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On the Smoothness of a Class of Weak Solutions to the Navier–Stokes equations

Hugo Beirão da Veiga

Abstract. We improve regularity criteria for weak solutions to the Navier–Stokes equations stated in references [1], [3] and [12], by using in the proof given in [3], a new idea introduced by H. O. Bae and H. J. Choe in [1]. This idea allows us, in one of the main hypothesis (see eq. (1.7)), to replace the velocity u by its projection $\bar{u}$ into an arbitrary hyperplane of $\mathbb{R}^n$; see Theorem A. For simplicity, we state our results for space dimension $n \leq 4$, since if $n \geq 5$ the proofs become more technical and additional hypotheses are needed. However, for the interested reader, we will present the formal calculations for arbitrary dimension n.

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1. Introduction

In the sequel we consider the Navier–Stokes equations in $(a, b) \times \mathbb{R}^n$, $n \leq 4$, namely

$$\left.\begin{array}{l} \partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0, \\ \nabla \cdot u = 0. \end{array}\right\} \tag{1.1}$$

We assume, for simplicity, that the external forces f are potential-like. However, it is not difficult to include non-potential external forces, under appropriate assumptions.

The problem treated here goes back to the classical works [15] and [16], and to their further development as, for instance, the well known result stating that $L^q(a, b; L^p), 2/q + n/p \leq 1,\ n < p$, is a regularity class for weak solutions to the Navier–Stokes equations (for a very simple proof see [3]). However, it remains open the case $p = n, q = +\infty$. In some sense the results proved in this note are along this line of research.

The well known symbols $L^p(\mathbb{R}^n),\ 1 \leq p \leq +\infty,\ H^k(\mathbb{R}^n)$, k positive integer, and $L^p(a, b; X)$, where X is a Banach space, stand for classical functional Lebesgue and Sobolev spaces, and will not be defined here. In the sequel, we shall omit the symbol $\mathbb{R}^n$. The canonical norm in L^p is denoted by $\| \cdot \|_p$. In our notation we will not distinguish between spaces consisting of scalar or vector functions. For instance we denote the space $(L^p)^n = L^p \times \ldots \times L^p$ (n times) simply by L^p. The same convention applies to norms.

$C(\alpha, \beta; X)$, X a Banach space, denotes the functional space consisting of continuous funtions on $[\alpha, \beta]$ with values in X. $C_w(\alpha, \beta; X)$ denotes the linear subspace of $L^\infty(\alpha, \beta; X)$ consisting of all the weakly continuous functions in $[\alpha, \beta)$ with values in X.

We say that u is a weak solution in (α, β) to the Navier–Stokes equations (1.1) if

$$u \in C_w(\alpha, \beta; L^2) \cap L^2(\alpha, \beta; H^1) \tag{1.2}$$

satisfies (1.1) in the usual distributional sense in (α, β) for some distribution $p(t, x)$.

We say that a weak solution in (α, β) is a strong solution if, moreover,

$$\left.\begin{aligned} &u \in L^2(\alpha, \beta; H^2), \\ &\partial_t u \in L^2(\alpha, \beta; L^2). \end{aligned}\right\} \tag{1.3}$$

In particular, it follows from (1.3) that strong solutions satisfy

$$\left.\begin{aligned} &u \in C(\alpha, \beta; H^1), \\ &\nabla u \in C(\alpha, \beta; L^2), \\ &\partial_t \nabla u \in L^2(\alpha, \beta; H^{-1}). \end{aligned}\right\} \tag{1.4}$$

Since $H^1 \hookrightarrow L^6$ if $n = 3$ and $H^1 \hookrightarrow L^4$ if $n = 4$ it follows from (1.4) that strong solutions u belong to $C(\alpha, \beta; L^n)$ and satisfy $\nabla u \in L^2(\alpha, \beta; L^n)$. Note that $|\nabla u|^3$ and the product $|u|\,|\nabla u|\,|\nabla^2 u|$ are integrable over $(\alpha,\ \beta) \times \mathbb{R}^n$, if $n \leq 4$.

