(\alpha) : \quad \alpha \in \Omega \Big\}.$$

Note that $\Omega_0 = \Omega$. Our problem is now reduced to find necessary and sufficient conditions on the initial data $u^0(\alpha)$ in order that the solution u of our problem (2.2) satisfy

$$\nabla \cdot u = 0 \quad \text{in } \Lambda_T. \tag{2.3}$$

One has the following result

Theorem 2. *Let $u^0 \in C^1(\overline{\Omega})$ be a divergence-free n-vector field in Ω, for $\Omega \subset \mathbf{R}^n$, $n \geq 2$. Then the solution $u(t,x)$ to problem* (2.2) *(which exists and is unique, at least for $t \in [0,T[$) solves problem* (1.3) *if and only if the eigenvalues of the Jacobian matrix Du^0 vanishes on Ω.*

Theorem 3. *Assume in Theorem 2 that $n = 2$ and that Ω is simply connected. Then the solution $u(t,x)$ to problem* (2.2) *solves problem* (1.3)*, i.e. u^0 satisfies the hypothesis of Theorem 2, if and only if*

$$u_1^0 = \frac{\partial \Phi}{\partial x_2}, \qquad u_2^0 = -\frac{\partial \Phi}{\partial x_1} \tag{2.4}$$

for some $\Phi(x_1,x_2)$ satisfying (1.2) *on Ω, i.e. for some $\Phi(x_1,x_2)$ such that the surface $x_3 = \Phi(x_1,x_2)$ has vanishing Gaussian curvature.*

Proof of Theorem 2. Let $u(t,x)$ be the above solution to problem (2.2). Since $\alpha = \alpha(t,x)$ is invertible (at least for each fixed $t \in [0,T[$) one easily gets from $u(t,x(t,\alpha)) = u^0(\alpha)$ that

$$(D_x u)(t,x) = (D_\alpha u^0)_{|\alpha=\alpha(x,t)} \cdot D_x\alpha(t,x). \tag{2.5}$$

On the other hand, since $\alpha + t\,u_0(\alpha) = x$, it follows that

$$\left[I + t\,(D_\alpha u^0)_{|\alpha=\alpha(x,t)}\right] \cdot (D_x\alpha(x,t)) = I.$$

Hence, for each fixed t,

$$D_x\alpha(t,x) = \left[I + t\,(D_\alpha u^0)_{|\alpha=\alpha(x,t)}\right]^{-1}. \tag{2.6}$$

From equations (2.5) and (2.6) it follows that

$$D_x u = D_\alpha u^0 \cdot \left(I + t\,D_\alpha u^0\right)^{-1}. \tag{2.7}$$

Next we introduce the constraint (2.3). By assumption, $\nabla_\alpha \cdot u^0(\alpha) = 0$, where the subscript "$_\alpha$" denotes that the corresponding derivatives are taken with respect to the to α-variables. Applying the divergence operator $\nabla_\alpha\cdot$ to both sides of the equation $u(t,x(t,\alpha)) = u^0(\alpha)$ by the use of (2.1) it readily follows that

$$(\nabla \cdot u)_{|x=\alpha+t\,u^0(\alpha)} + t\,\mathrm{tr}\left[(D_x u)_{|x=\alpha+t\,u^0(\alpha)} \cdot (D_\alpha u^0)(\alpha)\right] = 0,$$

where $\mathrm{tr}\,A$ denotes the trace of the matrix A. Hence, $(\nabla_x \cdot u)(t,x) \equiv 0$ if and only if

$$\mathrm{tr}\left[(D_x u)(t,x(t,\alpha) \cdot D_\alpha u^0(\alpha)\right] = 0. \tag{2.8}$$

With the help of (2.7) we show that (2.8) can be written in the form

$$\mathrm{tr}\left[D_\alpha u^0(\alpha) \cdot \left(I + t\,D_\alpha u^0\right)^{-1} \cdot D_\alpha u^0(\alpha)\right] = 0, \tag{2.9}$$

for each $\alpha \in \Omega$. Hence (2.9) is a necessary and sufficient condition (in terms of the initial data $u^0(x)$) for the validity of (2.3).

Next we show that this property holds in $[0, T[$, for some $T > 0$, if and only if the eigenvalues $\lambda_1, \dots, \lambda_n$ of $D_\alpha u^0$ vanish on Ω. By assumption $\operatorname{tr} D_\alpha u^0 = \nabla_\alpha \cdot u^0$ vanishes on Ω.

Before going on with the proof of Theorem 2 we establish the following auxiliary result.