In the sequel $\bar{u} = (u_1, \ldots, u_{n-1})$ denotes the projection of the vector field $u(t, x) \in \mathbb{R}^n$ into the hyperplane V generated by the first $n - 1$ vectors of a fixed basis of $\mathbb{R}^n$. Note that, due to the rotational invariance of the Navier–Stokes equations, the above hyperplane can be chosen in the most appropriate way. $|B|$ denotes the n–dimensional measure of the set B.

Set

$$A(t, k) = \{x \in \mathbb{R}^n : |v(t, x)| \geq k\} \tag{1.5}$$

for each t in which $v(t)$ is defined and for each real positive k. In the sequel our main assumption is the following.

Hypothesis A. *We say that a vector field $v(t, x)$ satisfies the hypothesis* A *at τ, with respect to the positive constant Λ, if there is a positive constant ϵ_0 such that*

$$v \in L^\infty(\tau - \epsilon_0, \tau; L^n), \tag{1.6}$$

and a real nonnegative function $k(t)$ defined and square integrable on $(\tau - \epsilon_0, \tau)$ such that

$$\int_{A(t,k(t))} |v(t, x)|^n dx \leq \Lambda^n, \ \text{a.e. in} \ \ (\tau - \epsilon_0, \tau). \tag{1.7}$$

A main point here is the possibility of using suitably the free choice of the square integrable function $k(t)$ (in this regard see also [10], eq. (1.4)). The above hypothesis was introduced by us in reference [3]. In [3] we also proved the following result.

Proposition 1.1. *Assume that v satisfies (1.6), that v is left continuous in $(\tau - \epsilon_0, \tau]$ with respect to the weak topology in L^n and, moreover, that*

$$\limsup_{t \to \tau - 0} \|v(t)\|_n^n < \|v(\tau)\|_n^n + 4^{1-n}\Lambda^n. \tag{1.8}$$

Then the hypothesis A holds at τ with respect to the constant Λ. In this particular case, the square integrable function $k(t)$ is simply a suitable constant k defined for $t \in (\tau - k^{-1}, \tau]$.

For the proof of this result we refer the reader to the proof of the Proposition 2.1 in reference [3].

Our main result is the following.

Theorem A. *There is a positive constant $C(n)$, depending only on the dimension n, such that the following statement holds. Let u be a weak solution to the Navier–Stokes equations (1.1) in (a, b) and let $\tau \in (a, b]$. Assume that $\bar{u} = (u_1, \ldots, u_{n-1})$ satisfies the hypothesis A at τ with respect to the constant*

$$\Lambda = 4^{1-\frac{1}{n}} C(n)^{\frac{1}{n}} \nu$$

where C(n) is defined in equation (2.1) below. Then, there is an $\epsilon > 0$ such that u is a strong solution of (1.1) in $(\tau - \epsilon, \tau + \epsilon)$.

In particular the result holds if the assumption (1.7) on $\bar{u}$ is replaced by the assumption (1.8) on $\bar{u}$.

Note that Theorem A implies that u is smooth in $(\tau - \epsilon, \tau + \epsilon) \times \mathbb{R}^n$. We remark that the constant $C(n)$ can be easily estimated.

Corollary 1.1. *Let u be a weak solution to (1.1) in (a, b). Assume that $u(a) \in H^1$, that $\nabla \cdot u(a) = 0$, that*

$$\bar{u} \in L^\infty(a, b; L^n), \tag{1.9}$$

and that there is a real positive function $k(t)$, defined and square integrable in (a, b), such that

$$\int_{A(t,k(t))} |\bar{u}(t, x)|^n dx \le \Lambda^n, \ \text{a.e. in} \ \ (a, b). \tag{1.10}$$

Then u is a strong solution in (a, b). In particular u is a strong solution in (a,b) if, instead of (1.10), we assume that (1.8) holds for $\bar{u}$ at each $\tau \in (a, b]$.

If we assume that $u(a) \in L^2$ instead of $u(a) \in H^1$, the solution is strong in (a', b) for each $a' > a$.

The above result also covers Theorem 1 in reference [1]. In fact, this theorem states two main results. The first one shows, essentially, that there is an $\epsilon_0 > 0$ such that if the $L^\infty(a, b; L^n)$ norm of $\bar{u}$ is less than ϵ_0 then u is a strong solution in (a, b). Clearly the above assumption implies that (1.8) holds in $(a, b]$, hence the result is covered by corollary 1.1. The second one can be summarized in the following.

Corollary 1.2 (Bae and Choe)**.** *Assume that u is as in Corollary 1.1 with the assumptions* (1.9) *and* (1.10) *replaced by*

$$\bar{u} \in L^q(a, b; L^p)$$

where $2/q + n/p \leq 1,\ n < p$. Then u is a strong solution.

This is a classical result if $\bar{u}$ is replaced by u. Let us show that this result follows also as an immediate consequence of the above Corollary 1.1.

Proof. Set

$$I \equiv \int_{A(t,k(t))} |\bar{u}(t, x)|^n dx.$$

By Hölder's inequality

$$I \leq \|\bar{u}(t)\|_p^n \ |A(t, k(t))|^{\frac{p-n}{p}}.$$

Obviously,

$$\int_{\{|v| \geq k\}} |v|^s dx \geq k^s |\{|v| \geq k\}|,\ 1 \leq s < +\infty.$$

Hence

$$I \leq \|\bar{u}(t)\|_p^p k(t)^{n-p}.$$

It follows that the Corollary 1.1 applies if there is a square integrable function $k(t)$ such that

$$\|\bar{u}(t)\|_p^p k(t)^{n-p} \leq \Lambda^n, \quad \text{a.e. in} \ \ (a, b),$$

or equivalently, if

$$k(t) = \Lambda^{\frac{n}{n-p}} \|\bar{u}(t)\|_p^{\frac{p}{p-n}}$$

belongs to $L^2(a, b)$. This latter statement holds, since $\frac{p}{p-n} = \frac{q}{2}$. □

Other results follow from Theorem A as, for instance, the fact that u is smooth if $\bar{u}$ is of bounded variation with values in L^n. This follows from the classical result that establishes the existence of left and right limits for functions of bounded variation in (a, b). This, clearly, implies (1.8). This fact was first remarked in reference [12], Corollaries 2 and 3.

A crucial point in our proof is the estimate

$$\left| \int_{\mathbb{R}^n} \nabla[(u \cdot \nabla)u] \cdot \nabla u dx \right| \leq c_1(n) \int_{\mathbb{R}^n} |\bar{u}| \, |\nabla u| \, |\nabla^2 u| dx, \tag{1.11}$$

for a.a. $t \in (\alpha, \ \beta)$, where $n \leq 4$ and u in a strong solution in $(\alpha, \ \beta)$. This estimate is due to H. O. Bae and H. J. Choe, see [1]. For the reader's convenience we present here its proof. Note that the estimate (1.11) is obvious if $\bar{u}$ is replaced by u.

Theorem A, under the hypothesis A on u, was proved in reference [3] and, under the stronger hypothesis (1.8), in references [3] and [12]. In [12] the reader also finds other very interesting related results, references, and remarks.

In reference [3] we consider the case of a bounded domain Ω, however the proof works also if $\Omega = \mathbb{R}^n$. On the contrary, in the present case, in replacing u by $\bar{u}$ the absence of boundary conditions is crucial in proving the estimate (1.11).

In order to extend the proofs to the case $n \geq 5$ some more refined background results, together with suitable additional assumptions (since $H^1 \hookrightarrow L^n$ is false), are needed. It is useful, in particular, to resort to uniqueness of solutions in $L^\infty(a, b; L^n)$ and to existence of local regular solutions for initial data in L^n. These results are due to many authors. In particular we quote here [7], [8], [9], [11], [12], [13], [14], [17], [18], [19], [20] and references therein. More recent developments can be found also in [10]. In particular, in [10] it is shown that there is a positive constant λ (see [10], Eq. (1.4)) such that (in our notation) if

$$\left[\sup_{R \geq k(t)} R |A(t, R)|^{1/n} \right]^n \leq \lambda^n \quad \text{a.e. in} \quad (a, b), \tag{1.12}$$

for some square integrable function $k(t)$, then the solution u is smooth ([10], Theorem 3).

It is interesting to compare the assumption (1.12) with the hypothesis A in reference [3], since they lead to similar results. If $f(x)$ is a nonnegative function defined on a measurable set B then

$$\int_B f(x) dx = \int_0^{+\infty} |\{x \in B : f(x) > t\}| dt.$$

It readily follows that, for each t,

$$\int_{A(t,k(t))} |u(t,x)|^n dx = \int_{k(t)}^{+\infty} [R |A(t,R)|^{1/n}]^n \frac{dR}{R} + k(t)^n |A(t, k(t))|. \tag{1.13}$$

Hence, the hypothesis A is equivalent to assuming that the right hand side of (1.13) is bounded by Λ^n, which is related to (1.12).