Lemma 4. *Let B be a (numerical) $n \times n$ square matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. The following properties* (2.10) *and* (2.11) *are equivalent:*

$$tr\, B = 0, \quad \textit{and} \quad tr\, \left[B(I - t\,B)^{-1} B \right] = 0, \quad \forall\, t \in [0, T[, \tag{2.10}$$

for some $T > 0$;

$$\lambda_k = 0, \quad \forall\, k = 1, \dots n. \tag{2.11}$$

Proof of Lemma 4. Denote by $\|B\|$ the norm of the matrix B as a linear transformation on $\mathbf{R}^n$. For $t\,\|B\| < 1$, one has, by using the classical Neumann expansion,

$$\operatorname{tr}\, \left[B(I - t\,B)^{-1} B \right] = \frac{1}{t^2} \sum_{k=2}^{\infty} (\operatorname{tr} B^k) t^k.$$

This shows that (2.10) is equivalent to

$$\operatorname{tr} B^k = 0, \quad \forall\, k \geq 1. \tag{2.12}$$

Next, from Cayley and Hamilton's theorem it follows that (2.12) holds for each $k \geq 1$ if it holds for $1 \leq k \leq n$. Let $\overline{B}$ be an upper triangular matrix, similar to B. Since the eigenvalues of similar matrices coincide and, moreover, $\overline{B}^k$ is similar to B^k, it follows that B satisfies the property (2.12) for the desired values k if and only if the triangular matrix $\overline{B}$ satisfies the same property. This means that its eigenvalues vanish, i.e. $\lambda_1^k + \cdots + \lambda_n^k = 0$, $k = 1, \dots, n$. This, in turn, is equivalent to (2.11). The proof of the Lemma is completed. □

Finally, we apply the above Lemma to the matrices $B = -D_\alpha u^0$, for each fixed $\alpha \in \Omega$. Recall that $\operatorname{tr} D_\alpha u^0 = 0$. The above Lemma shows that the necessary and sufficient condition (2.9) (to establish (2.3)) is equivalent to requiring that all the eigenvalues of the Jacobian matrix Du^0 vanish on Ω. Note that the positive value T is independent of α since $\|D_\alpha u^0\|$ is uniformly bounded in Ω. This proves Theorem 2 is proved. □

Remark 5. Note that if assumption (2.10) holds for some positive T then it holds for all $t \geq 0$. This is due to the equivalence to property (2.11). In fact, from the characteristic equation $\lambda^n = 0$, it follows that $B^n = 0$, hence $(I - t\,B)^{-1} = \sum_{k=0}^{n-1} t^k\, B^k$, for each t.

Proof of Theorem 3. The eigenvalues of $Du^0(x)$ vanish in Ω if and only if

$$\frac{\partial u_1^0}{\partial x_1} + \frac{\partial u_2^0}{\partial x_2} = 0$$

and

$$\frac{\partial u_1^0}{\partial x_1}\frac{\partial u_2^0}{\partial x_2} - \frac{\partial u_2^0}{\partial x_1}\frac{\partial u_1^0}{\partial x_2} = 0.$$

If Ω is simply connected, the first equation means that (2.4) holds for some real function $\Phi(x_1, x_2)$. Hence, the second equation can be written in the form

$$\frac{\partial^2 \Phi}{\partial x_1^2}\frac{\partial^2 \Phi}{\partial x_2^2} - \left(\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}\right)^2 = 0 \quad \text{in } \Omega,$$

which, in turn means that the Cartesian surface $x_3 = \Phi(x_1.x_2)$ has vanishing Gaussian curvature. □

Remark 6. One can show that the P-ruling passing through a given point $x_0 \in \Omega$ has just the direction of the (unique) eigenvector of the Jacobian matrix $(Du^0)(x_0)$.

Finally, let us give an explicit form to the velocity field u^0 that corresponds to a given twisted curve γ, represented in $\mathbf{R}^3$ by $(x(\tau), y(\tau), z(\tau))$, $a < \tau < b$. In the x, y-plane the forward (half) P-ruling passing through the point $(x(\tau), y(\tau))$ consists of points $(x(\tau) + sx'(\tau), y(\tau) + sy'(\tau))$, $s > 0$ ($s < 0$ corresponds to the "backward" P-rulings). Assume we make a suitable restriction on the values of s (for instance, of type $0 < \alpha(\tau) < s < \beta(\tau)$), that ensures that distinct P-rulings do not intersect (now the initial domain Ω is well defined). By our results it follows that the P-ruling corresponding to the value τ of the parameter moves with a constant velocity $u^0(\tau)$ given by

$$u^0(\tau) = \frac{1}{x'y'' - y'x''}(x'z'' - z'x'', y'z'' - z'y''),$$

where the prime denotes differentiation with respect to τ.

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