Finally, we remark that the hyperplane V may depend on t. For each t, let $e_1(t), \ldots, e_n(t)$ be an orthonormal basis of $\mathbb{R}^n$ such that each $e_i(t)$ is a continuous function of t with respect to a fixed basis. Let

$$\begin{aligned} u(t,x) &= \sum_{i=1}^{n} u_i(t,x)\, e_i(t), \\ \tilde{u}(t,x) &= \sum_{i=1}^{n-1} u_i(t,x)\, e_i(t). \end{aligned} \tag{1.14}$$

The results proved in this paper also hold if $\overline{u}$ is replaced by $\tilde{u}$, as is easily seen. The continuity of the moving basis is useful when (1.8) is assumed for $\tilde{u}$ (in order to get the left continuity of $\tilde{u}$ with respect to the weak topology). This continuity assumption can be substantially weakened if one directly uses the assumption A for $\tilde{u}$.

2. Proof of Theorem A

In order to prove Theorem A, it is clearly sufficient to prove it in the following form.

Theorem 2.1. *Let u be a weak solution u of* (1.1) *in* $(\tau-\epsilon_0,\tau)$ *and a strong solution in* $(\tau-\epsilon_0,\tau')$*, for each* $\tau'<\tau$*. Assume, moreover, that* $\bar{u}$ *satisfies* (1.6) *and* (1.7)*. In* (1.7) Λ *is defined as in Theorem A and* $C(n)$ *is defined by*

$$C(n) = 1/(\sqrt{2}4^{1-\frac{1}{n}} c_0(n) c_1(n))^n, \tag{2.1}$$

where $c_0(n)$ *is the constant in equation* (2.10) *and* $c_1(n)$ *that in equation* (1.11)*.*

Under the above hypothesis u is a strong solution in $(\tau-\epsilon_0,\tau)$*. In particular the result holds if* (1.7) *is replaced by* (1.8)*.*

Proof. We set

$$|\nabla u|^2 = \sum_{i,k} |\partial_k u_i|^2, \qquad |\nabla^2 u|^2 = \sum_{i,j,k} |\partial^2_{kj} u_i|^2,$$

where summations, without otherwise stated, are taken from 1 to n. The symbol ∂_k means differentiation with respect to x_k, and $\partial_{kj} = \partial_k\partial_j$.

Differentiating both sides of equation (1.1) with respect to x_k, taking the scalar product with $\partial_k u$, adding over k and, finally, integrating by parts over $\mathbb{R}^n$, we show that

$$\frac{1}{2}\frac{1}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \nu \int_{\mathbb{R}^n} |\nabla^2 u|^2 dx = -\int_{\mathbb{R}^n} \nabla[(u\cdot\nabla)u]\cdot\nabla u dx \tag{2.2}$$

where obvious integrations by parts have been done. Since $\nabla\cdot u = 0$ it readily follows that

$$-\int_{\mathbb{R}^n} \nabla[(u\cdot\nabla)u]\cdot\nabla u dx = \sum_{i,j,k} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j) dx.$$

Next, we prove estimate (1.11). Following [1], we consider separately the three cases $i \neq n$; $i = n$ and $j \neq n$; $i = j = n$.

If $i \neq n$ one has

$$\int_{\mathbb{R}^n} (\partial_k u_i)(\partial_k u_j)(\partial_i u_j) dx = - \int_{\mathbb{R}^n} u_i \partial_k [(\partial_k u_j)(\partial_i u_j)] dx. \tag{2.3}$$

If $i = n$ but $j \neq n$,

$$\begin{aligned} \int_{\mathbb{R}^n} (\partial_k u_n)(\partial_i u_j)(\partial_k u_j) dx = &- \int_{\mathbb{R}^n} (\Delta u_n)(\partial_i u_j) u_j dx \\ &- \int_{\mathbb{R}^n} (\partial_k u_n)(\partial^2_{ik} u_j) u_j dx. \end{aligned} \tag{2.4}$$

Finally, since

$$\partial_n u_n = - \sum_{\ell \neq n} \partial_\ell u_\ell,$$

it readily follows that

$$\int_{\mathbb{R}^n} (\partial_k u_n)(\partial_n u_n)(\partial_k u_n) dx = 2 \sum_{\ell \neq n} \int_{\mathbb{R}^n} u_\ell (\partial_k u_n)(\partial^2_{k\ell} u_n) dx. \tag{2.5}$$

From (2.3), (2.4), (2.5) the estimate (1.11) follows. Note that the constant $c_1(n)$ can be easily estimated.

From (1.11) and (2.2) one gets

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla u|^2 + \nu \int_{\mathbb{R}^n} |\nabla^2 u|^2 dx \leq c_1(n) \int_{\mathbb{R}^n} |\bar{u}| \, |\nabla u| \, |\nabla^2 u| dx.$$

By Cauchy–Schwarz inequality we find

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \nu \int_{\mathbb{R}^n} |\nabla^2 u|^2 dx \leq c_1^2 \nu^{-1} \int_{\mathbb{R}^n} |\bar{u}|^2 \, |\nabla u|^2 \, dx \tag{2.6}$$

where $c_1 = c_1(n)$.

From now on k denotes a constant such that (1.7) holds for the function $\bar{u}$ when $t \in (\tau - \epsilon_0, \tau)$. From (2.6) we get

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \nu \int_{\mathbb{R}^n} |\nabla^2 u|^2 dx \leq \\ &\leq c_1^2 \nu^{-1} k^2(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx + c_1^2 \nu^{-1} \int_{A(t,k(t))} |\bar{u}|^2 \, |\nabla u|^2 dx. \end{aligned} \tag{2.7}$$

By Hölder's inequality

$$\int_{A(t,k)} |\bar{u}|^2 \, |\nabla u|^2 dx \leq \|\nabla u\|^2_{2^*} \left(\int_{A(t,k)} |\bar{u}|^n dx \right)^{2/n} \tag{2.8}$$

where $A(t,k) = A(t,k(t))$ and $2^* = 2n/(n-2)$. Since, by a well known Sobolev embedding theorem,

$$\|v\|_{2^*} \le n^{-(1/n)}[2(n-1)/(n-2)] \sum_i \|\partial_i v\|_2 \tag{2.9}$$

one gets

$$\|\nabla u\|_{2^*} \le c_0(n) \|\nabla^2 u\|_2. \tag{2.10}$$

From (2.7), (2.8) and (2.10) it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \nu \int_{\mathbb{R}^n} |\nabla^2 u|^2 dx \le$$

$$\le c_1^2 \nu^{-1} k^2(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx + c_1^2 \nu^{-1} c_0^2 \, \|\nabla^2 u\|_2^2 \left(\int_{A(t,k)} |\bar{u}|^n dx \right)^{2/n}.$$

By using (1.7) and (2.1) one has

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{\nu}{2} \int_{\mathbb{R}^n} |\nabla^2 u|^2 dx \le c_1^2 \nu^{-1} k^2(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx \tag{2.11}$$

for $t \in (\tau - \epsilon_0, \tau)$. Theorem 2.1 then readily follows by integration with respect to t for $n = 3$. If $n = 4$, from $u \in L^2(\tau - \epsilon_0, \tau; H^2)$ it follows that $u \in L^2(\tau - \epsilon_0, \tau; W^{1,4})$. Since, for $n = 4$, $W^{1,4} \hookrightarrow L^\infty$ is false, we are not allowed to a priori assume that u belongs to $L^q(\tau - \epsilon_0, \tau; L^p)$ with $2/q + n/p = 1$ (which is a well know criterion for regularity of solutions to the Navier–Stokes equations). However $u \in L^2(\tau - \epsilon_0, \tau; W^{1,n})$ is sufficient to guarantee the smoothness of solutions u, as proved by us (in a quite simple way) in reference [2]. □

The proof of Theorem A is now straightforward. First of all, it is sufficient to prove the thesis of the theorem with respect to $(\tau - \epsilon, \tau]$ since $u(\tau) \in H^1$ allows the continuation of the regular solution for $t > \tau$. Let $u(\tau - \epsilon') \in H^1$ for a fixed ϵ', $0 < \epsilon' < \epsilon_0$. Then u coincides on $[\tau - \epsilon', \tau')$ with the unique (local) strong solution with initial data $u(\tau - \epsilon')$, where $[\tau - \epsilon', \tau')$ is the maximal interval of existence of this strong solution. Theorem 2.1 shows that it can not be $\tau' < \tau$. □

References

[1] H.-O. Bae and H.-J. Choe, A regularity criterion for the Navier–Stokes equations (to appear).

[2] H. Beirão da Veiga, A new regularity class for the Navier–Stokes equations in $\mathbb{R}^n$, *Chin. Ann. of Math.* **16 B**, 4 (1995), 407–412.

[3] H. Beirão da Veiga, Remarks on the smoothness of the $L^\infty(0,T;L^3)$ solutions to the $3-D$ Navier–Stokes equations, *Port. Math.* **54** (1997), 381–391.

[4] H. Beirão da Veiga, Concerning the regularity of the solutions to the Navier–Stokes equations via the truncation method; part I, *Diff. Int. Eq.* **10** (1997), 1149–1156.

[5] H. Beirão da Veiga, Concerning the regularity of the solutions to the Navier–Stokes equations via the truncation method; part II, in: *Équations aux Dérivées Partielles et Applications*, Articles dédiés à J.-L. Lions á l'occasion de son 70 anniversaire; 127–138, Gauthier-Villars, Paris, 1998.

[6] D. Choe and H.–J. Choe, Regularity of solutions to the Navier–Stokes equations, *Electronic J. Diff. Eq.* Vol. 1999 (1999), No. 05, pp. 1–7.

[7] Y. Giga and T. Miyakawa, Solutions in L^r to the Navier–Stokes initial value problem, *Arch. Rat. Mech. Anal.* **89** (1985), 267–281.

[8] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system, *J. Diff. Eq.* **62** (1986), 182–212.

[9] T. Kato, Strong L^p–solution of the Navier–Stokes equations in $\mathbb{R}^m$, with applications to weak solutions, *Math. Z.* **187** (1984), 471–480.

[10] H. Kozono, Uniqueness and Regularity of weak solutions to the Navier–Stokes equations, *Lecture Notes in Num. Appl. Anal.* **16** (1998), 161–208.

[11] H. Kozono and H. Sohr, Remark on uniqueness of weak solutions to the Navier–Stokes equations, *Analysis* **16** (1996), 255–271.

[12] H. Kozono and H. Sohr, Regularity criterion on weak solutions to the Navier–Stokes equations, *Advances in Diff. Eq.* **2** (1997), 535–554.

[13] H. Kozono and M. Yamazaki, Local and global unique solvability of the Navier–Stokes exterior problem with Cauchy data in the space L^n, *Houston J. Math.* **21** (1995), 755–799.

[14] K. Masuda, Weak solutions of the Navier–Stokes equations, *Tohoku Math. J.* **36** (1984), 623–646.

[15] G. Prodi, Un teorema di unicità per le equazioni di Navier–Stokes, *Ann. Mat. Pura Appl.* **48** (1959), 173–182.

[16] S. Serrin, The initial value problem for the Navier–Stokes equations, in: R. E. Langer, Ed., *Nonlinear problems*, 69–98, Univ. Wisconsin Press, 1963.

[17] H. Sohr, Zur Regularitätstheorie der instationären Gleichungen von Navier–Stokes, *Math. Z.* **184** (1983), 359–375.

[18] H. Sohr and W. von Wahl, On the singular set and the uniqueness of weak solutions of the Navier–Stokes equations, *Manuscripta Math.* **49** (1984), 27–59.

[19] M. Struwe, On partial regularity results for weak solutions of the Navier–Stokes equations, *Comm. Pure Appl. Math.* **41** (1988), 437–458.

[20] W. von Wahl, Regularity of weak solutions of the Navier–Stokes equations, in: F. E. Browder Ed., *Proc. Symposia in Pure Mathematics* **45**, 497–503, Amer. Math. Soc., Providence, Rhode Island, 1986.

H. Beirão da Veiga
Dipartimento di Matematica Applicata
Università di Pisa
Via Bonanno Pisano 25/B
I–56126 Pisa
Italy
e-mail: bveiga@dma.unipi.it

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